Riemannian Geometry

February 4, 2021

LMAC and MMA

1st Test	–	Questions **1.** and **2.**	–	90 minutes
2nd Test	–	Questions **3.** and **4.**	–	90 minutes
Exam	–	All questions	–	3 hours

Show your calculations

1. Consider $SL(2,\mathbb{R}) = \{A \in M_{2\times 2} : \ \det A = 1\}$.

a) Calculate $\mathfrak{sl}(2,\mathbb{R})$ and show that it is spanned by (3)

$$B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \qquad (*)$$

b) For $F \in \mathfrak{sl}(2,\mathbb{R})$, let X^F be the left invariant vector field corresponding to F. Check that (2)

$$[B,C] = D, \quad [D,B] = 2B, \quad [D,C] = -2C.$$

Calculate $[X^B, X^C]_g$, for $g = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$.

c) Calculate the flow $\phi_t(g_0)$ of X^B, where $g_0 = \begin{bmatrix} p_0 & q_0 \\ r_0 & s_0 \end{bmatrix}$. (3)

d) Calculate $\exp(tX^B)$. (2)

e) As you know, the fact that X^B is left-invariant is equivalent to $L_g \circ \phi_t = \phi_t \circ L_g$. Check this equality directly. (3)

f) Using the definition of Lie derivative, calculate $L_{X^B} X^C$. (3)

2. Let M be an $(n+1)$-dimensional compact oriented manifold with nonempty boundary ∂M. Let $X \in \mathcal{X}(M)$ be such that $X|_{\partial M}$ is tangent to ∂M. Let $\omega \in \Omega^n(M)$. Show that the $(n+1)$-form $d(L_X \omega)$ has a zero in M. (4)

3. Let B, C and D be as in (*). We put a left invariant metric on $SL(2,\mathbb{R})$ so that $\{B, C, D\}$ is orthonormal. Therefore, given $F \in \mathfrak{sl}(2,\mathbb{R})$, the map $\psi_s(g) = L_{\exp(sF)}(g)$ is an isometry.

a) Show that the (Killing) vector field defined by $X_g = \frac{d}{ds}\psi_s(g)\big|_{s=0}$ is right invariant. (2)

b) Recall that a vector field X is Killing iff it satifies the Killing equation (3)

$$(\nabla_Y X, Z) + (Y, \nabla_Z X) = 0, \quad \text{for all } Y, Z \in \mathcal{X}(M).$$

Show that X^B is not Killing. Suggestion: take $Y = X^C$ and $Z = X^D$ and use the Koszul formula.

c) Given an orthonormal frame $\{E_1, E_2, E_3\}$, show that (3)

$$\nabla_{E_i} E_j = \frac{1}{2}\sum_{k=1}^{3}(c_{ijk} + c_{kij} + c_{kji})E_k, \quad \text{where } [E_i, E_j] = \sum_{k=1}^{3} c_{ijk}E_k.$$

Use this formula to calculate $\nabla_{X^B} X^B$.

d) Given that (3)

$$\begin{aligned} \nabla_{X^C} X^B &= -\tfrac{1}{2}X^D, & \nabla_{X^C} X^C &= -2X^D, \\ \nabla_{X^D} X^C &= \tfrac{1}{2}X^B, & \nabla_{X^D} X^D &= 0, \end{aligned}$$

calculate the sectional curvature of the plane spanned by X^C and X^D.

4. Let $f : M \to M$ be an isometry and S be the set of fixed points of f. Let $p \in M$ be fixed.

a) Suppose $V \in T_pM$. Prove that $(df)_p(V) = V$ iff the geodesic with initial velocity V is contained in S. (3)

b) Show that there exists $B_\epsilon(p)$ such that the intersection of $B_\epsilon(p)$ with the set of geodesics whose initial velocities V satisfy $(df)_pV = V$ is a submanifold N of M. (3)

c) Justify that there exists $\epsilon > 0$ such that if $q \in B_\epsilon(p) \setminus N$, then $q \notin S$. Conclude that S is a submanifold of M (whose connected components might have different dimensions). (3)