Riemannian Geometry

January 29, 2020

LMAC and MMA

1st Test	–	Question 1	–	90 minutes
2nd Test	–	Question 2	–	90 minutes
Exam	–	All questions	–	3 hours

Show your calculations

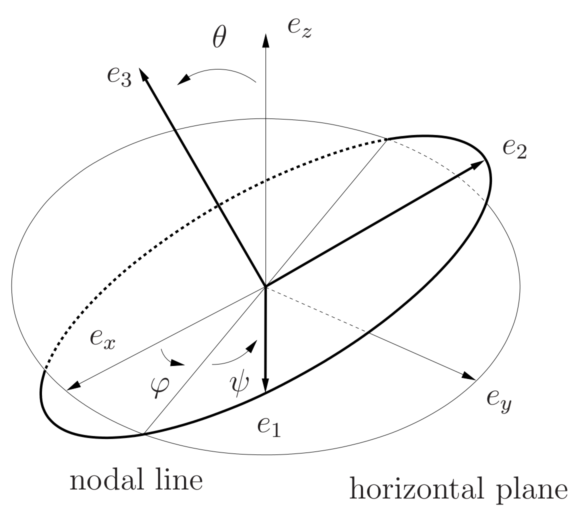
1. Consider the parameterization $S :]0,\pi[\times]0,2\pi[\times]0,2\pi[\to SO(3)$, defined by

$$\begin{aligned} S(\theta,\varphi,\psi) &= R_\varphi R_\theta R_\psi \\ &= \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \end{aligned}$$

and consider the volume form

$$\omega = \frac{1}{8}\sin\theta\, d\theta \wedge d\varphi \wedge d\psi.$$

The variables θ, φ and ψ are called the Euler angles. Their geometric interpretation is sketched in the figure below: if the rotation carries the canonical basis (e_x, e_y, e_z) to (e_1, e_2, e_3), then θ is the angle between e_3 and e_z, φ is the angle between the line of intersection of the planes spanned by $\{e_1, e_2\}$ and $\{e_x, e_y\}$ (called the nodal line) and the x-axis, and ψ is the angle between e_1 and the nodal line.



a) Let $X = 2\partial_\theta$. Use Cartan's formula and Stokes' Theorem to calculate (2)

$$\int_{]0,\frac{\pi}{2}[\times]0,2\pi[\times]0,2\pi[} L_X\omega.$$

Justify your answer.

b) Identify linear isomorphism $\Omega : so(3) \to \mathbb{R}^3$ such that (2)

$$A\xi = \Omega(A) \times \xi$$

for all $\xi \in \mathbb{R}^3$ and $A \in so(3)$. Check that $\Omega([A, B]) = \Omega(A) \times \Omega(B)$.

c) Clearly, we have that $R(v \times w) = Rv \times Rw$. Show that (2)

$$(R^{-1}\Omega^{-1}(v)R)\, w = \Omega^{-1}(R^{-1}v)\, w,$$

for all $w \in \mathbb{R}^3$.

d) Let $(\theta(\cdot), \varphi(\cdot), \psi(\cdot))$ be a curve. Check that $R_{-\varphi}\dot{R}_\varphi = \dot{\varphi}\,\Omega^{-1}(e_3)$. (1)
What are $R_{-\theta}\dot{R}_\theta$ and $R_{-\psi}\dot{R}_\psi$?

e) For $\dot{S} \in T_S SO(3)$, let A be such that $\dot{S} = SA$. Show that (2)

$$\begin{aligned}\Omega(A) &= \big(\dot{\theta}\cos\psi + \dot{\varphi}\sin\theta\sin\psi\big)e_1 + \big(-\dot{\theta}\sin\psi + \dot{\varphi}\sin\theta\cos\psi\big)e_2 \\ &\quad + \big(\dot{\varphi}\cos\theta + \dot{\psi}\big)e_3.\end{aligned}$$

f) Determine the left invariant vector fields $X = X^\theta\partial_\theta + X^\varphi\partial_\varphi + X^\psi\partial_\psi$ (1)
corresponding to $\Omega^{-1}(e_1)$, $\Omega^{-1}(-e_2)$ and $\Omega^{-1}(e_3)$.

2. Consider the metric on $SO(3)$ given in Euler coordinates by

$$g = \frac{1}{4}(d\theta^2 + \sin^2\theta\, d\varphi^2) + \frac{1}{4}(\cos\theta\, d\varphi + d\psi)^2.$$

a) Check that the coframe (1)

$$\begin{aligned}\omega^1 &= \frac{1}{2}(\cos\psi\, d\theta + \sin\theta\sin\psi\, d\varphi), \\ \omega^2 &= \frac{1}{2}(\sin\psi\, d\theta - \sin\theta\cos\psi\, d\varphi), \\ \omega^3 &= \frac{1}{2}(\cos\theta\, d\varphi + d\psi)\end{aligned}$$

is orthonormal. The dual frame is

$$\begin{aligned}X &= 2\left(\cos\psi\,\partial_\theta + \frac{\sin\psi}{\sin\theta}\,\partial_\varphi - \sin\psi\cot\theta\,\partial_\psi\right), \\ Y &= 2\left(\sin\psi\,\partial_\theta - \frac{\cos\psi}{\sin\theta}\,\partial_\varphi + \cos\psi\cot\theta\,\partial_\psi\right), \\ Z &= 2\,\partial_\psi.\end{aligned}$$

b) Determine the constant a such that (1)

$$\begin{aligned} d\omega^1 &= a\,\omega^2 \wedge \omega^3, \\ d\omega^2 &= -a\,\omega^1 \wedge \omega^3, \\ d\omega^3 &= a\,\omega^1 \wedge \omega^2. \end{aligned}$$

c) Determine the connection forms ω^1_2, ω^1_3 and ω^2_3. (1)

d) Determine the curvature tensor. (2)

e) Use the connection forms to calculate the covariant derivatives $\nabla_X X$, $\nabla_Y X$, $\nabla_Z X$, $\nabla_X Y$, $\nabla_Y Y$ and $\nabla_Z Y$. (1)

f) Define (1)

$$W = \sin\psi\, X - \cos\psi\, Y \quad \text{and} \quad N = \cos\psi\, X + \sin\psi\, Y.$$

The frame (W, Z, N) is orthonormal. Check that W and Z are tangent to the torus, T, where θ is a fixed constant.

g) Compute the second fundamental form, B, of T. For what values of θ is the mean curvature of T equal to zero? (2)

h) Let Π be the plane spanned by W and Z. Check the formula (1)

$$K^T(\Pi) - K^{SO(3)}(\Pi) = (B(W,W), B(Z,Z)) - \|B(W,Z)\|^2$$

in the present situation.