Riemannian Geometry, Fall 2019/20
Instituto Superior Técnico

September 26, 2019

1. Consider the torus T^2 equal to the quotient of $[0,1]^2$ by the equivalence relation
$$(x,y) \sim (x+1,y) \sim (x,y+1).$$

a) For $p = (x,0)$, with $0 < x < 1$, give a parameterization of a neighborhood of p.
b) Give a parameterization of a neighborhood of $(0,0)$.

2. Consider the topological manifold $\mathbb{R}P^2$ equal to the quotient of $[0,1]^2$ by the equivalence relation
$$(x,y) \sim (x+1,1-y) \sim (1-x,y+1).$$
Give a parameterization of a neighborhood of $(0,0)$.

3. Consider the paraboloid
$$P = \{(x,y,z) \in \mathbb{R}^3 : z = x^2 + y^2\}.$$

a) Show that the functions ϕ and ψ, defined by
$$\phi(x,y) = (x,y,x^2+y^2), \qquad \psi(r,\theta) = (r\cos\theta, r\sin\theta, r^2),$$
define the same differential structure on $P \setminus \{(0,0,0)\}$.
b) Does there exist a parameterization φ of a neighborhood in P of $(0,0,0)$ such that the parameterizations φ together with ψ form an atlas for P, with φ incompatible with ϕ?

4. Let $a > 0$. Consider the catenoid C parameterized by
$$(a\cosh v \, \cos u, a\cosh v \, \sin u, av)$$
and the helicoid H parameterized by
$$(w\cos z, w\sin z, az).$$
Consider the function $f : C \to H$, defined by
$$f(x,y,z) = \left(a\frac{x}{\sqrt{x^2+y^2}} \sinh\frac{z}{a}, a\frac{y}{\sqrt{x^2+y^2}} \sinh\frac{z}{a}, a\arctan\frac{y}{x} \right),$$
where $\arctan\frac{y}{x}$ denotes the argument of $x + iy$.

a) Check that, indeed, f has range in H.
b) Determine the representation of f in local coordinates. Is f differentiable?

5. Consider $\mathbb{R}P^2$ parameterized by

$$\varphi_1(y,z) = [1,y,z], \quad \varphi_2(x,z) = [x,1,z], \quad \varphi_3(x,y) = [x,y,1],$$

and $f : \mathbb{R}P^2 \to \mathbb{R}P^2$, defined by

$$f([x,y,z]) = [y,x,z].$$

Verify that f is differentiable.

October 9, 2019

6. Consider the immersion $f : \mathbb{R}^2 \to \mathbb{R}^3$, defined by

$$f(x,y) = (x,y,x^2+y^2).$$

a) Give local coordinates around $p = (0,0)$ and $f(p) = (0,0,0)$ on which f is the canonical immersion.
b) Give local coordinates around $p = (1,2)$ and $f(p) = (1,2,5)$ on which f is the canonical immersion, with p having coordinates $(0,0)$ and $f(p)$ having coordinates $(0,0,0)$.

7. Consider the submersion $f : \mathbb{R}^3 \to \mathbb{R}$, defined by

$$f(x,y,z) = x - yz.$$

Give coordinates around $p = (0,0,0)$ and $f(p) = 0$ on which f is the canonical projection.

8. Consider the vector fields defined on $\mathbb{R}^2$ by

$$X(x,y) = (x+y)\partial_y,$$

$$Y(x,y) = (-2x+y)\partial_x + (x-2y)\partial_y.$$

a) Compute the bracket $[X,Y]$.
b) Compute the flow of X, $\phi_t(x_0,y_0)$, and the flow of Y, $\psi_t(x_0,y_0)$.
c) Compute $(d\phi_{-t})_{\phi_t(x_0,y_0)} Y_{\phi_t(x_0,y_0)}$.
d) Compute the Lie derivative $L_X Y$ using your answer to **c)**.

9. Consider the vector fields defined on $\mathbb{R}$ by

$$X(x) = x\partial_x, \qquad Y(x) = x^2\partial_x.$$

a) Compute the bracket $[X, Y]$.
b) Compute the flow of X, $\phi_t(x_0)$, and the flow of Y, $\psi_t(x_0)$.
c) Compute $\eta_t(x_0) = (\psi_{-t} \circ \phi_{-t} \circ \psi_t \circ \phi_t)(x_0)$.
d) Verify that

$$\frac{1}{2}\frac{d^2}{dt^2}\eta_t(x_0)\bigg|_{t=0} = [X, Y](x_0).$$

10. Let J be the $(2n) \times (2n)$ matrix

$$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}.$$

Consider the group

$$Sp(2n, \mathbb{R}) = \left\{A \in \mathcal{M}_{(2n)\times(2n)} : AJA^T = J\right\}.$$

a) Prove that $Sp(2n, \mathbb{R})$ is a submanifold of $\mathbb{R}^{(2n)^2}$. What is the dimension of $Sp(2n, \mathbb{R})$?
b) Compute the Lie algebra $\mathfrak{sp}(2n, \mathbb{R})$. Compute directly the dimension of $\mathfrak{sp}(2n, \mathbb{R})$.
c) Check directly that for $B \in \mathfrak{sp}(2n, \mathbb{R})$, we have $e^B \in Sp(2n, \mathbb{R})$.
d) Compute the tangent space to $Sp(2n, \mathbb{R})$ at J, $T_J Sp(2n, \mathbb{R})$.
e) Can you guess a $B \in \mathfrak{sp}(2n, \mathbb{R})$ for which $e^B = J$? (Note: $J^2 = -I$.)

`October 23, 2019`

11. Consider the manifold $M = \mathbb{C}^3 \setminus \{0\}$ and the Lie group $G = (\mathbb{C} \setminus \{0\}, \cdot)$ acting on M by $\lambda\cdot(z^1, z^2, z^3) = (\lambda z^1, \lambda z^2, \lambda z^3)$. Argue that M/G is a manifold and compute its dimension.

12. Construct a two sheeted covering of the Klein bottle [given by the quotient of $[0, 1]^2$ by the equivalence relation

$$(x, y) \sim (x + 1, y) \sim (1 - x, y + 1)]$$

by the Torus T^2 [given by the quotient of $[0, 1]^2$ by the equivalence relation

$$(x, y) \sim (x + 1, y) \sim (x, y + 1)].$$

13. Let V belong to $\mathfrak{so}(3)$.

a) Compute X_A^V, the left invariant vector field generated by V at $A \in SO(3)$.
b) Characterize the tangent space to $SO(3)$ at A, and verify directly that X_A^V belongs to $T_A SO(3)$.
c) Let $\psi_t = F(\,\cdot\,, t)$ for F the flow of X^V. What is $\psi_t(A)$?

14. Suppose φ and $\psi : G \to H$ are two homomorphisms of Lie groups such that $\varphi_* = \psi_*$, with G connected. Show that $\varphi = \psi$.

15.

a) Compute $\mathrm{Alt}\,(dx^1 \otimes dx^2)$ and $\mathrm{Alt}\,(dx^1 \otimes dx^2 \otimes dx^3)$.
b) Expand $dx^1 \wedge dx^2 \wedge dx^3$ using tensor products.
c) Let u, v, w belong to $\mathbb{R}^5$. Write $dx^2 \wedge dx^3 \wedge dx^5\,(u, v, w)$ as a determinant.
d) Let X belong to $\mathbb{R}^5$. Simplify $\iota(X)\, dx^1 \wedge dx^3 \wedge dx^5$.

`November 8, 2019`

16. Consider the 2-covariant tensor field $g \in \mathcal{T}^2(\mathbb{R}^3)$ defined by

$$g = dx \otimes dx + dy \otimes dy + dz \otimes dz,$$

let i be the inclusion ot S^2 in $\mathbb{R}^3$, and r be the parameterization of S^2 given by

$$r(\varphi, \theta) = (\cos\theta \sin\varphi, \sin\theta \sin\varphi, \cos\varphi).$$

a) Calculate $h = r^* i^* g$.
b) Calculate $\int_{r^{-1}(S^2)} \sqrt{\det h}\, d\varphi \wedge d\theta$.
c) Consider the one form $\omega = \cos\varphi\, d\theta$ and the region R of S^2 such that $0 < \theta < \theta_0$ and $\varphi_0 < \varphi < \frac{\pi}{2}$, where θ_0 and φ_0 are fixed. Calculate directly $\int_{\partial R} (r^{-1})^* \omega$ and $\int_R d((r^{-1})^* \omega)$, verifying the equality of Stokes' Theorem.

17. Consider the two torus T^2 with cartesian equation

$$\left(\sqrt{x^2 + y^2} - R\right)^2 + z^2 = r^2$$

and the parameterization of a neighborhood of T^2 given by

$$p(\rho, \theta, \varphi) = ((R + \rho\cos\varphi)\cos\theta, (R + \rho\cos\varphi)\sin\theta, \rho\sin\varphi).$$

a) Calculate $\omega := p^*(dx \wedge dy \wedge dz)$.
b) Calculate $\eta := \iota\left(\frac{\partial}{\partial\rho}\right)\omega$.

c) Calculate $\int_{p^{-1}(T^2)} \eta$.

18. Consider the vector field

$$X = -\frac{y}{x^2+y^2}\partial_x + \frac{x}{x^2+y^2}\partial_y.$$

and $\omega = dx \wedge dy$.

a) Compute $L_X\omega$ using

$$L_X(\omega_1 \wedge \omega_2) = (L_X\omega_1) \wedge \omega_2 + \omega_1 \wedge (L_X\omega_2)$$

and

$$d(L_X\omega) = L_X(d\omega).$$

b) Compute $L_X\omega$ using Cartan's formula.
c) Calculate X and ω in polar coordinates.
d) Compute $L_X\omega$ using polar coordinates and $L_X\omega = \frac{d}{dt}\phi_t^*\omega\big|_{t=0}$, where ϕ_t is the flow of X.

19. Let $\alpha > 1$. Verify directly the equality of the Divergence Theorem,

$$\int_\Omega \operatorname{div}\left(p_*(\rho^\alpha\partial_\rho)\right) dx \wedge dy \wedge dz = \int_{T^2} (p_*(\rho^\alpha\partial_\rho)) \cdot \nu\, \overline{\eta},$$

where Ω is the interior of the torus T^2 of exercise **17**, p and ρ are as in the same exercise, ν is the unit outer normal to the torus, and $\overline{\eta} = \iota(\nu)\, dx \wedge dy \wedge dz$ is the area form on T^2.

`November 22, 2019`

20. Consider the sphere S^2 with the metric induced by the euclidean metric on $\mathbb{R}^3$, i.e., using the parameterization $r(\theta,\varphi) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$, with metric

$$ds^2 = d\theta^2 + \sin^2\theta\, d\varphi^2,$$

and with Levi-Civita connection.

a) Write the equation for the geodesics.
b) What are the nonzero Christoffel symbols?
c) Let $0 \le \theta_0 < \frac{\pi}{2}$. Consider the parallel $c(\varphi) = (\theta_0, \varphi)$. Let V be a vector field defined on c which is parallel along c. Write down the differential equations satisfied by V.

d) Solve the equations in **c)** knowing that $V(\theta_0, 0) = V_0^\theta \partial_\theta + V_0^\varphi \partial_\varphi$. In particular, compute $V(\theta_0, 2\pi)$.
e) Indicate the covariant derivatives $\nabla_{\dot{c}}\, d\theta$ and $\nabla_{\dot{c}}\, d\varphi$. Use them to compute the covariant derivative of the metric along c.

21. Consider the hyperbolic plane $H^2 = \{(x, y) \in \mathbb{R}^2 : y > 0\}$ with the metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2}$$

and Levi-Civita connection.

a) Write the equation for the geodesics.
b) What are the nonzero Christoffel symbols?
c) Let $y_0 > 0$. Consider the line $c(x) = (x, y_0)$. Let V be a vector field defined on c which is parallel along c. Write down the differential equations satisfied by V.
d) Solve the equations in **c)** knowing that $V(0, y_0) = V_0^x \partial_x + V_0^y \partial_y$.
e) Indicate the covariant derivatives $\nabla_{\dot{c}}\, dx$ and $\nabla_{\dot{c}}\, dy$. Use them to compute the covariant derivative of the metric along c.

22. Consider the hyperbolic plane H^2 with the metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2}$$

and Levi-Civita connection.

a) Let $f : H^2 \to \mathbb{R}$. Compute $\operatorname{grad} f$.
b) What is the Riemannian volume element, ω on H^2? Calculate the Lie derivative

$$L_{\operatorname{grad} f}\, \omega.$$

What is the divergence of the gradient of f, i.e., the Laplacian of f?

`December 10, 2019`

23. Consider the sphere S^2 with the metric

$$ds^2 = d\theta^2 + \sin^2\theta\, d\varphi^2$$

and Levi-Civita connection.

a) Using the results of **20.b)**, calculate $R(\partial_\theta, \partial_\varphi)\partial_\theta$ and $R(\partial_\theta, \partial_\varphi)\partial_\varphi$. Write the Riemann tensor and the curvature tensor.

b) Calculate the curvature of S^2.

24. Consider the hyperbolic plane H^2 with the metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2}$$

and Levi-Civita connection.

a) Using the results of **21.b)**, calculate $R(\partial_x, \partial_y)\partial_x$ and $R(\partial_x, \partial_y)\partial_y$. Write the Riemann tensor and the curvature tensor.
b) Calculate the curvature of H^2.

25. Show that the curvature tensor of a 3-dimensional Riemannian manifold is entirely determined by its Ricci tensor.

26. Consider the cylinder $C :=]0, \infty[\times S^1$ with metric

$$ds^2 = dr^2 + \sinh^2 r \, d\theta^2.$$

a) Compute ω_r^θ, Ω_r^θ and $R_{r\theta r\theta}$. What is the curvature of C?
b) Verify the equality of the Gauss-Bonnet Theorem,

$$\int_\Omega K + \int_{\partial\Omega} k_g = 2\pi \sum_i I_{p_i},$$

when $\Omega =]r_0, r_1[\times S^1$.

27. Let S be a compact orientable surface with a Riemannian metric with constant negative Gauss curvature. Let γ be a geodesic without self-intersections that separates S in two surfaces S_1 and S_2 with boundary, i.e.

$$S = S_1 \cup S_2, \quad \partial S_1 = \partial S_2 = \gamma.$$

Show that the quotient of the areas of S_1 and S_2 is a rational number.

28. Consider the surface in $\mathbb{R}^3$ represented parametrically by

$$(u, v) \mapsto f(u, v) = (f^1(u, v), f^2(u, v), f^3(u, v))$$

with the metric induced by the Euclidean metric, $E_1 = \frac{f_u}{\|f_u\|}$, $E_2 = \frac{f_v}{\|f_v\|}$ and $N = \frac{E_1 \times E_2}{\|E_1 \times E_2\|}$.

a) Check that

$$
\begin{aligned}
p &:= (-\nabla_{E_1} N, E_1) &=& \frac{1}{\|f_u\|^2}(N, f_{uu}) &=:& \frac{l}{\|f_u\|^2}, \\
q &:= (-\nabla_{E_1} N, E_2) &=& \frac{1}{\|f_u\|\|f_v\|}(N, f_{uv}) &=& \frac{m}{\|f_u\|\|f_v\|}, \\
r &:= (-\nabla_{E_2} N, E_2) &=& \frac{1}{\|f_v\|^2}(N, f_{vv}) &=& \frac{n}{\|f_v\|^2}.
\end{aligned}
$$

b) Let $\Delta = 1 - (E_1, E_2)^2$. Check that the second fundamental form of the surface is

$$
II(u, v) = \frac{1}{\Delta} \begin{bmatrix} p & q \\ q & r \end{bmatrix} \begin{bmatrix} 1 & -(E_1, E_2) \\ -(E_1, E_2) & 1 \end{bmatrix}.
$$

c) Check that the curvature of the surface is

$$
K = \frac{ln - m^2}{\|f_u\|^2\|f_v\|^2 - (f_u, f_v)^2}.
$$