Riemannian Geometry

Exam - January 30, 2018

LMAC and MMA

Solutions

1.

a) G is a subgroup of $\mathcal{M}_{2\times 2}(\mathbb{C})$. Indeed, suppose $A \in G$ and $B \in G$. Then $AB \in G$ since

$$(AB)^*J(AB) = B^*(A^*JA)B = B^*JB = J,$$

and $A^{-1} \in G$ as

$$A^*JA = J \Rightarrow \ (A^*)^{-1}A^*JAA^{-1} = (A^{-1})^*JA^{-1} \ \Rightarrow \ J = (A^{-1})^*JA^{-1}.$$

b) We denote by $\mathcal{S}_{2\times 2}(\mathbb{C})$ the space of skew-hermitean 2×2 matrices. This is a 4-dimensional space. Let $f : \mathcal{M}_{2\times 2} \to \mathcal{S}_{2\times 2}$ be defined by

$$f(A) = A^*JA.$$

This function is smooth and

$$Df(A)(B) = B^*JA + A^*JB.$$

Suppose $A \in f^{-1}(J)$ and $C \in T_J\mathcal{S}_{2\times 2}(\mathbb{C}) \equiv \mathcal{S}_{2\times 2}(\mathbb{C})$. Choosing $B = -\frac{1}{2}AJC$, we get

$$\begin{aligned} Df(A)\left(-\frac{1}{2}AJC\right) &= \frac{1}{2}C^*JA^*JA - \frac{1}{2}A^*JAJC \\ &= -\frac{1}{2}CJ^2 - \frac{1}{2}J^2C \ = \ C. \end{aligned}$$

This shows that f is a submersion at A. Since A is arbitrary in $f^{-1}(J)$, J is a regular value of f. It follows that $G = f^{-1}(J)$ is a submanifold of $\mathcal{M}_{2\times 2}(\mathbb{C})$ of dimension $8 - 4 = 4$. Hence, G is a Lie group.

c)

$$T_IG = \ker Df(I) = \{B \in \mathcal{M}_{2\times 2} : B^*J + JB = 0\}.$$

A basis for T_IG is $\{B_1, B_2, B, \tilde{B}\}$, where

$$B_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

One easily checks that $[B_1, B] = 2B$, $[B_1, \tilde{B}] = 2\tilde{B}$ and $[B, \tilde{B}] = B_1$. Obviously, the bracket of B_2 with any other matrix is 0.

d) X and $\tilde{X}$ are the left invariant vector fields that take the values B and $\tilde{B}$ at the identity, respectively. Therefore, $[X, \tilde{X}]$ is the left invariant vector field that takes the value equal to the bracket of B and $\tilde{B}$ at the identity, i.e.

$$[X, \tilde{X}]_A = A[B, \tilde{B}] = AB_1.$$

2.

a) $\partial_x = \left(1, 0, -\frac{y}{x^2+y^2}\right)$ and $\partial_y = \left(0, 1, \frac{x}{x^2+y^2}\right)$. The unit normal to $\mathcal{H}$ with positive third component is

$$n = \frac{\partial_x \times \partial_y}{\|\partial_x \times \partial_y\|} = \frac{\left(\frac{y}{\sqrt{x^2+y^2}}, -\frac{x}{\sqrt{x^2+y^2}}, \sqrt{x^2+y^2}\right)}{\sqrt{x^2+y^2+1}}.$$

b)

$$\begin{aligned}
\iota(n)(dx \wedge dy \wedge dz) &= (\iota(n)dx) \wedge dy \wedge dz - dx \wedge (\iota(n)dy) \wedge dz \\
&\quad + dx \wedge dy \wedge (\iota(n)dz) \\
&= \frac{y}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+1}}\, dy \wedge dz \\
&\quad + \frac{x}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+1}}\, dx \wedge dz \\
&\quad + \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+1}}\, dx \wedge dy \;=\; \omega.
\end{aligned}$$

c)

$$\begin{aligned}
\eta \;=\; p^*\omega \;&=\; \frac{1}{\sqrt{r^2+1}} \left(\sin\theta \begin{vmatrix} \sin\theta & r\cos\theta \\ 0 & 1 \end{vmatrix} + \right. \\
&\quad + \cos\theta \begin{vmatrix} \cos\theta & -r\sin\theta \\ 0 & 1 \end{vmatrix} + \\
&\quad \left. + r \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} \right) dr \wedge d\theta \\
&= \sqrt{r^2+1}\, dr \wedge d\theta.
\end{aligned}$$

d)

$$
\begin{aligned}
L_X\eta &= L_{\left(\frac{\partial f}{\partial r}\,\partial_r+\frac{1}{r^2+1}\frac{\partial f}{\partial\theta}\,\partial_\theta\right)}\sqrt{r^2+1}\,dr\wedge d\theta \\
&= \frac{\partial f}{\partial r}\frac{r}{\sqrt{r^2+1}}\,dr\wedge d\theta \\
&\quad +\sqrt{r^2+1}\frac{\partial^2 f}{\partial r^2}\,dr\wedge d\theta \\
&\quad +\frac{1}{\sqrt{r^2+1}}\frac{\partial^2 f}{\partial\theta^2}\,dr\wedge d\theta \\
&= (\operatorname{div}X)\,\eta,
\end{aligned}
$$

where

$$
\operatorname{div}X = \frac{\partial f}{\partial r}\frac{r}{r^2+1}+\frac{\partial^2 f}{\partial r^2}+\frac{1}{r^2+1}\frac{\partial^2 f}{\partial\theta^2}
$$

e) Using Cartan's formula and Stokes' Theorem, we have

$$
\begin{aligned}
\int_S L_X\eta &= \int_S[d(\iota(X)\eta)+\iota(X)d\eta] = \int_{\partial S}\iota(X)\eta \\
&= \int_{\partial S}\iota\left(\frac{\partial f}{\partial r}\,\partial_r+\frac{1}{r^2+1}\frac{\partial f}{\partial\theta}\,\partial_\theta\right)\sqrt{r^2+1}\,dr\wedge d\theta \\
&= \int_{\partial S}-\frac{1}{\sqrt{r^2+1}}\frac{\partial f}{\partial\theta}\,dr+\frac{\partial f}{\partial r}\sqrt{r^2+1}\,d\theta \\
&= \int_{\theta_0}^{\theta_1}\frac{\partial f}{\partial r}(r_1,\theta)\sqrt{r_1^2+1}\,d\theta-\int_{\theta_0}^{\theta_1}\frac{\partial f}{\partial r}(r_0,\theta)\sqrt{r_0^2+1}\,d\theta \\
&\quad +\int_{r_0}^{r_1}\frac{1}{\sqrt{r^2+1}}\frac{\partial f}{\partial\theta}(r,\theta_1)\,dr-\int_{r_0}^{r_1}\frac{1}{\sqrt{r^2+1}}\frac{\partial f}{\partial\theta}(r,\theta_0)\,dr.
\end{aligned}
$$

3.

a) Since $(x,y,z)=(r\cos\theta,r\sin\theta,\theta)$, clearly

$$
\begin{cases}
dx = \cos\theta\,dr-r\sin\theta\,d\theta, \\
dy = \sin\theta\,dr+r\cos\theta\,d\theta, \\
dz = d\theta.
\end{cases}
$$

So the metric induced on the helicoid by the euclidean metric,

$$
ds^2=dx^2+dy^2+dz^2,
$$

is

$$
ds^2=dr^2+(r^2+1)\,d\theta^2.
$$

b) The lagrangian is

$$L(r,\theta,\dot r,\dot\theta) = \frac{1}{2}\left(\dot r^2 + (r^2+1)\dot\theta^2\right).$$

The equations for the geodesics are

$$\begin{aligned} \frac{d}{dt}\frac{\partial L}{\partial \dot r} - \frac{\partial L}{\partial r} &= 0, \\ \frac{d}{dt}\frac{\partial L}{\partial \dot\theta} - \frac{\partial L}{\partial \theta} &= 0. \end{aligned}$$

These equations are equivalent to

$$\begin{aligned} \ddot r - r\dot\theta^2 &= 0, \\ \frac{d}{dt}[(r^2+1)\dot\theta] = 0 \Leftrightarrow \ddot\theta + \frac{2r}{r^2+1}\dot r\dot\theta &= 0. \end{aligned}$$

We read that the nonzero Christoffel symbols are

$$\Gamma^r_{\theta\theta} = -r, \qquad \Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{r}{r^2+1}.$$

c) We know that

$$\nabla_{\partial_r}\partial_r = 0, \qquad \nabla_{\partial_r}\partial_\theta = \nabla_{\partial_\theta}\partial_r = \frac{r}{r^2+1}\partial_\theta, \qquad \nabla_{\partial_\theta}\partial_\theta = -r\partial_r.$$

Using these equalities, we get

$$\begin{aligned} R(\partial_r,\partial_\theta)\partial_r &= \nabla_{\partial_r}\nabla_{\partial_\theta}\partial_r - \nabla_{\partial_\theta}\nabla_{\partial_r}\partial_r - \nabla_{[\partial_r,\partial_\theta]}\partial_r \\ &= \nabla_{\partial_r}\left(\frac{r}{r^2+1}\partial_\theta\right) \\ &= \frac{1-r^2}{(r^2+1)^2}\partial_\theta + \frac{r^2}{(r^2+1)^2}\partial_\theta \\ &= \frac{1}{(r^2+1)^2}\partial_\theta. \end{aligned}$$

Thus

$$R(\partial_r,\partial_\theta,\partial_r,\partial_\theta) = (R(\partial_r,\partial_\theta)\partial_r,\partial_\theta) = \frac{1}{r^2+1}.$$

The curvature of $\mathcal{H}$ is

$$K = -\frac{R(\partial_r,\partial_\theta,\partial_r,\partial_\theta)}{(\partial_r,\partial_r)(\partial_\theta,\partial_\theta) - (\partial_r,\partial_\theta)^2} = -\frac{1}{(r^2+1)^2}.$$

d) Since V is parallel along c and $\dot{c} = \partial_r$,

$$\nabla_{\partial_r}(V^r \partial_r + V^\theta \partial_\theta) = \frac{\partial V^r}{\partial r}\partial_r + \frac{\partial V^\theta}{\partial r}\partial_\theta + V^\theta \frac{r}{r^2+1}\partial_\theta = 0.$$

The components of V satisfy

$$\begin{cases} \dfrac{\partial V^r}{\partial r} = 0, \\ \dfrac{\partial V^\theta}{\partial r} + \dfrac{r}{r^2+1} V^\theta = 0. \end{cases}$$

The solution of this system is

$$\begin{cases} V^r(r,\theta_0) = V^r(r_0,\theta_0), \\ V^\theta(r,\theta_0) = \dfrac{\sqrt{r_0^2+1}}{\sqrt{r^2+1}} V^\theta(r_0,\theta_0). \end{cases}$$

So, we have

$$V(r,\theta) = V^r(r_0,\theta_0)\partial_r + \frac{\sqrt{r_0^2+1}}{\sqrt{r^2+1}} V^\theta(r_0,\theta_0)\partial_\theta.$$

Note that the angle between V and $\dot{c}$ is constant (because c is a geodesic) and that the length of V is constant.

e) The reparemeterization of c by arclength is $\tilde{c}(\theta) = \left(r_0, \frac{\theta}{\sqrt{r_0^2+1}}\right)$ since $\|\dot{\tilde{c}}\| = \left\|\frac{\partial_\theta}{\sqrt{r_0^2+1}}\right\| = 1$. The acceleration of $\tilde{c}$ is

$$\nabla_{\dot{\tilde{c}}}\dot{\tilde{c}} \;=\; \nabla_{\frac{\partial_\theta}{\sqrt{r_0^2+1}}}\frac{\partial_\theta}{\sqrt{r_0^2+1}} \;=\; -\frac{r_0}{r_0^2+1}\partial_r.$$

The frame $\left(\dot{\tilde{c}}, -\partial_r\right)$ is orthonormal and positively oriented. Hence, the geodesic curvature of $\tilde{c}$ is $\frac{r_0}{r_0^2+1}$.

f) We have

$$\partial_r = (\cos\theta, \sin\theta, 0) = \tilde{\partial}_r,$$
$$\partial_\theta = (-r\sin\theta, r\cos\theta, 1) = \tilde{\partial}_\theta + \tilde{\partial}_z.$$

Since the metric is

$$ds^2 = dr^2 + r^2\, d\theta^2 + dz^2,$$

$\tilde{\partial}_r$, $\frac{1}{r}\tilde{\partial}_\theta$ and $\tilde{\partial}_z$ are unit vectors. We note that $\tilde{\partial}_r \times \frac{1}{r}\tilde{\partial}_\theta = \tilde{\partial}_z$ and $\tilde{\partial}_r \times \tilde{\partial}_z = -\frac{1}{r}\tilde{\partial}_\theta$. We have

$$E_r = \frac{\partial_r}{\|\partial_r\|} = \tilde{\partial}_r,$$

$$E_\theta = \frac{\partial_\theta}{\|\partial_\theta\|} = \frac{\tilde{\partial}_\theta + \tilde{\partial}_z}{\|\tilde{\partial}_\theta + \tilde{\partial}_z\|} = \frac{\tilde{\partial}_\theta + \tilde{\partial}_z}{\sqrt{r^2+1}}.$$

The unit normal n to $\mathcal{H}$ with a positive z component is

$$n = E_r \times E_\theta = \frac{-\tilde{\partial}_\theta + r^2\tilde{\partial}_z}{r\sqrt{r^2+1}}.$$

g)

$$\begin{aligned}
\tilde{\nabla}_{E_r} n &= \tilde{\nabla}_{\tilde{\partial}_r}\left(\frac{-\tilde{\partial}_\theta + r^2\tilde{\partial}_z}{r\sqrt{r^2+1}}\right) \\
&= \frac{1}{r^2\sqrt{r^2+1}}\tilde{\partial}_\theta + \frac{1}{\sqrt{(r^2+1)^3}}\tilde{\partial}_\theta - \frac{1}{r^2\sqrt{r^2+1}}\tilde{\partial}_\theta \\
&\quad + \frac{1}{\sqrt{r^2+1}}\tilde{\partial}_z - \frac{r^2}{\sqrt{(r^2+1)^3}}\tilde{\partial}_z \\
&= \frac{1}{\sqrt{(r^2+1)^3}}\tilde{\partial}_\theta + \frac{1}{\sqrt{(r^2+1)^3}}\tilde{\partial}_z, \\
\tilde{\nabla}_{E_\theta} n &= \tilde{\nabla}_{\frac{\tilde{\partial}_\theta + \tilde{\partial}_z}{\sqrt{r^2+1}}}\left(\frac{-\tilde{\partial}_\theta + r^2\tilde{\partial}_z}{r\sqrt{r^2+1}}\right) \\
&= \frac{1}{r^2+1}\partial_r.
\end{aligned}$$

h) Using the results obtained above, we get

$$\begin{aligned}
\left(-\tilde{\nabla}_{E_r} n, E_r\right) &= 0, \\
\left(-\tilde{\nabla}_{E_r} n, E_\theta\right) &= -\frac{1}{(r^2+1)^2}(\tilde{\partial}_\theta + \tilde{\partial}_z, \tilde{\partial}_\theta + \tilde{\partial}_z) = -\frac{1}{r^2+1}, \\
\left(-\tilde{\nabla}_{E_\theta} n, E_r\right) &= -\frac{1}{r^2+1}, \\
\left(-\tilde{\nabla}_{E_\theta} n, E_\theta\right) &= 0.
\end{aligned}$$

The second fundamental form of $\mathcal{H}$ is

$$\mathrm{II}(r,\theta) = \begin{bmatrix} 0 & -\frac{1}{r^2+1} \\ -\frac{1}{r^2+1} & 0 \end{bmatrix}.$$

i) The principal directions are the eigenvectors of the second fundamental form and the principal curvatures are the eigenvalues of the second fundamental form. These are $E_r + E_\theta$ with principal curvature $-\frac{1}{r^2+1}$ and $E_r - E_\theta$ with principal curvature $\frac{1}{r^2+1}$ The mean curvature of $\mathcal{H}$ is the average of the principal curvatures, or half the trace of the second fundamental form. In this case the mean curvature is 0.