Riemannian Geometry

Exam - January 29, 2020

LMAC and MMA

Solutions

1.

a) As $X = 2\partial_\theta$, according to Cartan's formula, we have

$$L_X\omega = d\iota(X)\omega + \iota(X)d\omega = d\iota(X)\omega = d\left(\frac{1}{4}\sin\theta d\varphi \wedge d\psi\right).$$

Thus, according to Stokes' Theorem, we have

$$\begin{aligned}\int\limits_{\left]0,\frac{\pi}{2}\right[\times]0,2\pi[\times]0,2\pi[} L_X\omega &= \int\limits_{]0,2\pi[\times]0,2\pi[} \frac{1}{4}\sin\left(\frac{\pi}{2}\right) d\varphi \wedge d\psi \\ &- \int\limits_{]0,2\pi[\times]0,2\pi[} \frac{1}{4}\sin 0\, d\varphi \wedge d\psi = \pi^2.\end{aligned}$$

To justify this computation rigorously, one should first integrate on $\left[\epsilon, \frac{\pi}{2}\right] \times [0, 2\pi] \times [0, 2\pi]$ and pass to the limit as $\epsilon \searrow 0$. Indeed, the coordinates degenerate at $\theta = 0$.

b) Recall that $\mathfrak{so}(3) = \{A \in M_{3\times 3} : A^T = -A\}$. If

$$A = \begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix} \text{ and } \xi = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \text{ then } A\xi = \begin{bmatrix} -cy + bz \\ cx - az \\ -bx + ay \end{bmatrix}.$$

On the other hand

$$(a, b, c) \times (x, y, z) = (-cy + bz, cx - az, -bx + ay).$$

Therefore, for A as above, $A\xi = \Omega(A) \times \xi$ for

$$\Omega(A) = (a, b, c).$$

When A is as above and

$$B = \begin{bmatrix} 0 & -\bar{c} & \bar{b} \\ \bar{c} & 0 & -\bar{a} \\ -\bar{b} & \bar{a} & 0 \end{bmatrix},$$

we have

$$[A,B] = \begin{bmatrix} 0 & b\bar{a}-a\bar{b} & c\bar{a}-a\bar{c} \\ -b\bar{a}+a\bar{b} & 0 & c\bar{b}-b\bar{c} \\ -c\bar{a}+a\bar{c} & -c\bar{b}+b\bar{c} & 0 \end{bmatrix},$$

whereas

$$\Omega(A)\times\Omega(B) = (a,b,c)\times(\bar{a},\bar{b},\bar{c}) = (b\bar{c}-c\bar{b}, c\bar{a}-a\bar{c}, a\bar{b}-b\bar{a}).$$

The first entry of this vector is $[A,B]_{3,2}$, the second entry of this vector is $[A,B]_{1,3}$, and the third entry of this vector is $[A,B]_{2,1}$. This shows that $\Omega([A,B]) = \Omega(A)\times\Omega(B)$.

c) Using **b)** and $R(v\times w) = Rv\times Rw$, we obtain

$$\begin{aligned} (R^{-1}\Omega^{-1}(v)R)\,w &= R^{-1}(\Omega^{-1}(v)(R\,w)) \;=\; R^{-1}(v\times(R\,w)) \\ &= (R^{-1}v)\times(R^{-1}(R\,w)) \;=\; (R^{-1}v)\times w \\ &= \Omega^{-1}(R^{-1}v)\,w \end{aligned}$$

for all $w\in\mathbb{R}^3$. This means that

$$R^{-1}\Omega^{-1}(v)R = \Omega^{-1}(R^{-1}v).$$

d) We have

$$\begin{aligned} R_{-\varphi}\dot{R}_{\varphi} &= \dot{\varphi}\begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -\sin\varphi & -\cos\varphi & 0 \\ \cos\varphi & -\sin\varphi & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ &= \dot{\varphi}\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \dot{\varphi}\,\Omega^{-1}(e_3). \end{aligned}$$

Similarly, it follows that $R_{-\theta}\dot{R}_{\theta} = \dot{\theta}\,\Omega^{-1}(e_1)$ and $R_{-\psi}\dot{R}_{\psi} = \dot{\psi}\,\Omega^{-1}(e_3)$.

e) For $\dot{S}\in T_S SO(3)$, let A be such that $\dot{S} = SA$. Since $S = R_\varphi R_\theta R_\psi$, we have

$$\begin{aligned} A = S^{-1}\dot{S} &= R_{-\psi}R_{-\theta}R_{-\varphi}\dot{R}_\varphi R_\theta R_\psi + R_{-\psi}R_{-\theta}R_{-\varphi}R_\varphi\dot{R}_\theta R_\psi \\ &\quad + R_{-\psi}R_{-\theta}R_{-\varphi}R_\varphi R_\theta\dot{R}_\psi \\ &= R_{-\psi}R_{-\theta}R_{-\varphi}\dot{R}_\varphi R_\theta R_\psi + R_{-\psi}R_{-\theta}\dot{R}_\theta R_\psi + R_{-\psi}\dot{R}_\psi \\ &= \dot{\varphi}\,R_{-\psi}R_{-\theta}\Omega^{-1}(e_3)R_\theta R_\psi + \dot{\theta}\,R_{-\psi}\Omega^{-1}(e_1)R_\psi + \dot{\psi}\,\Omega^{-1}(e_3) \\ &= \dot{\varphi}\,\Omega^{-1}(R_{-\psi}R_{-\theta}e_3) + \dot{\theta}\,\Omega^{-1}(R_{-\psi}e_1) + \dot{\psi}\,\Omega^{-1}(e_3). \end{aligned}$$

Hence, we have

$$\begin{aligned}\Omega(A) &= \dot\varphi\, R_{-\psi}R_{-\theta}e_3 + \dot\theta\, R_{-\psi}e_1 + \dot\psi e_3\\ &= \dot\varphi(\sin\theta\sin\psi e_1 + \sin\theta\cos\psi e_2 + \cos\theta e_3)\\ &\quad +\dot\theta(\cos\psi e_1 - \sin\psi e_2) + \dot\psi e_3\\ &= \big(\dot\theta\cos\psi + \dot\varphi\sin\theta\sin\psi\big)e_1 + \big(-\dot\theta\sin\psi + \dot\varphi\sin\theta\cos\psi\big)e_2\\ &\quad +\big(\dot\varphi\cos\theta + \dot\psi\big)e_3.\end{aligned}$$

f) Suppose X is a left invariant vector field and $S(\theta(\,\cdot\,),\varphi(\,\cdot\,),\psi(\,\cdot\,))$ is an integral curve of X, i.e. $X_S = \dot S$. Then A is constant. Indeed, since X is left invariant, $X_S = SA$ with $A = X_I$. This is the same as $\dot S = SA$. Now, in local coordinates $X = \dot\theta\partial_\theta + \dot\varphi\partial_\varphi + \dot\psi\partial_\psi$. Notice that

$$\Omega(A) = \begin{bmatrix} \cos\psi & \sin\theta\sin\psi & 0\\ -\sin\psi & \sin\theta\cos\psi & 0\\ 0 & \cos\theta & 1\end{bmatrix}\begin{bmatrix}\dot\theta\\ \dot\varphi\\ \dot\psi\end{bmatrix} = M\begin{bmatrix}X^\theta\\ X^\varphi\\ X^\psi\end{bmatrix}.$$

Of course, when $A = \Omega^{-1}(e_1)$, we have $\Omega(A) = e_1$, when $A = \Omega^{-1}(-e_2)$, we have $\Omega(A) = -e_2$, and when $A = \Omega^{-1}(e_3)$, we have $\Omega(A) = e_3$. Thus, the left invariant vector fields corresponding to $\Omega^{-1}(e_1)$, $\Omega^{-1}(-e_2)$ and $\Omega^{-1}(e_3)$ are $X_1 = M^{-1}e_1$, $X_2 = -M^{-1}e_2$ and $X_3 = M^{-1}e_3$, respectively. As

$$M^{-1} = \begin{bmatrix} \cos\psi & -\sin\psi & 0\\ \frac{\sin\psi}{\sin\theta} & \frac{\cos\psi}{\sin\theta} & 0\\ -\sin\psi\cot\theta & -\cos\psi\cot\theta & 1\end{bmatrix},$$

it follows that

$$\begin{aligned} X_1 &= \cos\psi\,\partial_\theta + \frac{\sin\psi}{\sin\theta}\,\partial_\varphi - \sin\psi\cot\theta\,\partial_\psi,\\ X_2 &= \sin\psi\,\partial_\theta - \frac{\cos\psi}{\sin\theta}\,\partial_\varphi + \cos\psi\cot\theta\,\partial_\psi,\\ X_3 &= \partial_\psi.\end{aligned}$$

2.

a) The metric and its inverse are

$$g = \frac{1}{4}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & \cos\theta\\ 0 & \cos\theta & 1\end{bmatrix},\quad g^{-1} = 4\begin{bmatrix}1 & 0 & 0\\ 0 & \csc^2\theta & -\csc\theta\cot\theta\\ 0 & -\csc\theta\cot\theta & \csc^2\theta\end{bmatrix}.$$

The inner product of ω^i and ω^j is

$$(\omega^i, \omega^j) = (\omega^i)_k g^{kl} (\omega^j)_l.$$

Let $X_1 = X$, $X_2 = Y$ and $X_3 = Z$. Since

$$\begin{aligned}
g^{-1}\omega^1 &= 2\begin{bmatrix} \cos\psi \\ \frac{\sin\psi}{\sin\theta} \\ -\sin\psi\cot\theta \end{bmatrix} = X_1, \\
g^{-1}\omega^2 &= 2\begin{bmatrix} \sin\psi \\ -\frac{\cos\psi}{\sin\theta} \\ \cos\psi\cot\theta \end{bmatrix} = X_2 \\
g^{-1}\omega^3 &= 2\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = X_3,
\end{aligned}$$

and $\omega^i(X_j) = \delta^i_j$, we have $\omega^i g^{-1}\omega^j = \delta^i_j$. The base $(\omega^1, \omega^2, \omega^3)$ is orthonormal.

b) One readily calculates

$$d\omega^1 = \frac{1}{2}(\sin\psi\, d\theta \wedge d\psi + \cos\theta \sin\psi\, d\theta \wedge d\varphi - \sin\theta\cos\psi\, d\varphi \wedge d\psi)$$

and

$$\omega^2 \wedge \omega^3 = \frac{1}{4}(\sin\psi\, d\theta \wedge d\psi + \cos\theta \sin\psi\, d\theta \wedge d\varphi - \sin\theta\cos\psi\, d\varphi \wedge d\psi).$$

So, the constant a is equal to 2.

c) According to Cartan's structure equations, we have

$$\begin{aligned}
2\,\omega^2 \wedge \omega^3 &= d\omega^1 &&= \omega^2 \wedge \omega^1_2 + \omega^3 \wedge \omega^1_3, \\
-2\,\omega^1 \wedge \omega^3 &= d\omega^2 &&= \omega^1 \wedge \omega^2_1 + \omega^3 \wedge \omega^2_3 \\
& &&= -\,\omega^1 \wedge \omega^1_2 + \omega^3 \wedge \omega^2_3, \\
2\,\omega^1 \wedge \omega^2 &= d\omega^3 &&= \omega^1 \wedge \omega^3_1 + \omega^2 \wedge \omega^3_2 \\
& &&= -\,\omega^1 \wedge \omega^1_3 - \,\omega^3 \wedge \omega^2_3.
\end{aligned}$$

By inspection, we see that the connection forms are

$$\begin{aligned}
\omega^1_2 &= \omega^3, \\
\omega^1_3 &= -\,\omega^2, \\
\omega^2_3 &= \omega^1.
\end{aligned}$$

d) On the one hand, we have

$$\begin{aligned} d\omega_2^1 &= d\omega^3 = 2\,\omega^1\wedge\omega^2, \\ d\omega_3^1 &= -d\omega^2 = 2\,\omega^1\wedge\omega^3, \\ d\omega_3^2 &= d\omega^1 = 2\,\omega^2\wedge\omega^2. \end{aligned}$$

On the other hand, according to Cartan's structure equations, we have

$$\begin{aligned} d\omega_2^1 &= \Omega_2^1 + \omega_2^3\wedge\omega_3^1 = \Omega_2^1 + \omega^1\wedge\omega^2, \\ d\omega_3^1 &= \Omega_3^1 + \omega_3^2\wedge\omega_2^1 = \Omega_3^1 + \omega^1\wedge\omega^3, \\ d\omega_3^2 &= \Omega_3^2 + \omega_3^1\wedge\omega_1^2 = \Omega_3^2 + \omega^2\wedge\omega^3. \end{aligned}$$

Therefore, the curvature forms are given by

$$\begin{aligned} \Omega_2^1 &= \omega^1\wedge\omega^2, \\ \Omega_3^1 &= \omega^1\wedge\omega^3, \\ \Omega_3^2 &= \omega^2\wedge\omega^3. \end{aligned}$$

The Riemann tensor is given by

$$\begin{aligned} R = \; & R_{kl2}{}^1\omega^k\otimes\omega^l\otimes\omega^2\otimes X_1 + R_{kl1}{}^2\omega^k\otimes\omega^l\otimes\omega^1\otimes X_2 \\ & + R_{kl3}{}^1\omega^k\otimes\omega^l\otimes\omega^3\otimes X_1 + R_{kl1}{}^3\omega^k\otimes\omega^l\otimes\omega^1\otimes X_3 \\ & + R_{kl3}{}^2\omega^k\otimes\omega^l\otimes\omega^3\otimes X_2 + R_{kl2}{}^3\omega^k\otimes\omega^l\otimes\omega^2\otimes X_3. \end{aligned}$$

As the frame is orthonormal, the curvature tensor is given by

$$\begin{aligned} R &= R_{kl21}\omega^k\otimes\omega^l\otimes\omega^2\wedge\omega^1 + R_{kl31}\omega^k\otimes\omega^l\otimes\omega^3\wedge\omega^1 \\ &\quad + R_{kl32}\omega^k\otimes\omega^l\otimes\omega^3\wedge\omega^2 \\ &= R_{kl2}{}^1\omega^k\otimes\omega^l\otimes\omega^2\wedge\omega^1 + R_{kl3}{}^1\omega^k\otimes\omega^l\otimes\omega^3\wedge\omega^1 \\ &\quad + R_{kl3}{}^2\omega^k\otimes\omega^l\otimes\omega^3\wedge\omega^2 \\ &= \Omega_2^1\otimes\omega^2\wedge\omega^1 + \Omega_3^1\otimes\omega^3\wedge\omega^1 + \Omega_3^2\otimes\omega^3\wedge\omega^2 \\ &= \omega^1\wedge\omega^2\otimes\omega^2\wedge\omega^1 + \omega^1\wedge\omega^3\otimes\omega^3\wedge\omega^1 + \omega^2\wedge\omega^3\otimes\omega^3\wedge\omega^2. \end{aligned}$$

We see that $R_{1212} = R_{1313} = R_{2323} = -1$ and that $R_{ijkl} = 0$ if three of the indices are different. The sectional curvature of $SO(3)$ is constant, equal to 1.

e) The covariant derivatives of $X = X_1$ and $Y = X_2$ are

$$\begin{aligned}
\nabla_{X_1} X_1 &= \omega^1(\nabla_{X_1} X_1) X_1 + \omega^2(\nabla_{X_1} X_1) X_2 + \omega^3(\nabla_{X_1} X_1) X_3 \\
&= \omega^1_1(X_1) X_1 + \omega^2_1(X_1) X_2 + \omega^3_1(X_1) X_3 \\
&= 0 - \omega^3(X_1) X_2 + \omega^2(X_1) X_3 \\
&= 0, \\
\nabla_{X_2} X_1 &= 0 - \omega^3(X_2) X_2 + \omega^2(X_2) X_3 \\
&= X_3, \\
\nabla_{X_3} X_1 &= 0 - \omega^3(X_3) X_2 + \omega^2(X_3) X_3 \\
&= -X_2, \\
\nabla_{X_1} X_2 &= \omega^1_2(X_1) X_1 + \omega^2_2(X_1) X_2 + \omega^3_2(X_1) X_3 \\
&= \omega^3(X_1) X_1 + 0 - \omega^1(X_1) X_3 \\
&= -X_3, \\
\nabla_{X_2} X_2 &= \omega^3(X_2) X_1 + 0 - \omega^1(X_2) X_3 \\
&= 0, \\
\nabla_{X_3} X_2 &= \omega^3(X_3) X_1 + 0 - \omega^1(X_3) X_3 \\
&= X_1.
\end{aligned}$$

f) The vector fields W and Z are tangent to the torus because they have no ∂_θ component:

$$\begin{aligned}
W &= \sin\psi\, X - \cos\psi\, Y \;=\; 2\left(\csc\theta\,\partial_\varphi - \cot\theta\,\partial_\psi\right), \\
Z &= 2\partial_\psi.
\end{aligned}$$

g) The covariant derivatives of N are

$$\begin{aligned}
\nabla_W N &= \nabla_{\sin\psi X_1 - \cos\psi X_2}(\cos\psi X_1 + \sin\psi X_2) \\
&= \sin\psi(X_1 \cdot \cos\psi) X_1 + \sin\psi\cos\psi \nabla_{X_1} X_1 \\
&\quad + \sin\psi(X_1 \cdot \sin\psi) X_2 + \sin^2\psi \nabla_{X_1} X_2 \\
&\quad - \cos\psi(X_2 \cdot \cos\psi) X_1 - \cos^2\psi \nabla_{X_2} X_1 \\
&\quad - \cos\psi(X_2 \cdot \sin\psi) X_2 - \cos\psi\sin\psi \nabla_{X_2} X_2 \\
&= 2\sin^3\psi\cot\theta X_1 + 0 - 2\sin^2\psi\cos\psi\cot\theta X_2 \\
&\quad - \sin^2\psi X_3 + 2\sin\psi\cos^2\psi\cot\theta X_1 \\
&\quad - \cos^2\psi X_3 - 2\cos^3\psi\cot\theta X_2 \\
&= 2\sin\psi\cot\theta X_1 - 2\cos\psi\cot\theta X_2 - X_3 \\
&= 2\cot\theta W - Z,
\end{aligned}$$

$$\begin{aligned}
\nabla_Z N &= \nabla_{2\partial_\psi}(\cos\psi X_1 + \sin\psi X_2) \\
&= -2\sin\psi X_1 + \cos\psi \nabla_{X_3} X_1 + 2\cos\psi X_2 + \sin\psi \nabla_{X_3} X_2 \\
&= -2\sin\psi X_1 - \cos\psi X_2 + 2\cos\psi X_2 + \sin\psi X_1 \\
&= -(\sin\psi X_1 - \cos\psi X_2) \\
&= -W.
\end{aligned}$$

The second fundamental form of T is

$$\begin{bmatrix} -(\nabla_W N, W) & -(\nabla_Z N, W) \\ -(\nabla_W N, Z) & -(\nabla_Z Z, N) \end{bmatrix} = \begin{bmatrix} -2\cot\theta & 1 \\ 1 & 0 \end{bmatrix}$$

The mean curvature of T is $-\cot\theta$. For $\theta = \frac{\pi}{2}$ the mean curvature of T is equal to zero.

h) We have seen that $K^{SO(3)}(\Pi) = 1$. Moreover,

$$\begin{aligned}
(B(W,W), B(Z,Z)) &= ((\nabla_W W, N)N, (\nabla_Z Z, N)N) \\
&= (\nabla_W N, W)(\nabla_Z N, Z) \ = \ 0, \\
\|B(W,Z)\|^2 &= \|(\nabla_W Z, N)N)\|^2 \ = \ (\nabla_W N, Z)^2 \ = \ 1.
\end{aligned}$$

Hence, according to the formula

$$K^T(\Pi) - K^{SO(3)}(\Pi) = (B(W,W), B(Z,Z)) - \|B(W,Z)\|^2,$$

we have $K^T(\Pi) = 0$. The metric induced on T is

$$g = \frac{1}{4}\sin^2\theta\, d\varphi^2 + \frac{1}{4}(\cos\theta\, d\varphi + d\psi)^2,$$

with θ fixed. We change to coordinates $(\sigma, \chi) = (\sin\theta\,\varphi, \cos\theta\,\varphi + \psi)$. This is equivalent to $(\varphi, \psi) = (\csc\theta\,\sigma, -\cot\theta\,\sigma + \chi)$, so that (σ, χ) are indeed coordinates. Then, the metric becomes the flat metric

$$g = \frac{1}{4}(d\sigma^2 + d\chi^2).$$