Riemannian Geometry, Spring 2021/22 Instituto Superior Técnico

March 17, 2022 (due April 4)

1. Consider the torus T^2 equal to the quotient of $[0,1]^2$ by the equivalence relation

$$(x, y) \sim (x+1, y) \sim (x, y+1).$$

Give a parameterization of a neighborhood of (0, 0).

2. Consider the topological manifold $\mathbb{R}P^2$ equal to the quotient of $[0, 1]^2$ by the equivalence relation

$$(x, y) \sim (x + 1, 1 - y) \sim (1 - x, y + 1).$$

Give a parameterization of a neighborhood of (0, 0).

3. Consider the paraboloid

$$P = \{(x, y, z) \in \mathbb{R}^3 : z = x^2 + y^2\}.$$

a) Show that the parameterizations $\phi : \mathbb{R}^2 \to P$ and $\psi : (0, \infty) \times (0, 2\pi)$, defined by

$$\phi(x,y) = (x, y, x^2 + y^2), \qquad \psi(r,\theta) = (r\cos\theta, r\sin\theta, r^2),$$

are compatible.

- **b)** Can you find a parameterization φ of a neighborhood of the point (0,0,0) of P such that $\{\varphi,\psi\}$ is an atlas for P, with φ incompatible with ϕ ?
- 4. Consider $\mathbb{R}P^2$, the set of lines through the origin in \mathbb{R}^3 , parameterized by

$$\varphi_1(y,z) = [1, y, z], \quad \varphi_2(x,z) = [x, 1, z], \quad \varphi_3(x,y) = [x, y, 1],$$

and $f : \mathbb{R}P^2 \to \mathbb{R}P^2$, defined by

$$f([x, y, z]) = [x, z, y].$$

Verify that f is differentiable.

5. Consider the function $f: S^n \setminus \{N\} \to S^n \setminus \{N\}$ given in the stereographic projection from the north pole coordinates by

$$\pi_N \circ f \circ \pi_N^{-1}(y) = y \|y\|^{\alpha}.$$

This function can be extended to a continuous function from S^n to S^n . For what values of α is the extended function once differentiable?

6. Consider the immersion $f : \mathbb{R}^2 \to \mathbb{R}^3$, defined by

$$f(x,y) = (2y, x+y, x^2 + y^2)$$

Give coordinates in \mathbb{R}^3 such that the representation of f is the canonical immersion.

7. Consider the immersion $f: \left] - \frac{\pi}{2}, \frac{\pi}{2} \right[\rightarrow \mathbb{R}^2$, defined by

$$f(\theta) = (\cos\theta, \sin\theta).$$

Give coordinates in $\{(x, y) \in \mathbb{R}^2 : |y| < 1\}$ such that the representation of f is the canonical immersion.

8. Consider $f : \{(x, y, z) \in \mathbb{R}^3 : x > 0\} \to \mathbb{R}^2$, defined by

$$f(x, y, z) = (y, x^2 + y^2 + z).$$

Give coordinates on which f is the canonical projection.

March 27, 2022 (due April 11)

9. Let 0 < r < R. Consider the torus T parameterized by

$$(\theta, \varphi) \mapsto ((R + r\sin\theta)\cos\varphi, (R + r\sin\theta)\sin\varphi, r\cos\theta),$$

for $(\theta, \varphi) \in (0, 2\pi) \times (0, 2\pi)$, and the sphere S^2 parameterized by

$$(\tilde{\theta}, \tilde{\varphi}) \mapsto (\sin \tilde{\theta} \cos \tilde{\varphi}, \sin \tilde{\theta} \sin \tilde{\varphi}, \cos \tilde{\theta}),$$

for $(\tilde{\theta}, \tilde{\varphi}) \in (0, \pi) \times (0, 2\pi)$. Consider the map $n : T \to S^2$, which assigns to each point of T the exterior unit normal to the T at that point. Give the local representation of n in these coordinates.

10. Consider the map $(x, y, z) \mapsto \left(-\frac{yz}{x^2+y^2}, \frac{xz}{x^2+y^2}, \frac{\pi z^2}{x^2+y^2} + \arctan \frac{y}{x}\right)$, which sends p = (1, 1, 1) to $q = \left(-\frac{1}{2}, \frac{1}{2}, \frac{3\pi}{4}\right)$. Check that the map has rank 2 and find coordinates around p and q such that the map is locally represented by $(\xi^1, \xi^2, \xi^3) \to (\xi^1, \xi^2, 0)$.

April 3, 2022 (due April 18)

11. Consider the vector fields defined on \mathbb{R}^2 by

$$X(x,y) = x\partial_y,$$

$$Y(x,y) = (x+5y)\partial_x + (5x+y)\partial_y.$$

- **a)** Compute the bracket [X, Y].
- **b)** Compute the flow of X, $\phi_t(x_0, y_0)$, and the flow of Y, $\psi_t(x_0, y_0)$.
- c) Compute $(d\phi_{-t})_{\phi_t(x_0,y_0)} Y_{\phi_t(x_0,y_0)}$.
- d) Compute the Lie derivative $L_X Y$ using your answer to c).

12. Consider the vector fields X and Y on the manifold M, with flows ϕ_t and ψ_t , respectively. Let f be smooth function from M to \mathbb{R} , p belong to M, and

$$c_p(t) = \psi_{-t} \circ \phi_{-t} \circ \psi_t \circ \phi_t(p).$$

- **a)** Calculate $\frac{d}{dt}f(c_p(t))$ and its value at t = 0.
- **b)** Calculate $\frac{d^2}{dt^2}f(c_p(t))$. Verify that $\frac{d^2}{dt^2}f(c_p(t))\Big|_{t=0} = 2[X,Y] \cdot f$.

13. Use **12.b**) to show that [A, B] = AB - BA for $A, B \in \mathfrak{gl}(n)$. (In this way you have calculated the bracket without using coordinates as we did in class.)

14. Show that SU(n) is a submanifold of U(n).

15. Let J be the $(2n) \times (2n)$ matrix

$$J = \left[\begin{array}{cc} 0 & I \\ -I & 0 \end{array} \right].$$

Consider the group

$$Sp(2n,\mathbb{R}) = \left\{ A \in \mathcal{M}_{(2n) \times (2n)} : AJA^T = J \right\}.$$

- a) Prove that $Sp(2n, \mathbb{R})$ is a submanifold of $\mathbb{R}^{(2n)^2}$. What is the dimension of $Sp(2n, \mathbb{R})$?
- **b)** Compute the Lie algebra $sp(2n, \mathbb{R})$. Compute directly the dimension of $sp(2n, \mathbb{R})$.
- c) Check directly that for $B \in sp(2n, \mathbb{R})$, we have $e^B \in Sp(2n, \mathbb{R})$.
- **d)** Compute the tangent space to $Sp(2n, \mathbb{R})$ at $J, T_J Sp(2n, \mathbb{R})$.
- e) Can you guess a $B \in sp(2n, \mathbb{R})$ for which $e^B = J$? (Note: $J^2 = -I$.)

16. Consider the manifold $M = \mathbb{C}^3 \setminus \{0\}$ and the Lie group $G = (\mathbb{C} \setminus \{0\}, \cdot)$ acting on M by $\lambda \cdot (z^1, z^2, z^3) = (\lambda z^1, \lambda z^2, \lambda z^3)$. Argue that M/G is a manifold and compute its dimension.

17. Suppose $\omega \in \Omega^2(M)$ and $\eta \in \Omega^1(M)$. Check that

$$(\omega \wedge \eta)(X, Y, Z) = \omega(X, Y)\eta(Z) + \omega(Y, Z)\eta(X) + \omega(Z, X)\eta(Y).$$

18. Consider the 2-covariant tensor field $g \in \mathcal{T}^2(\mathbb{R}^3)$ defined by

$$g = dx \otimes dx + dy \otimes dy + dz \otimes dz,$$

let i be the inclusion of S^2 in $\mathbb{R}^3,$ and r be the parameterization of S^2 given by

$$r(\varphi, \theta) = (\cos \theta \sin \varphi, \sin \theta \sin \varphi, \cos \varphi).$$

- a) Calculate $h = r^* i^* g$.
- **b)** Calculate $\int_{r^{-1}(S^2)} \sqrt[]{\det h} \, d\varphi \wedge d\theta$.
- c) Consider the one form $\omega = \cos \varphi \, d\theta$ and the region R of S^2 such that $0 < \theta < \theta_0$ and $\varphi_0 < \varphi < \frac{\pi}{2}$, where θ_0 and φ_0 are fixed. Calculate directly $\int_{\partial R} (r^{-1})^* \omega$ and $\int_R d((r^{-1})^* \omega)$, verifying the equality of Stokes' Theorem.
- **19.** Consider the two torus T^2 with cartesian equation

$$\left(\sqrt{x^2 + y^2} - R\right)^2 + z^2 = r^2$$

and the parameterization of a neighborhood of T^2 given by

 $p(\rho, \theta, \varphi) = ((R + \rho \cos \varphi) \cos \theta, (R + \rho \cos \varphi) \sin \theta, \rho \sin \varphi).$

- a) Calculate $\omega := p^*(dx \wedge dy \wedge dz)$.
- **b)** Calculate $\eta := \iota \left(\frac{\partial}{\partial \rho}\right) \omega$.
- c) Calculate $\int_{p^{-1}(T^2)} \eta$.
- 20. As special cases of Stokes' Theorem proved in class, obtain
 - a) Stokes' Theorem as stated in Calculus courses.
 - b) The Divergence Theorem.

May 13, 2022 (due May 30)

21. Consider the vector field

$$X = -\frac{y}{x^2 + y^2}\partial_x + \frac{x}{x^2 + y^2}\partial_y.$$

and $\omega = dx \wedge dy$.

a) Compute $L_X \omega$ using

$$L_X(\omega_1 \wedge \omega_2) = (L_X \omega_1) \wedge \omega_2 + \omega_1 \wedge (L_X \omega_2)$$

and

$$d(L_X\omega) = L_X(d\omega).$$

- **b)** Compute $L_X \omega$ using Cartan's formula.
- c) Calculate X and ω in polar coordinates.
- **d)** Compute $L_X \omega$ using polar coordinates and $L_X \omega = \frac{d}{dt} \phi_t^* \omega \big|_{t=0}$, where ϕ_t is the flow of X.

May 20, 2022 (due June 6)

22. Consider the hyperbolic plane $H^2 = \{(x, y) \in \mathbb{R}^2 : y > 0\}$ with the metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2}$$

and Levi-Civita connection.

a) Write the equation for the geodesics using the fact that they are the Euler-Lagrange equations for the Lagrangian

$$L(x,v) = \frac{1}{2}g_{ij}(x)v^i v^j.$$

- b) Write down the nonzero Christoffel symbols using your answer to a).
- c) Let $y_0 > 0$. Consider the line $c(x) = (x, y_0)$. Let V be a vector field defined on c which is parallel along c. Write down the differential equations satisfied by V.
- **d)** Solve the equations in **c)** knowing that $V(0, y_0) = V_0^x \partial_x + V_0^y \partial_y$.
- e) Indicate the covariant derivatives $\nabla_{\dot{c}} dx$ and $\nabla_{\dot{c}} dy$. Use them to verify that the covariant derivative of the metric along c is zero.
- **f)** Let $f: H^2 \to \mathbb{R}$. Compute grad f.

g) What is the Riemannian volume element, ω on H^2 ? Calculate the Lie derivative

 $L_{\operatorname{grad} f} \omega$.

What is the divergence of the gradient of f, i.e., the Laplacian of f?

23. Consider a Riemannian manifold M and a smooth function $f: M \to \mathbb{R}$ such that $\|\text{grad } f\| \equiv 1$. By differentiating both sides of $\dot{x} = \text{grad } f(x)$ with respect to t, show that the integral curves of grad f are geodesics.

24. Let G be a Lie group with Lie algebra g.

a) Suppose that $\beta : g \times g \to g$ is a bilinear map. Prove there exists a unique connection $\nabla = \nabla^{\beta}$ on G which satisfies the following condition: if $v, w \in g$ and X^v, X^w denote the corresponding left-invariant vector fields then

$$\nabla_{X^v} X^w = X^{\beta(v,w)}$$

b) Prove that this connection is left-invariant in the sense that

$$(L_g)_* \nabla_X Y = \nabla_{(L_g)_* X} (L_g)_* Y, \quad \forall X, Y \in \mathcal{X}(G), \ g \in G.$$

Deduce that the parallel transport determined by this connection is left invariant in the sense that if Y is parallel along a curve c then $(L_g)_*Y$ is parallel along $L_q \circ c$.

c) Prove that any connection ∇ on G determines a bilinear on map g via

$$\beta(v,w) = (\nabla_{X^v} X^w)_e.$$

Hence, there is a bijective correspondence between bilinear maps from $g \times g$ to g and left invariant connections on G.

25. Consider the hyperbolic plane H^2 with the metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2}$$

and Levi-Civita connection.

- **a)** Using the results of **22.b)**, calculate $R(\partial_x, \partial_y)\partial_x$ and $R(\partial_x, \partial_y)\partial_y$. Write the Riemann tensor and the curvature tensor. Example: $\nabla_{\partial_x} \nabla_{\partial_y} \partial_x = \nabla_{\partial_x} (\Gamma^x_{yx} \partial_x + \Gamma^y_{yx} \partial_y) = \dots$ **b)** Calculate the curvature of H^2 .

26. Show that the curvature tensor of a 3-dimensional Riemannian manifold is entirely determined by its Ricci tensor.

June 10, 2022 (due June 24)

27. Consider the cylinder $C :=]0, \infty[\times S^1$ with metric

$$ds^2 = dr^2 + \sinh^2 r \, d\theta^2$$

- **a)** Compute ω_r^{θ} , Ω_r^{θ} and $R_{r\theta r\theta}$. What is the curvature of C?
- b) Verify the equality of the Gauss-Bonnet Theorem,

$$\int_{\Omega} K + \int_{\partial \Omega} k_g = 2\pi \sum_i I_{p_i},$$

when $\Omega =]r_0, r_1[\times S^1]$.