Riemannian Geometry, Fall 2020/21
Instituto Superior Técnico

`October 2, 2020 (due October 19)`

1. Consider the torus T^2 equal to the quotient of $[0,1]^2$ by the equivalence relation

$$(x,y) \sim (x+1,y) \sim (x,y+1).$$

Give a parameterization of a neighborhood of $(0,0)$.

2. Consider the topological manifold $\mathbb{R}P^2$ equal to the quotient of $[0,1]^2$ by the equivalence relation

$$(x,y) \sim (x+1,1-y) \sim (1-x,y+1).$$

Give a parameterization of a neighborhood of $(0,0)$.

3. Consider the paraboloid

$$P = \{(x,y,z) \in \mathbb{R}^3 : z = x^2 + y^2\}.$$

a) Show that the parameterizations $\phi : \mathbb{R}^2 \to P$ and $\psi : (0,\infty) \times (0,2\pi)$, defined by

$$\phi(x,y) = (x,y,x^2+y^2), \qquad \psi(r,\theta) = (r\cos\theta, r\sin\theta, r^2),$$

are compatible.

b) Can you find a parameterization φ of a neighborhood of the point $(0,0,0)$ of P such that $\{\varphi,\psi\}$ form an atlas for P, with φ incompatible with ϕ?

4. Let $a > 0$. Consider the catenoid C parameterized by

$$(a\cosh v\,\cos u, a\cosh v\,\sin u, av)$$

and the helicoid H parameterized by

$$(w\cos z, w\sin z, az).$$

Consider the function $f : C \to H$, defined by

$$f(x,y,z) = \left(a\frac{x}{\sqrt{x^2+y^2}}\sinh\frac{z}{a}, a\frac{y}{\sqrt{x^2+y^2}}\sinh\frac{z}{a}, a\arctan\frac{y}{x} \right),$$

where $\arctan\frac{y}{x}$ denotes the argument of $x+iy$.

a) Check that, indeed, f has range in H.
b) Determine the representation of f in local coordinates. Is f differentiable?

5. Consider $\mathbb{R}P^2$ parameterized by

$$\varphi_1(y,z) = [1,y,z], \quad \varphi_2(x,z) = [x,1,z], \quad \varphi_3(x,y) = [x,y,1],$$

and $f : \mathbb{R}P^2 \to \mathbb{R}P^2$, defined by

$$f([x,y,z]) = [y,x,z].$$

Verify that f is differentiable.

6. Consider the immersion $f : \mathbb{R}^2 \to \mathbb{R}^3$, defined by

$$f(x,y) = (x, y + x^2, x^2 + y^2).$$

Give local coordinates around $p = (0,0)$ and $f(p) = (0,0,0)$ on which f is the canonical immersion.

7. Consider the submersion $f : \mathbb{R}^3 \to \mathbb{R}$, defined by

$$f(x,y,z) = x - xyz.$$

Give coordinates around $p = (0,0,0)$ and $f(p) = 0$ on which f is the canonical projection.

8. Let $0 < r < R$. Consider the torus T parameterized by

$$(\theta,\varphi) \mapsto ((R + r\sin\theta)\cos\varphi, (R + r\sin\theta)\sin\varphi, r\cos\theta),$$

for $(\theta,\varphi) \in (0,2\pi) \times (0,2\pi)$, and the sphere S^2 parameterized by

$$(\tilde\theta,\tilde\varphi) \mapsto (\sin\tilde\theta\cos\tilde\varphi, \sin\tilde\theta\sin\tilde\varphi, \cos\tilde\theta),$$

for $(\tilde\theta,\tilde\varphi) \in (0,\pi) \times (0,2\pi)$. Consider the map $n : T \to S^2$, which assigns to each point of T the exterior unit normal to the T at that point. Give the local representation of n in these coordinates.

`October 8, 2020 (due October 26)`

9. Consider the vector fields defined on $\mathbb{R}^2$ by

$$X(x,y) = (x+y)\partial_y,$$

$$Y(x,y) = (-2x+y)\partial_x + (x-2y)\partial_y.$$

a) Compute the bracket $[X, Y]$.
b) Compute the flow of X, $\phi_t(x_0, y_0)$, and the flow of Y, $\psi_t(x_0, y_0)$.
c) Compute $(d\phi_{-t})_{\phi_t(x_0,y_0)} Y_{\phi_t(x_0,y_0)}$.
d) Compute the Lie derivative $L_X Y$ using your answer to **c)**.

10. Consider the vector fields X and Y on the manifold M, with flows ϕ_t and ψ_t, respectively. Let f be smooth function from M to $\mathbb{R}$, p belong to M, and

$$c_p(t) = \psi_{-t} \circ \phi_{-t} \circ \psi_t \circ \phi_t(p).$$

a) Calculate $\frac{d}{dt} f(c_p(t))$ and its value at $t = 0$.
b) Calculate $\frac{d^2}{dt^2} f(c_p(t))$. Verify that $\frac{d^2}{dt^2} f(c_p(t))\Big|_{t=0} = 2[X, Y] \cdot f$.

`October 15, 2020 (due November 2)`

11. Use **10.b)** to show that $[A, B] = AB - BA$ for $A, B \in \mathfrak{gl}(n)$. (In this way you have calculated the bracket without using coordinates as we did in class.)

12. Show that $SU(n)$ is a submanifold of $U(n)$.

13. Let J be the $(2n) \times (2n)$ matrix

$$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}.$$

Consider the group

$$Sp(2n, \mathbb{R}) = \left\{ A \in \mathcal{M}_{(2n)\times(2n)} : AJA^T = J \right\}.$$

a) Prove that $Sp(2n, \mathbb{R})$ is a submanifold of $\mathbb{R}^{(2n)^2}$. What is the dimension of $Sp(2n, \mathbb{R})$?
b) Compute the Lie algebra $\mathfrak{sp}(2n, \mathbb{R})$. Compute directly the dimension of $\mathfrak{sp}(2n, \mathbb{R})$.
c) Check directly that for $B \in \mathfrak{sp}(2n, \mathbb{R})$, we have $e^B \in Sp(2n, \mathbb{R})$.
d) Compute the tangent space to $Sp(2n, \mathbb{R})$ at J, $T_J Sp(2n, \mathbb{R})$.
e) Can you guess a $B \in \mathfrak{sp}(2n, \mathbb{R})$ for which $e^B = J$? (Note: $J^2 = -I$.)

14. Consider the manifold $M = \mathbb{C}^3 \setminus \{0\}$ and the Lie group $G = (\mathbb{C} \setminus \{0\}, \cdot)$ acting on M by $\lambda\cdot(z^1, z^2, z^3) = (\lambda z^1, \lambda z^2, \lambda z^3)$. Argue that M/G is a manifold and compute its dimension.

15. Construct a two sheeted covering of the Klein bottle [given by the quotient of $[0,1]^2$ by the equivalence relation

$$(x,y) \sim (x+1,y) \sim (1-x,y+1)]$$

by the Torus T^2 [given by the quotient of $[0,1]^2$ by the equivalence relation

$$(x,y) \sim (x+1,y) \sim (x,y+1)].$$

`October 27, 2020 (due November 10)`

16. Suppose φ and $\psi : G \to H$ are two homomorphisms of Lie groups, with G connected, such that the vector fields $\varphi_* X$ and $\psi_* X$ are equal, for all X. Show that $\varphi = \psi$.

17.

a) Expand $dx^1 \wedge dx^2 \wedge dx^3$ using tensor products.
b) Let u, v, w belong to $T\mathbb{R}^5$. Write $dx^2 \wedge dx^3 \wedge dx^5\,(u,v,w)$ as a determinant.
c) Let X belong to $T\mathbb{R}^5$. Simplify $\iota(X)\, dx^1 \wedge dx^3 \wedge dx^5$.

`October 31, 2020 (due November 16)`

18. Consider the 2-covariant tensor field $g \in \mathcal{T}^2(\mathbb{R}^3)$ defined by

$$g = dx \otimes dx + dy \otimes dy + dz \otimes dz,$$

let i be the inclusion ot S^2 in $\mathbb{R}^3$, and r be the parameterization of S^2 given by

$$r(\varphi, \theta) = (\cos\theta \sin\varphi, \sin\theta \sin\varphi, \cos\varphi).$$

a) Calculate $h = r^* i^* g$.
b) Calculate $\int_{r^{-1}(S^2)} \sqrt{\det h}\, d\varphi \wedge d\theta$.
c) Consider the one form $\omega = \cos\varphi\, d\theta$ and the region R of S^2 such that $0 < \theta < \theta_0$ and $\varphi_0 < \varphi < \frac{\pi}{2}$, where θ_0 and φ_0 are fixed. Calculate directly $\int_{\partial R} (r^{-1})^* \omega$ and $\int_R d((r^{-1})^* \omega)$, verifying the equality of Stokes' Theorem.

19. Consider the two torus T^2 with cartesian equation

$$\left(\sqrt{x^2+y^2} - R\right)^2 + z^2 = r^2$$

and the parameterization of a neighborhood of T^2 given by

$$p(\rho, \theta, \varphi) = ((R + \rho\cos\varphi)\cos\theta, (R + \rho\cos\varphi)\sin\theta, \rho\sin\varphi).$$

a) Calculate $\omega := p^*(dx \wedge dy \wedge dz)$.
b) Calculate $\eta := \iota\left(\frac{\partial}{\partial\rho}\right)\omega$.
c) Calculate $\int_{p^{-1}(T^2)} \eta$.

`November 9, 2020 (due November 24)`

20. Consider the vector field

$$X = -\frac{y}{x^2+y^2}\partial_x + \frac{x}{x^2+y^2}\partial_y.$$

and $\omega = dx \wedge dy$.

a) Compute $L_X\omega$ using

$$L_X(\omega_1 \wedge \omega_2) = (L_X\omega_1) \wedge \omega_2 + \omega_1 \wedge (L_X\omega_2)$$

and

$$d(L_X\omega) = L_X(d\omega).$$

b) Compute $L_X\omega$ using Cartan's formula.
c) Calculate X and ω in polar coordinates.
d) Compute $L_X\omega$ using polar coordinates and $L_X\omega = \frac{d}{dt}\phi_t^*\omega\big|_{t=0}$, where ϕ_t is the flow of X.

`November 11, 2020 (due November 30)`

21. Consider the hyperbolic plane $H^2 = \{(x,y) \in \mathbb{R}^2 : y > 0\}$ with the metric

$$ds^2 = \frac{dx^2+dy^2}{y^2}$$

and Levi-Civita connection.

a) Write the equation for the geodesics.
b) What are the nonzero Christoffel symbols?
c) Let $y_0 > 0$. Consider the line $c(x) = (x, y_0)$. Let V be a vector field defined on c which is parallel along c. Write down the differential equations satisfied by V.
d) Solve the equations in **c)** knowing that $V(0,y_0) = V_0^x\partial_x + V_0^y\partial_y$.
e) Indicate the covariant derivatives $\nabla_{\dot c}\, dx$ and $\nabla_{\dot c}\, dy$. Use them to compute the covariant derivative of the metric along c.

22. Consider the hyperbolic plane H^2 with the metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2}$$

and Levi-Civita connection.

a) Let $f : H^2 \to \mathbb{R}$. Compute $\operatorname{grad} f$.

b) What is the Riemannian volume element, ω on H^2? Calculate the Lie derivative

$$L_{\operatorname{grad} f}\, \omega.$$

What is the divergence of the gradient of f, i.e., the Laplacian of f?

`November 25, 2020 (due December 10)`

23. Consider a Riemannian manifold M and a smooth function $f : M \to \mathbb{R}$ such that $\|\operatorname{grad} f\| \equiv 1$. By differentiating both sides of $\dot{x} = \operatorname{grad} f(x)$ with respect to t, show that the integral curves of $\operatorname{grad} f$ are geodesics.

24. Let G be a Lie group with Lie algebra $\mathfrak{g}$.

a) Suppose that $\beta : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is a bilinear map. Prove there exists a unique connection ∇^β on G which satisfies the following condition: if $v, w \in \mathfrak{g}$ and X^v, X^w denote the corresponding left-invariant vector fields then

$$\nabla_{X^v} X^w = X^{\beta(v,w)}.$$

b) Prove that this connection is left-invariant in the sense that

$$(L_g)_\star \nabla_X Y = \nabla_{(L_g)_\star X} (L_g)_\star Y, \quad \forall X, Y \in \mathcal{X}(G),\ g \in G.$$

Deduce that the parallel transport determined by this connection is left invariant in the sense that if Y is parallel along a curve c then $(L_g)_\star Y$ is parallel along $L_g \circ c$.

c) Prove that any connection ∇ on G determines a bilinear on map $\mathfrak{g}$ via

$$\beta(v, w) = \left(\nabla_{X^v} X^w\right)_e.$$

Hence, there is a bijective correspondence between bilinear maps from $\mathfrak{g} \times \mathfrak{g}$ to $\mathfrak{g}$ and left invariant connections on G.

`December 8, 2020 (due December 22)`

25. Consider the hyperbolic plane H^2 with the metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2}$$

and Levi-Civita connection.

a) Using the results of **21.b)**, calculate $R(\partial_x, \partial_y)\partial_x$ and $R(\partial_x, \partial_y)\partial_y$. Write the Riemann tensor and the curvature tensor.
b) Calculate the curvature of H^2.

26. Show that the curvature tensor of a 3-dimensional Riemannian manifold is entirely determined by its Ricci tensor.

27. Consider the cylinder $C :=]0, \infty[\times S^1$ with metric

$$ds^2 = dr^2 + \sinh^2 r\, d\theta^2.$$

a) Compute ω_r^θ, Ω_r^θ and $R_{r\theta r\theta}$. What is the curvature of C?
b) Verify the equality of the Gauss-Bonnet Theorem,

$$\int_\Omega K + \int_{\partial\Omega} k_g = 2\pi \sum_i I_{p_i},$$

when $\Omega =]r_0, r_1[\times S^1$.

`December 16, 2020 (not to hand in)`

28. Consider the surface in $\mathbb{R}^3$ represented parametrically by

$$(u, v) \mapsto f(u, v) = (f^1(u, v), f^2(u, v), f^3(u, v))$$

with the metric induced by the Euclidean metric, $E_1 = \frac{f_u}{\|f_u\|}$, $E_2 = \frac{f_v}{\|f_v\|}$ and $N = \frac{E_1 \times E_2}{\|E_1 \times E_2\|}$.

a) Check that

$$\begin{aligned} p &:= (-\nabla_{E_1} N, E_1) &=& \frac{1}{\|f_u\|^2}(N, f_{uu}) &=:& \frac{l}{\|f_u\|^2}, \\ q &:= (-\nabla_{E_1} N, E_2) &=& \frac{1}{\|f_u\|\|f_v\|}(N, f_{uv}) &=& \frac{m}{\|f_u\|\|f_v\|}, \\ r &:= (-\nabla_{E_2} N, E_2) &=& \frac{1}{\|f_v\|^2}(N, f_{vv}) &=& \frac{n}{\|f_v\|^2}. \end{aligned}$$

b) Let $\Delta = 1 - (E_1, E_2)^2$. Check that the second fundamental form of the surface in the basis (E_1, E_2) is

$$II(u,v) = \frac{1}{\Delta}\begin{bmatrix} 1 & -(E_1, E_2) \\ -(E_1, E_2) & 1 \end{bmatrix}\begin{bmatrix} p & q \\ q & r \end{bmatrix}.$$

c) Check that the curvature of the surface is

$$K = \frac{ln - m^2}{\|f_u\|^2\|f_v\|^2 - (f_u, f_v)^2}.$$