# Riemannian Geometry, Spring 2021/22 <br> Instituto Superior Técnico 

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March 17, 2022 (due April 4)
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1. Consider the torus $T^{2}$ equal to the quotient of $[0,1]^{2}$ by the equivalence relation

$$
(x, y) \sim(x+1, y) \sim(x, y+1) .
$$

Give a parameterization of a neighborhood of $(0,0)$.
2. Consider the topological manifold $\mathbb{R} P^{2}$ equal to the quotient of $[0,1]^{2}$ by the equivalence relation

$$
(x, y) \sim(x+1,1-y) \sim(1-x, y+1) .
$$

Give a parameterization of a neighborhood of $(0,0)$.
3. Consider the paraboloid

$$
P=\left\{(x, y, z) \in \mathbb{R}^{3}: z=x^{2}+y^{2}\right\} .
$$

a) Show that the parameterizations $\phi: \mathbb{R}^{2} \rightarrow P$ and $\psi:(0, \infty) \times(0,2 \pi)$, defined by

$$
\phi(x, y)=\left(x, y, x^{2}+y^{2}\right), \quad \psi(r, \theta)=\left(r \cos \theta, r \sin \theta, r^{2}\right),
$$

are compatible.
b) Can you find a parameterization $\varphi$ of a neighborhood of the point $(0,0,0)$ of $P$ such that $\{\varphi, \psi\}$ is an atlas for $P$, with $\varphi$ incompatible with $\phi$ ?
4. Consider $\mathbb{R} P^{2}$, the set of lines through the origin in $\mathbb{R}^{3}$, parameterized by

$$
\varphi_{1}(y, z)=[1, y, z], \quad \varphi_{2}(x, z)=[x, 1, z], \quad \varphi_{3}(x, y)=[x, y, 1],
$$

and $f: \mathbb{R} P^{2} \rightarrow \mathbb{R} P^{2}$, defined by

$$
f([x, y, z])=[x, z, y] .
$$

Verify that $f$ is differentiable.
5. Consider the function $f: S^{n} \backslash\{N\} \rightarrow S^{n} \backslash\{N\}$ given in the stereographic projection from the north pole coordinates by

$$
\pi_{N} \circ f \circ \pi_{N}^{-1}(y)=y\|y\|^{\alpha} .
$$

This function can be extended to a continuous function from $S^{n}$ to $S^{n}$. For what values of $\alpha$ is the extended function once differentiable?
6. Consider the immersion $f: \mathbb{R}^{2} \rightarrow \mathbb{R}^{3}$, defined by

$$
f(x, y)=\left(2 y, x+y, x^{2}+y^{2}\right) .
$$

Give coordinates in $\mathbb{R}^{3}$ such that the representation of $f$ is the canonical immersion.
7. Consider the immersion $f:]-\frac{\pi}{2}, \frac{\pi}{2}\left[\rightarrow \mathbb{R}^{2}\right.$, defined by

$$
f(\theta)=(\cos \theta, \sin \theta)
$$

Give coordinates in $\left\{(x, y) \in \mathbb{R}^{2}:|y|<1\right\}$ such that the representation of $f$ is the canonical immersion.
8. Consider $f:\left\{(x, y, z) \in \mathbb{R}^{3}: x>0\right\} \rightarrow \mathbb{R}^{2}$, defined by

$$
f(x, y, z)=\left(y, x^{2}+y^{2}+z\right) .
$$

Give coordinates on which $f$ is the canonical projection.

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March 27, 2022 (due April 11)
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9. Let $0<r<R$. Consider the torus $T$ parameterized by

$$
(\theta, \varphi) \mapsto((R+r \sin \theta) \cos \varphi,(R+r \sin \theta) \sin \varphi, r \cos \theta),
$$

for $(\theta, \varphi) \in(0,2 \pi) \times(0,2 \pi)$, and the sphere $S^{2}$ parameterized by

$$
(\tilde{\theta}, \tilde{\varphi}) \mapsto(\sin \tilde{\theta} \cos \tilde{\varphi}, \sin \tilde{\theta} \sin \tilde{\varphi}, \cos \tilde{\theta})
$$

for $(\tilde{\theta}, \tilde{\varphi}) \in(0, \pi) \times(0,2 \pi)$. Consider the map $n: T \rightarrow S^{2}$, which assigns to each point of $T$ the exterior unit normal to the $T$ at that point. Give the local representation of $n$ in these coordinates.
10. Consider the map $(x, y, z) \mapsto\left(-\frac{y z}{x^{2}+y^{2}}, \frac{x z}{x^{2}+y^{2}}, \frac{\pi z^{2}}{x^{2}+y^{2}}+\arctan \frac{y}{x}\right)$, which sends $p=(1,1,1)$ to $q=\left(-\frac{1}{2}, \frac{1}{2}, \frac{3 \pi}{4}\right)$. Check that the map has rank 2 and find coordinates around $p$ and $q$ such that the map is locally represented by $\left(\xi^{1}, \xi^{2}, \xi^{3}\right) \rightarrow\left(\xi^{1}, \xi^{2}, 0\right)$.
11. Consider the vector fields defined on $\mathbb{R}^{2}$ by

$$
\begin{gathered}
X(x, y)=x \partial_{y} \\
Y(x, y)=(x+5 y) \partial_{x}+(5 x+y) \partial_{y}
\end{gathered}
$$

a) Compute the bracket $[X, Y]$.
b) Compute the flow of $X, \phi_{t}\left(x_{0}, y_{0}\right)$, and the flow of $Y, \psi_{t}\left(x_{0}, y_{0}\right)$.
c) Compute $\left(d \phi_{-t}\right)_{\phi_{t}\left(x_{0}, y_{0}\right)} Y_{\phi_{t}\left(x_{0}, y_{0}\right)}$.
d) Compute the Lie derivative $L_{X} Y$ using your answer to $\mathbf{c}$ ).
12. Consider the vector fields $X$ and $Y$ on the manifold $M$, with flows $\phi_{t}$ and $\psi_{t}$, respectively. Let $f$ be smooth function from $M$ to $\mathbb{R}, p$ belong to $M$, and

$$
c_{p}(t)=\psi_{-t} \circ \phi_{-t} \circ \psi_{t} \circ \phi_{t}(p) .
$$

a) Calculate $\frac{d}{d t} f\left(c_{p}(t)\right)$ and its value at $t=0$.
b) Calculate $\frac{d^{2}}{d t^{2}} f\left(c_{p}(t)\right)$. Verify that $\left.\frac{d^{2}}{d t^{2}} f\left(c_{p}(t)\right)\right|_{t=0}=2[X, Y] \cdot f$.

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April 10, 2022 (due May 9)
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13. Use 12.b) to show that $[A, B]=A B-B A$ for $A, B \in g l(n)$. (In this way you have calculated the bracket without using coordinates as we did in class.)
14. Show that $S U(n)$ is a submanifold of $U(n)$.
15. Let $J$ be the $(2 n) \times(2 n)$ matrix

$$
J=\left[\begin{array}{cc}
0 & I \\
-I & 0
\end{array}\right] .
$$

Consider the group

$$
S p(2 n, \mathbb{R})=\left\{A \in \mathcal{M}_{(2 n) \times(2 n)}: A J A^{T}=J\right\}
$$

a) Prove that $S p(2 n, \mathbb{R})$ is a submanifold of $\mathbb{R}^{(2 n)^{2}}$. What is the dimension of $S p(2 n, \mathbb{R})$ ?
b) Compute the Lie algebra $s p(2 n, \mathbb{R})$. Compute directly the dimension of $s p(2 n, \mathbb{R})$.
c) Check directly that for $B \in \operatorname{sp}(2 n, \mathbb{R})$, we have $e^{B} \in S p(2 n, \mathbb{R})$.
d) Compute the tangent space to $S p(2 n, \mathbb{R})$ at $J, T_{J} S p(2 n, \mathbb{R})$.
e) Can you guess a $B \in s p(2 n, \mathbb{R})$ for which $e^{B}=J$ ? (Note: $J^{2}=-I$.)
16. Consider the manifold $M=\mathbb{C}^{3} \backslash\{0\}$ and the Lie group $G=(\mathbb{C} \backslash\{0\}, \cdot)$ acting on $M$ by $\lambda \cdot\left(z^{1}, z^{2}, z^{3}\right)=\left(\lambda z^{1}, \lambda z^{2}, \lambda z^{3}\right)$. Argue that $M / G$ is a manifold and compute its dimension.

May 8, 2022 (due May 23)
17. Suppose $\omega \in \Omega^{2}(M)$ and $\eta \in \Omega^{1}(M)$. Check that

$$
(\omega \wedge \eta)(X, Y, Z)=\omega(X, Y) \eta(Z)+\omega(Y, Z) \eta(X)+\omega(Z, X) \eta(Y)
$$

18. Consider the 2-covariant tensor field $g \in \mathcal{T}^{2}\left(\mathbb{R}^{3}\right)$ defined by

$$
g=d x \otimes d x+d y \otimes d y+d z \otimes d z
$$

let $i$ be the inclusion ot $S^{2}$ in $\mathbb{R}^{3}$, and $r$ be the parameterization of $S^{2}$ given by

$$
r(\varphi, \theta)=(\cos \theta \sin \varphi, \sin \theta \sin \varphi, \cos \varphi)
$$

a) Calculate $h=r^{*} i^{*} g$.
b) Calculate $\int_{r^{-1}\left(S^{2}\right)} \sqrt{\operatorname{det} h} d \varphi \wedge d \theta$.
c) Consider the one form $\omega=\cos \varphi d \theta$ and the region $R$ of $S^{2}$ such that $0<\theta<\theta_{0}$ and $\varphi_{0}<\varphi<\frac{\pi}{2}$, where $\theta_{0}$ and $\varphi_{0}$ are fixed. Calculate directly $\int_{\partial R}\left(r^{-1}\right)^{*} \omega$ and $\int_{R} d\left(\left(r^{-1}\right)^{*} \omega\right)$, verifying the equality of Stokes' Theorem.
19. Consider the two torus $T^{2}$ with cartesian equation

$$
\left(\sqrt{x^{2}+y^{2}}-R\right)^{2}+z^{2}=r^{2}
$$

and the parameterization of a neighborhood of $T^{2}$ given by

$$
p(\rho, \theta, \varphi)=((R+\rho \cos \varphi) \cos \theta,(R+\rho \cos \varphi) \sin \theta, \rho \sin \varphi)
$$

a) Calculate $\omega:=p^{*}(d x \wedge d y \wedge d z)$.
b) Calculate $\eta:=\iota\left(\frac{\partial}{\partial \rho}\right) \omega$.
c) Calculate $\int_{p^{-1}\left(T^{2}\right)} \eta$.
20. As special cases of Stokes' Theorem proved in class, obtain
a) Stokes' Theorem as stated in Calculus courses.
b) The Divergence Theorem.

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May 13, 2022 (due May 30)
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21. Consider the vector field

$$
X=-\frac{y}{x^{2}+y^{2}} \partial_{x}+\frac{x}{x^{2}+y^{2}} \partial_{y} .
$$

and $\omega=d x \wedge d y$.
a) Compute $L_{X} \omega$ using

$$
L_{X}\left(\omega_{1} \wedge \omega_{2}\right)=\left(L_{X} \omega_{1}\right) \wedge \omega_{2}+\omega_{1} \wedge\left(L_{X} \omega_{2}\right)
$$

and

$$
d\left(L_{X} \omega\right)=L_{X}(d \omega) .
$$

b) Compute $L_{X} \omega$ using Cartan's formula.
c) Calculate $X$ and $\omega$ in polar coordinates.
d) Compute $L_{X} \omega$ using polar coordinates and $L_{X} \omega=\left.\frac{d}{d t} \phi_{t}^{*} \omega\right|_{t=0}$, where $\phi_{t}$ is the flow of $X$.

$$
\text { May 20, } 2022 \text { (due June 6) }
$$

22. Consider the hyperbolic plane $H^{2}=\left\{(x, y) \in \mathbb{R}^{2}: y>0\right\}$ with the metric

$$
d s^{2}=\frac{d x^{2}+d y^{2}}{y^{2}}
$$

and Levi-Civita connection.
a) Write the equation for the geodesics using the fact that they are the Euler-Lagrange equations for the Lagrangian

$$
L(x, v)=\frac{1}{2} g_{i j}(x) v^{i} v^{j} .
$$

b) Write down the nonzero Christoffel symbols using your answer to a).
c) Let $y_{0}>0$. Consider the line $c(x)=\left(x, y_{0}\right)$. Let $V$ be a vector field defined on $c$ which is parallel along $c$. Write down the differential equations satisfied by $V$.
d) Solve the equations in c) knowing that $V\left(0, y_{0}\right)=V_{0}^{x} \partial_{x}+V_{0}^{y} \partial_{y}$.
e) Indicate the covariant derivatives $\nabla_{\dot{c}} d x$ and $\nabla_{\dot{c}} d y$. Use them to verify that the covariant derivative of the metric along $c$ is zero.
f) Let $f: H^{2} \rightarrow \mathbb{R}$. Compute grad $f$.
g) What is the Riemannian volume element, $\omega$ on $H^{2}$ ? Calculate the Lie derivative

$$
L_{\operatorname{grad}_{f}} \omega .
$$

What is the divergence of the gradient of $f$, i.e., the Laplacian of $f$ ?
23. Consider a Riemannian manifold $M$ and a smooth function $f: M \rightarrow \mathbb{R}$ such that $\|\operatorname{grad} f\| \equiv 1$. By differentiating both sides of $\dot{x}=\operatorname{grad} f(x)$ with respect to $t$, show that the integral curves of $\operatorname{grad} f$ are geodesics.

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May 25, 2022 (due June 15)
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24. Let $G$ be a Lie group with Lie algebra $g$.
a) Suppose that $\beta: g \times g \rightarrow g$ is a bilinear map. Prove there exists a unique connection $\nabla=\nabla^{\beta}$ on $G$ which satisfies the following condition: if $v, w \in g$ and $X^{v}, X^{w}$ denote the corresponding left-invariant vector fields then

$$
\nabla_{X^{v}} X^{w}=X^{\beta(v, w)} .
$$

b) Prove that this connection is left-invariant in the sense that

$$
\left(L_{g}\right)_{\star} \nabla_{X} Y=\nabla_{\left(L_{g}\right)_{\star} X}\left(L_{g}\right)_{\star} Y, \quad \forall X, Y \in \mathcal{X}(G), g \in G
$$

Deduce that the parallel transport determined by this connection is left invariant in the sense that if $Y$ is parallel along a curve $c$ then $\left(L_{g}\right)_{\star} Y$ is parallel along $L_{g} \circ c$.
c) Prove that any connection $\nabla$ on $G$ determines a bilinear on map $g$ via

$$
\beta(v, w)=\left(\nabla_{X^{v}} X^{w}\right)_{e} .
$$

Hence, there is a bijective correspondence between bilinear maps from $g \times g$ to $g$ and left invariant connections on $G$.

June 2, 2022 (due June 20)
25. Consider the hyperbolic plane $H^{2}$ with the metric

$$
d s^{2}=\frac{d x^{2}+d y^{2}}{y^{2}}
$$

and Levi-Civita connection.
a) Using the results of 22.b), calculate $R\left(\partial_{x}, \partial_{y}\right) \partial_{x}$ and $R\left(\partial_{x}, \partial_{y}\right) \partial_{y}$. Write the Riemann tensor and the curvature tensor.
Example: $\nabla_{\partial_{x}} \nabla_{\partial_{y}} \partial_{x}=\nabla_{\partial_{x}}\left(\Gamma_{y x}^{x} \partial_{x}+\Gamma_{y x}^{y} \partial_{y}\right)=\ldots$.
b) Calculate the curvature of $H^{2}$.
26. Show that the curvature tensor of a 3-dimensional Riemannian manifold is entirely determined by its Ricci tensor.

$$
\text { June 10, } 2022 \text { (due June 24) }
$$

27. Consider the cylinder $C:=] 0, \infty\left[\times S^{1}\right.$ with metric

$$
d s^{2}=d r^{2}+\sinh ^{2} r d \theta^{2}
$$

a) Compute $\omega_{r}^{\theta}, \Omega_{r}^{\theta}$ and $R_{r \theta r \theta}$. What is the curvature of $C$ ?
b) Verify the equality of the Gauss-Bonnet Theorem,

$$
\int_{\Omega} K+\int_{\partial \Omega} k_{g}=2 \pi \sum_{i} I_{p_{i}},
$$

when $\Omega=] r_{0}, r_{1}\left[\times S^{1}\right.$.

