Riemannian Geometry

February 26, 2020

LMAC and MMA

1st Test	–	Questions **1.** and **2.**	– 90 minutes
2nd Test	–	Questions **3.** and **4.**	– 90 minutes
Exam	–	All questions	– 3 hours

Show your calculations

1. Consider the cylinder $\mathcal{C} = S^1 \times [0, \alpha]$ with the 2-form

$$\eta = \operatorname{sech}^2 v \, d\theta \wedge dv.$$

a) Use Stokes' Theorem to calculate $\int_{\mathcal{C}} \eta$. (2)

b) Let $f : \mathcal{C} \to \mathbb{R}$, (2)

$$X = \operatorname{sech}^2 v \left(\frac{\partial f}{\partial \theta} \partial_\theta + \frac{\partial f}{\partial v} \partial_v \right)$$

and

$$\epsilon = \cosh^2 v \, d\theta \wedge dv$$

be the volume form on $\mathcal{C}$. Calculate $L_X \epsilon$ using the fact that the Lie derivative is a derivation and that it commutes with the exterior derivative.

c) Calculate $L_X \epsilon$ using Cartan's formula. What is the divergence of X? (2)

d) Calculate directly both sides of the equality (2)

$$\int_{\mathcal{C}} (\operatorname{div} X) \, \epsilon = \int_{\partial \mathcal{C}} (X, n) \, ds,$$

thereby verifying the Divergence Theorem in this case. Here the parenthesis denote the inner product in the metric $\cosh^2 v \, (d\theta^2 + dv^2)$.

2. Let $f : M \to N$ be a map between smooth manifolds and let S be a submanifold of N. The map f is said to be transverse to S if (2)

$$\forall_{p \in f^{-1}(S)} \ \operatorname{Im} (Df)_p + T_{f(p)} S = T_{f(p)} N.$$

Show that if f is transverse to S, then $f^{-1}(S)$ is a submanifold of M and determine its dimension.

Suggestion: Use the special case when S is a point, studied in class.

3. Consider $\mathcal{C}$, the catenoid in $\mathbb{R}^3$ with the Euclidean metric, parameterized by

$$r(u,v) = (\cos u \cosh v, \sin u \cosh v, v).$$

a) Compute the metric induced on $\mathcal{C}$, and check that the frame (2)

$$(\omega^u, \omega^v) = \cosh v\,(du, dv)$$

is orthonormal.

b) Using the frame in **a)** and Cartan's structure equations, compute the curvature of $\mathcal{C}$. (2)

c) The vectors (2)

$$n = \operatorname{sech} v\,(\cos u, \sin u, -\sinh v)$$

are unitary and normal to $\mathcal{C}$. Compute the representation of the second fundamental form of $\mathcal{C}$ in the frame

$$(X_u, X_v) = \operatorname{sech} v\,(\partial_u, \partial_v).$$

What is the mean curvature of $\mathcal{C}$?

4. Let G be a Lie group endowed with a bi-invariant Riemannian metric (i.e. such that L_g and R_g are isometries for all $g \in G$), and let $i : G \to G$ be the diffeomorphism defined by $i(g) = g^{-1}$.

a) Compute $(di)_e$ and show that (1)

$$(di)_g = (dR_{g^{-1}})_e (di)_e (dL_{g^{-1}})_g,$$

for all $g \in G$. Conclude that i is an isometry.

b) Let $v \in \mathfrak{g} = T_eG$ and c_v be the geodesic satisfying $c_v(0) = e$ and $\dot{c}_v(0) = v$. Show that if t is sufficiently small then $c_v(-t) = (c_v(t))^{-1}$. Conclude that c_v is defined in $\mathbb{R}$ and satisfies $c_v(t+s) = c_v(t)c_v(s)$ for all $t, s \in \mathbb{R}$. (1)
Suggestion: Recall that any two points in a totally normal neighborhood are connected by a unique geodesic in that neighborhood.

c) Show that the geodesics of G are the integral curves of left-invariant vector fields, and that the Lie group exponential and the geodesic exponential at the identity coincide. (1)

d) Let ∇ be the Levi-Civita connection of the bi-invariant metric, and X and Y be two left-invariant vector fields. Show that (1)

$$\nabla_X Y = \frac{1}{2}[X, Y].$$