Riemannian Geometry

1^{st} Test - November 16, 2016
LMAC and MMA

Duration: 90 minutes
Show your calculations

1. Let M be the Lie group $SL(2) = \{A \in M_{2\times 2} : \det A = 1\}$.

a) Show that M is diffeomorphic to $\mathbb{R}^2 \times S^1$. Suggestion: Write $A \in M$ as (3)

$$A = A(p,q,r,s) = \begin{bmatrix} p+q & r+s \\ r-s & p-q \end{bmatrix},$$

and use the map

$$\varphi(q,r,\theta) = (p,q,r,s) = \left(\sqrt{1+q^2+r^2}\cos\theta, q, r, \sqrt{1+q^2+r^2}\sin\theta\right)$$

defined on $\mathbb{R}^2 \times S^1$.

b) Compute the matrices (1)

$$B := A_*\varphi_*\left(\frac{\partial}{\partial q}\right)_{(0,0,0)} \qquad \text{and} \qquad C := A_*\varphi_*\left(\frac{\partial}{\partial \theta}\right)_{(0,0,0)}.$$

(Note that we have $A_* = A$, because A is linear.)

c) Compute the Lie algebra sl(2) and T_AM. (2)

d) Compute X^B and X^C, the left invariant vector fields corresponding to B and C respectively, i.e. such that $(X^B)_I = B$ and $(X^C)_I = C$. (2)

e) Compute the integral curves of X^B and X^C through I. Suggestion: Calculate C^2. (2)

f) Compute $[X^B, X^C]_I$. (1)

g) Is it true that both $X^B = A_*\varphi_*\frac{\partial}{\partial q}$ and $X^C = A_*\varphi_*\frac{\partial}{\partial \theta}$? Explain. (1)

2. Consider the region

$$\Omega = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2-z^2 < 1 \ \wedge \ 0 < z < \sinh 1\},$$

and consider the piece of hyperboloid

$$S = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2-z^2 = 1 \ \wedge \ 0 < z < \sinh 1\}$$

with the parametrization

$$\varphi(\gamma,\theta) = (\cosh\gamma\cos\theta, -\cosh\gamma\sin\theta, \sinh\gamma)$$

(defined on an appropriate set). Moreover, in $\mathbb{R}^3$ define ω to be the 2-form

$$\omega = x\, dy \wedge dz - y\, dx \wedge dz.$$

a) Compute $\varphi^*\omega$. (2)
b) Compute $\int_S \omega$, where S has the orientation induced by φ. (1)
c) Compute $d\omega$. (1)
d) Compute $\int_\Omega d\omega$, where Ω has the canonical orientation of $\mathbb{R}^3$. (1)
e) Use Stokes' Theorem to relate the results of **b)** and **d)**. (2)
f) Compute $L_X(dx \wedge dy \wedge dz)$, where the vector field X is given by (1)

$$X(x, y, z) = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}.$$