Riemannian Geometry

Exam - February 26, 2020

LMAC and MMA

Solutions

1.

a) Using Stokes' Theorem, we have

$$\int_{\mathcal{C}} \operatorname{sech}^2 v \, d\theta \wedge dv = -\int_{\mathcal{C}} d \tanh v \, d\theta = -\int_{\partial\mathcal{C}} \tanh v \, d\theta = 2\pi \tanh \alpha.$$

b) Using the fact that the Lie derivative is a derivation and that it commutes with the exterior derivative, we obtain

$$\begin{aligned}
L_X \cosh^2 v \, d\theta \wedge dv &= X \cdot (\cosh^2 v) \, d\theta \wedge dv \\
&\quad + \cosh^2 v \, d(L_X \theta) \wedge dv + \cosh^2 v \, d\theta \wedge d(L_X v) \\
&= \operatorname{sech}^2 v \frac{\partial f}{\partial v} 2 \cosh v \sinh v \, d\theta \wedge dv \\
&\quad + \cosh^2 v \, d\left(\operatorname{sech}^2 v \frac{\partial f}{\partial \theta}\right) \wedge dv \\
&\quad + \cosh^2 v \, d\theta \wedge d\left(\operatorname{sech}^2 v \frac{\partial f}{\partial v}\right) \\
&= \frac{2}{\cosh v} \sinh v \frac{\partial f}{\partial v} \, d\theta \wedge dv \\
&\quad + \frac{\partial^2 f}{\partial \theta^2} \, d\theta \wedge dv + \frac{\partial^2 f}{\partial v^2} \, d\theta \wedge dv \\
&\quad - 2 \cosh^2 v \frac{\sinh v}{\cosh^3 v} \frac{\partial f}{\partial v} \, d\theta \wedge dv \\
&= \frac{1}{\cosh^2 v} \left(\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial v^2} \right) \epsilon.
\end{aligned}$$

c) According to Cartan's formula, we have

$$L_X \epsilon = d(\iota(X)\, \epsilon) + \iota(X)\, (d\epsilon) = d(\iota(X)\, \epsilon).$$

Since,

$$\begin{aligned}
\iota(X)\, \epsilon &= \cosh^2 v \operatorname{sech}^2 v \frac{\partial f}{\partial \theta} \, dv - \cosh^2 v \operatorname{sech}^2 v \frac{\partial f}{\partial v} \, d\theta \\
&= \frac{\partial f}{\partial \theta} \, dv - \frac{\partial f}{\partial v} \, d\theta,
\end{aligned}$$

it follows that

$$L_X\epsilon \;=\; d(\iota(X)\,\epsilon) \;=\; \mathrm{sech}^2 v\left(\frac{\partial^2 f}{\partial\theta^2}+\frac{\partial^2 f}{\partial v^2}\right)\epsilon.$$

The divergence of X is

$$\mathrm{div}\,X = \mathrm{sech}^2 v\left(\frac{\partial^2 f}{\partial\theta^2}+\frac{\partial^2 f}{\partial v^2}\right).$$

d) The left-hand side of

$$\int_{\mathcal{C}}(\mathrm{div}\,X)\,\epsilon = \int_{\partial\mathcal{C}}(X,n)\,ds$$

is

$$\begin{aligned}\int_0^\alpha\int_{-\pi}^{\pi}\left(\frac{\partial^2 f}{\partial\theta^2}+\frac{\partial^2 f}{\partial v^2}\right)d\theta dv \;&=\; \int_0^\alpha\int_{-\pi}^{\pi}\frac{\partial^2 f}{\partial v^2}\,d\theta dv\\ &=\; \int_{-\pi}^{\pi}\left(\frac{\partial f}{\partial v}(\theta,\alpha)-\frac{\partial f}{\partial v}(\theta,0)\right)d\theta.\end{aligned}$$

A parameterization of the top side of $\partial\mathcal{C}$ is $r(\theta)=(\theta,\alpha)$, for which $\|r'(\theta)\|=\cosh\alpha$. Thus, on the top side of $\partial\mathcal{C}$, we have $ds=\cosh\alpha\,d\theta$. On the bottom of $\partial\mathcal{C}$, we have $ds=d\theta$.
On $[-\pi,\pi]\times\{0\}$, $n=-\partial_v$, $(X,n)=-\frac{\partial f}{\partial v}$, $(X,n)\,ds=-\frac{\partial f}{\partial v}$.
On $\{\pi\}\times[0,\alpha]$, $n=\mathrm{sech}\,v\,\partial_\theta$, $(X,n)=\mathrm{sech}\,v\,\frac{\partial f}{\partial\theta}$.
On $[-\pi,\pi]\times\{\alpha\}$, $n=\mathrm{sech}\,\alpha\,\partial_v$, $(X,n)=\mathrm{sech}\,\alpha\,\frac{\partial f}{\partial v}$, $(X,n)\,ds=\frac{\partial f}{\partial v}$.
On $\{-\pi\}\times[0,\alpha]$, $n=-\mathrm{sech}\,v\,\partial_\theta$, $(X,n)=-\mathrm{sech}\,v\,\frac{\partial f}{\partial\theta}$.
Taking into account that $\frac{\partial f}{\partial\theta}(-\pi,v)=\frac{\partial f}{\partial\theta}(\pi,v)$, the integrals on the left side and on right side of $\partial\mathcal{C}$ cancel. We conclude that

$$\int_{\partial\mathcal{C}}(X,n)\,ds=\int_{-\pi}^{\pi}\left(\frac{\partial f}{\partial v}(\theta,\alpha)-\frac{\partial f}{\partial v}(\theta,0)\right)d\theta.$$

This verifies the Divergence Theorem for the present situation.

2. We denote by m, n and k the dimensions of M, N and S, respectively. Let p_0 belong to $f^{-1}(S)$. We will define a neighborhood W of p_0 such that $f^{-1}(S)\cap W$ is a submanifold of W. This implies that $f^{-1}(S)$ is a submanifold of M. Let $q_0=f(p_0)$. Note that q_0 belongs to S. Since S is a submanifold of N, there exists a neighborhood V of q_0, and a chart φ^{-1} defined on V, such that $S\cap V=\varphi(\{(x^1,\ldots,x^n)\in\varphi^{-1}(V):x^{k+1}=\ldots=x^n=0\})$. Let $j:\varphi^{-1}(V)\to\mathbb{R}^{n-k}$ be defined by $j(x^1,\ldots,x^n)=(x^{k+1},\ldots,x^n)$. We observe that $S \cap V = (j \circ \varphi^{-1})^{-1}(0) = (\varphi \circ j^{-1})(0)$. Let $W = f^{-1}(V)$. W is open because f is continuous. Clearly, we have $f^{-1}(S) \cap W = (j \circ \varphi^{-1} \circ f)^{-1}(0)$. We define $g : W \to \mathbb{R}^{n-k}$ by $g = j \circ \varphi^{-1} \circ f$. We claim that 0 is a regular value of g. Indeed, let p belong to $g^{-1}(0)$. As

$$(Dg)_p = (Dj)_{\varphi^{-1}(f(p))}(D\varphi^{-1})_{f(p)}(Df)_p,$$

$$\mathrm{Im}\,(Df)_p + T_{f(p)}S = T_{f(p)}N,$$

$$(Dj)_{\varphi^{-1}(f(p))}(D\varphi^{-1})T_{f(p)}S = (Dj)_{\varphi^{-1}(f(p))}(\mathbb{R}^k \times \{0\}) = \{0\}$$

and

$$(Dj)_{\varphi^{-1}(f(p))}(D\varphi^{-1})T_{f(p)}N = (Dj)_{\varphi^{-1}(f(p))}\mathbb{R}^n = \mathbb{R}^{n-k},$$

we have that

$$\mathrm{Im}\,(Dg)_p = (Dj)_{\varphi^{-1}(f(p))}(D\varphi^{-1})(\mathrm{Im}\,(Df)_p) = \mathbb{R}^{n-k}.$$

Since p is arbitrary in $g^{-1}(0)$, this proves that 0 is a regular value of g. Therefore $g^{-1}(0)$ is a submanifold of W of dimension $m-(n-k) = m-n+k$. As argued above, this implies that $f^{-1}(S)$ is a submanifold of M of dimension $m - n + k$.

3.

a) As

$$\begin{aligned} dx &= -\sin u \cosh v\, du + \cos u \sinh v\, dv, \\ dy &= -\cos u \cosh v\, du + \sin u \sinh v\, dv, \\ dz &= dv, \end{aligned}$$

the metric induced on $\mathcal{C}$ is

$$\cosh^2 v\,(du^2 + dv^2).$$

The frame

$$(\omega^u, \omega^v) = \cosh v\,(du, dv)$$

is orthonormal because

$$g^{-1} = \mathrm{sech}^2 v\,(\partial_u^2 + \partial_v^2).$$

b) One easily computes that

$$\omega^u_v = \tanh v\, du$$

and

$$\Omega^u_v = -\text{sech}^2 v \, du \wedge dv.$$

Since

$$\Omega^v_u = R_{uvu}{}^v \, \omega^u \wedge \omega^v = R_{uvu}{}^v \cosh^2 v \, du \wedge dv,$$

we obtain

$$R_{uvu}{}^v = \text{sech}^4 v.$$

Because we are working in an orthonormal frame, we also have

$$R_{uvuv} = \text{sech}^4 v.$$

The curvature of $\mathcal{C}$ is

$$K = -R_{uvuv} = -\text{sech}^4 v.$$

c) We denote by ∇ the Levi-Civita connection of $\mathbb{R}^3$ with the Euclidean metric. As

$$n = \text{sech}\, v \, (\cos u, \sin u, -\sinh v),$$

we readily obtain that

$$\begin{aligned} \nabla_{X_u} n &= \nabla_{\text{sech}\, v \, \partial_u} n \\ &= \text{sech}^2 v \, (-\sin u, \cos u, 0), \\ \nabla_{X_v} n &= \nabla_{\text{sech}\, v \, \partial_v} n \\ &= \text{sech}\, v \, [-\text{sech}\, v \tanh v (\cos u, \sin u, 0) + (0, 0, -\text{sech}^2 v)]. \end{aligned}$$

Since

$$\begin{aligned} X_u &= (-\sin u, \cos u, 0), \\ X_v &= \text{sech}\, v \, (\cos u \sinh v, \sin u \sinh v, 1), \end{aligned}$$

the representation of the second fundamental form of $\mathcal{C}$ in the frame (X_u, X_v) is

$$\begin{bmatrix} (\nabla_{X_u} n, X_u) & (\nabla_{X_v} n, X_u) \\ (\nabla_{X_u} n, X_v) & (\nabla_{X_v} n, X_v) \end{bmatrix} = \begin{bmatrix} \text{sech}^2 v & 0 \\ 0 & -\text{sech}^2 v \end{bmatrix}.$$

Once again we obtain that the curvature of $\mathcal{C}$, the determinant of this matrix, is $-\text{sech}^4 v$. The principal curvatures of $\mathcal{C}$ are $\pm\text{sech}^2 v$. The mean curvature of $\mathcal{C}$ is zero, i.e. $\mathcal{C}$ is a minimal surface.

4. This exercise was solved in class. See also page 370 of Godinho and Natário.