Riemannian Geometry

1st Test - November 14, 2017

LMAC and MMA

Duration: 120 minutes

Show your calculations

1. Let Λ be the 3×3 matrix

$$\Lambda = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Consider the Lorentz group

$$O(1,2) = \left\{ A \in \mathcal{M}_{3\times 3} : A\Lambda A^T = \Lambda \right\}.$$

a) Verify that $O(1,2)$ is indeed a group. (1)

b) Prove that $O(1,2)$ is a submanifold of $\mathcal{M}_{3\times 3}$. Therefore, $O(1,2)$ is a Lie group. What is its dimension? (3)

c) Determine a basis for $T_I O(1,2)$ and the Lie algebra $\mathfrak{o}(1,2)$. (1)

d) Identify the tangent space to $O(1,2)$ at Λ, $T_\Lambda O(1,2)$. (1)

e) Compute the left-invariant vector field corresponding to (2)

$$B_1 := \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and $\exp(tB_1)$.

2. Consider the form ω defined on S^2 by contracting the volume form $dx \wedge dy \wedge dz$ on $\mathbb{R}^3$ with the unit normal to S^2, i.e.

$$\omega = \iota(x\partial_x + y\partial_y + z\partial_z)\, dx \wedge dy \wedge dz.$$

Let r be the parameterization of S^2 given by

$$r(\varphi, \theta) = (\cos\theta \sin\varphi, \sin\theta \sin\varphi, \cos\varphi),$$

and consider the orientation of S^2 induced by r.

a) Justify that ω is a volume form on S^2 and calculate it. (1)

b) Compute $\eta = r^*\omega$. (2)

c) Calculate $\int_{S^2} \omega$. (1)

d) Let $X = \partial_\varphi$. Compute $L_X \eta$. (2)

e) Using the definition of Lie derivative, justify that (2)

$$L_X \eta = r^* \left(L_{r_* X} \omega \right).$$

f) Let Ω be the subset of S^2 such that $0 < \varphi_0 < \varphi < \varphi_1 < \pi$. Calculate directly $\int_\Omega L_{r_* X} \omega$. (1)

g) Calculate directly $\int_{\partial\Omega} \iota(r_* X)\omega$, where $\partial\Omega$ has the orientation induced by the orientation of Ω. (1)

h) Can you relate the answers to the previous two questions? (2)