Riemannian Geometry

Exam - February 4, 2021

LMAC and MMA

Solutions

1.

a) If $f(A) = \det A$ and A is nonsingular, then

$$\begin{aligned} f'(A)B &= \frac{d}{dt}\det(A+tB)\Big|_{t=0} = \det A\,\frac{d}{dt}\det(I+tA^{-1}B)\Big|_{t=0} \\ &= \det A\,\mathrm{tr}\,(A^{-1}B). \end{aligned}$$

This implies that $\mathfrak{sl}(2,\mathbb{R}) = T_I SL(2,\mathbb{R}) = \ker f'(I)$ is the vector space of 2×2 traceless matrices with the bracket defined $[B,C] = BC - CB$. So, $\mathfrak{sl}(2,\mathbb{R})$ is clearly spanned by

$$B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

as if $F \in \mathfrak{sl}(2,\mathbb{R})$, then

$$F = \begin{bmatrix} x & y \\ z & -x \end{bmatrix}.$$

b) Direct calculation using $[F,G] = FG - GF$, shows that

$$[B,C] = D, \quad [D,B] = 2B, \quad [D,C] = -2C.$$

For

$$g = \begin{bmatrix} p & q \\ r & s \end{bmatrix},$$

we have

$$\begin{aligned} [X^B, X^C]_g &= [(L_g)_\star B, (L_g)_\star C]_g = (L_g)_\star [B,C]_e \\ &= (L_g)_\star D_e = (X^D)_g \\ &= \begin{bmatrix} p & q \\ r & s \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\ &= \begin{bmatrix} p & -q \\ r & -s \end{bmatrix}. \end{aligned}$$

c) To calculate the flow $\phi_t(g_0)$ of X^B, we solve $\dot{g} = (X^B)_g$ with initial condition $g(0) = g_0$, i.e. we solve

$$\begin{bmatrix} \dot{p} & \dot{q} \\ \dot{r} & \dot{s} \end{bmatrix} = \begin{bmatrix} p & q \\ r & s \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & p \\ 0 & r \end{bmatrix}$$

with

$$g(0) = \begin{bmatrix} p_0 & q_0 \\ r_0 & s_0 \end{bmatrix}.$$

We obtain

$$\phi_t(g_0) = \begin{bmatrix} p(t) & q(t) \\ r(t) & s(t) \end{bmatrix} = \begin{bmatrix} p_0 & q_0 + p_0 t \\ r_0 & s_0 + r_0 t \end{bmatrix}$$

d) Let

$$\begin{bmatrix} p_0 & q_0 \\ r_0 & s_0 \end{bmatrix} = e = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

We have

$$\exp(tX^B) = \phi_t(e) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}.$$

e) Suppose

$$g = \begin{bmatrix} p_0 & q_0 \\ r_0 & s_0 \end{bmatrix}, \qquad h = \begin{bmatrix} p_1 & q_1 \\ r_1 & s_1 \end{bmatrix}.$$

Then

$$\begin{aligned} L_g \circ \phi_t(h) &= \begin{bmatrix} p_0 & q_0 \\ r_0 & s_0 \end{bmatrix} \begin{bmatrix} p_1 & q_1 + p_1 t \\ r_1 & s_1 + r_1 t \end{bmatrix} \\ &= \begin{bmatrix} p_0 p_1 + q_0 r_1 & p_0(q_1 + p_1 t) + q_0(s_1 + r_1 t) \\ r_0 p_1 + s_0 r_1 & r_0(q_1 + p_1 t) + s_0(s_1 + r_1 t) \end{bmatrix} \\ &= \phi_t\left(\begin{bmatrix} p_0 p_1 + q_0 r_1 & p_0 q_1 + q_0 s_1 \\ r_0 p_1 + s_0 r_1 & r_0 q_1 + s_0 s_1 \end{bmatrix} \right) \\ &= \phi_t\left(\begin{bmatrix} p_0 & q_0 \\ r_0 & s_0 \end{bmatrix} \begin{bmatrix} p_1 & q_1 \\ r_1 & s_1 \end{bmatrix} \right) \\ &= \phi_t \circ L_g(h). \end{aligned}$$

f) Taking into account that

$$(X^C)_g = \begin{bmatrix} p & q \\ r & s \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} q & 0 \\ s & 0 \end{bmatrix}$$

and

$$(d\phi_{-t})_{\phi_t(g_0)}(X^C)_{\phi_t(g_0)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -t & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -t & 1 \end{bmatrix} \begin{bmatrix} q_0 + p_0 t \\ 0 \\ s_0 + r_0 t \\ 0 \end{bmatrix},$$

according to the definition of Lie derivative, we have

$$\left(L_{X^B} X^C\right)_{g_0} = \frac{d}{dt}(d\phi_{-t})_{\phi_t(g_0)}(X^C)_{\phi_t(g_0)}\Big|_{t=0} = \begin{bmatrix} p_0 \\ -q_0 \\ r_0 \\ -s_0 \end{bmatrix} \cong (X^D)_{g_0}.$$

2. First using Stoke's Theorem and then using Cartan's formula, we obtain

$$\int_M d(L_X\omega) = \int_{\partial M} L_X\omega = \int_{\partial M} \iota(X)\, d\omega + \int_{\partial M} d(\iota(X)\omega).$$

Again using Stokes' Theorem, we have

$$\int_{\partial M} d(\iota(X)\omega) = \int_{\partial\partial M} \iota(X)\omega = 0.$$

This is zero because $\partial\partial M = \emptyset$ (which is a consequence of $\partial\partial H = \emptyset$, for H a half-space). On the other hand, if $\rho : M \to [0,1]$ has support in the range $\varphi(U)$ of a parameterization φ of ∂M, φ compatible with the orientation of ∂M (induced by the orientation of M), then

$$\int_{\partial M} \rho\, \iota(X)\, d\omega = \int_U \varphi^\star(\rho\, \iota(X)\, d\omega).$$

The form $\varphi^\star(\rho\, \iota(X)\, d\omega)$ is identically equal to zero. Indeed, for $X_1, \dots, X_n \in \mathcal{X}(U)$, we have

$$\begin{aligned} \varphi^\star(\rho\, \iota(X)\, d\omega)(X_1, \dots, X_n) &= \rho\, \iota(X)\, d\omega(\varphi_\star X_1, \dots, \varphi_\star X_n) \\ &= \rho\, d\omega(X, \varphi_\star X_1, \dots, \varphi_\star X_n) = 0 \end{aligned}$$

because $\{X, \varphi_\star X_1, \dots, \varphi_\star X_n\}$ are linearly dependent at each point (they are $n+1$ vectors tangent to ∂M). This implies that

$$\int_{\partial M} \iota(X)\, d\omega = 0.$$

We have proved that

$$\int_M d(L_X\omega) = 0.$$

This shows that there exists a $p \in M$ and $n+1$ linearly independent $\{(X_1)_p, \ldots, (X_{n+1})_p\}$ such that

$$(d(L_X\omega))_p((X_1)_p, \ldots, (X_{n+1})_p) = 0.$$

But the $(n+1)$-alternating tensor $(d(L_X\omega))_p$ must be a multiple of the determinant. It therefore has to vanish at p.

3.

a) The vector field is right invariant because

$$\begin{aligned} X_g = \frac{d}{ds}\psi_s(g)\Big|_{s=0} &= \frac{d}{ds}L_{\exp(sF)}g\Big|_{s=0} \\ &= \frac{d}{ds}\exp(sF)g\Big|_{s=0} \\ &= \frac{d}{ds}R_g\exp(sF)\Big|_{s=0} \\ &= (dR_g)_e X_e. \end{aligned}$$

b) Assuming that X^B is Killing, using the Koszul formula, and noting that (X^α, X^β) is constant (because both these vector fields and the metric are left invariant), we arrive at the contradiction

$$\begin{aligned} 0 &= (\nabla_{X^C}X^B, X^D) + (X^C, \nabla_{X^D}X^B) \\ &= -\frac{1}{2}([X^B,X^D],X^C) - \frac{1}{2}([X^C,X^D],X^B) + \frac{1}{2}([X^C,X^B],X^D) \\ &\quad -\frac{1}{2}([X^B,X^C],X^D) - \frac{1}{2}([X^D,X^C],X^B) + \frac{1}{2}([X^D,X^B],X^C) \\ &= 0 + 0 - \frac{1}{2}(D,D) - \frac{1}{2}(D,D) + 0 + 0 = -(D,D) = -1. \end{aligned}$$

c) Given that

$$[E_i, E_j] = \sum_{k=1}^{3} c_{ijk}E_k,$$

from the Koszul formula we obtain

$$\begin{aligned} (\nabla_{E_i}E_j, E_k) &= -\frac{1}{2}(E_i,[E_j,E_k]) - \frac{1}{2}(E_j,[E_i,E_k]) + \frac{1}{2}(E_k,[E_i,E_j]) \\ &= -\frac{1}{2}c_{jki} - \frac{1}{2}c_{ikj} + \frac{1}{2}c_{ijk} \\ &= \frac{1}{2}(c_{ijk} + c_{kij} + c_{kji}). \end{aligned}$$

This implies that

$$\nabla_{E_i} E_j = \sum_{k=1}^{3} \left(\nabla_{E_i} E_j, E_k\right) E_k = \frac{1}{2} \sum_{k=1}^{3} (c_{ijk} + c_{kij} + c_{kji}) E_k.$$

This formula allows us to calculate $\nabla_{X^B} X^B$:

$$\begin{aligned}
\nabla_{X^B} X^B &= \frac{1}{2}\left(c_{BBB} + c_{BBB} + c_{BBB}\right) X^B \\
&\quad + \frac{1}{2}\left(c_{BBC} + c_{CBB} + c_{CBB}\right) X^C \\
&\quad + \frac{1}{2}\left(c_{BBD} + c_{DBB} + c_{DBB}\right) X^D \\
&= 2X^D.
\end{aligned}$$

d) Given that

$$\begin{aligned}
\nabla_{X^C} X^B &= -\tfrac{1}{2} X^D, & \nabla_{X^C} X^C &= -2X^D, \\
\nabla_{X^D} X^C &= \tfrac{1}{2} X^B, & \nabla_{X^D} X^D &= 0,
\end{aligned}$$

it follows that

$$\begin{aligned}
& R\left(X^C, X^D, X^C, X^D\right) \\
&\quad = \left(\nabla_{X^C} \nabla_{X^D} X^C - \nabla_{X^D} \nabla_{X^C} X^C - \nabla_{[X^C, X^D]} X^C, X^D\right) \\
&\quad = \left(\frac{1}{2} \nabla_{X^C} X^B + 2\nabla_{X^D} X^D - 2\nabla_{X^C} X^C, X^D\right) \\
&\quad = \left(-\frac{1}{4} X^D + 0 + 4X^D, X^D\right) \\
&\quad = \frac{15}{4}.
\end{aligned}$$

The sectional curvature of the plane spanned by X^C and X^D is $-\frac{15}{4}$.

4.

a) Let c be the geodesic with initial velocity V, $c(0) = p$ and $\dot{c}(0) = V$. Suppose c is contained in S. Then $f(c(t)) = c(t)$. Differentiating both sides of this equality with respect to t and setting $t = 0$ we obtain $(df)_p(V) = V$.
Suppose now that $(df)_p(V) = V$. Since f is an isometry, $\gamma = f \circ c$ is a geodesic. Its initial velocity is V and at $t = 0$ it is at p (because $(df)_p$ sends V to itself, which is a vector based at p). As the geodesic is uniquely determined by a point and its velocity at that point, $\gamma = c$.

b) Let $B_\epsilon(p)$ be such that $\exp_p$ is a diffeomorphism from $B_\epsilon(0) \subset T_pM$ to $B_\epsilon(p)$. The set of V's in T_pM such that $(df)_pV = V$ is a subspace of T_pM. The image by $\exp_p$ of the intersection of this subspace with $B_\epsilon(0)$ is a submanifold N of M.

c) Without loss of generality, we may assume that $B_\epsilon(p)$ is a totally normal neighborhood of p. Suppose, by contradiction, that $q \in B_\epsilon(p) \setminus N$ and q belongs to S. Then there exists a geodesic γ connecting p to q. Now, $f \circ \gamma$ is also a geodesic connecting p to q (since both p and q are fixed points of S). If $f \circ \gamma = \gamma$, then γ belongs to N (which contradicts $q \notin N$). If $f \circ \gamma$ is different from γ, then we contradict the uniqueness of geodesics connecting p to q (because the geodesic and its image have the same length). We conclude that there does not exist any $q \in B_\epsilon(p) \setminus N$ that belongs to S. In summary, each point in S has a neighborhood U such that $S \cap U$ is a manifold. We conclude that S is a submanifold of M (whose components might have different dimensions).

⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆

For the record:

$$\begin{array}{lllllllll}
\nabla_{X^B} X^B &=& 2X^D, & \nabla_{X^B} X^C &=& \frac{1}{2}X^D, & \nabla_{X^B} X^D &=& -2X^B - \frac{1}{2}X^C, \\
\nabla_{X^C} X^B &=& -\frac{1}{2}X^D, & \nabla_{X^C} X^C &=& -2X^D, & \nabla_{X^C} X^D &=& \frac{1}{2}X^B + 2X^C, \\
\nabla_{X^D} X^B &=& -\frac{1}{2}X^C, & \nabla_{X^D} X^C &=& \frac{1}{2}X^B, & \nabla_{X^D} X^D &=& 0.
\end{array}$$

$$\begin{aligned}
R\left(X^B, X^C, X^B, X^C\right) &= -\frac{13}{4}, \\
R\left(X^B, X^D, X^B, X^D\right) &= \frac{15}{4}, \\
R\left(X^C, X^D, X^C, X^D\right) &= \frac{15}{4}.
\end{aligned}$$