Riemannian Geometry

January 30, 2018

LMAC and MMA

1st Test	–	Questions 1 and 2	–	90 minutes
2nd Test	–	Question 3	–	90 minutes
Exam	–	All questions	–	3 hours

Show your calculations

1. Consider

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

and let

$$G = \{U \in \mathcal{M}_{2\times 2}(\mathbb{C}) : U^* J U = J\}.$$

a) Prove that G is a group. (1)

b) Prove that G is a Lie group. (1)

c) Determine a basis for $T_I G$ and the Lie algebra $\mathfrak{g}$. (1.5)

d) Let (1)

$$B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Consider the vector fields X and $\tilde{X}$ in $\mathfrak{X}(G)$, defined by

$$X_A = AB, \qquad \tilde{X}_A = A\tilde{B}.$$

Calculate $[X, \tilde{X}]$ and justify your answer.

2. Consider the helicoid

$$\mathcal{H} := \left\{(x, y, z) \in \mathbb{R}^3 : x > 0 \text{ and } z = \arctan\frac{y}{x}\right\}.$$

a) Determine n, the unit normal to $\mathcal{H}$ with positive third component. (1)

b) Compute $\omega := \iota(n)(dx \wedge dy \wedge dz)$. (1)

c) Consider the parameterization p of $\mathcal{H}$, (1)

$$p(r, \theta) = (r\cos\theta, r\sin\theta, \theta),$$

defined in $\Omega := \left\{(r, \theta) \in \mathbb{R}^2 : r > 0 \text{ and } \theta \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\right\}$. Compute the area form $\eta := p^*\omega$.

d) Consider the vector field (1.5)

$$X = \frac{\partial f}{\partial r}\,\partial_r + \frac{1}{r^2+1}\frac{\partial f}{\partial \theta}\,\partial_\theta,$$

where f is a smooth function defined in Ω. Calculate $L_X\eta$ and the divergence of X.

e) Let (1)

$$S = \left\{(r,\theta) \in \mathbb{R}^2 : 0 < r_0 < r < r_1 \text{ and } -\frac{\pi}{2} < \theta_0 < \theta < \theta_1 < \frac{\pi}{2}\right\},$$

with the canonical orientation. Use Stokes' Theorem to write $\int_S L_X\eta$ in terms of an integral on the boundary of S.

3. Consider $\mathbb{R}^3$ with the euclidean metric. Consider also the helicoid $\mathcal{H}$, parameterized by

$$p(r,\theta) = (r\cos\theta, r\sin\theta, \theta),$$

with $(r,\theta) \in \mathbb{R}^+ \times \mathbb{R}$.

a) Write the metric induced on $\mathcal{H}$. (1)

b) Write the equations for the geodesics and the Christoffel symbols. (1)

c) Compute $\nabla_{\partial_r}\partial_r$, $\nabla_{\partial_r}\partial_\theta$, $\nabla_{\partial_\theta}\partial_\theta$, $R(\partial_r,\partial_\theta)\partial_r$, $R(\partial_r,\partial_\theta,\partial_r,\partial_\theta)$ and the curvature of $\mathcal{H}$. (1.5)

d) Suppose $V = V^r\partial_r + V^\theta\partial_\theta$ is parallel along $c(r) = (r,\theta_0)$, with $V(r_0,\theta_0) = V_0^r\partial_r + V_0^\theta\partial_\theta$. Determine V. (1)

e) Reparameterize $c(\theta) = (r_0,\theta)$ by arclength and calculate its geodesic curvature. (1)

f) Let (r,θ,z) be cylindrical coordinates in $\mathbb{R}^3$, and $(\tilde{\partial}_r, \tilde{\partial}_\theta, \tilde{\partial}_z)$ be the associated frame. Write $E_r := \partial_r/\|\partial_r\|$, $E_\theta := \partial_\theta/\|\partial_\theta\|$ and the unit normal n to $\mathcal{H}$ with a positive z component in this frame. (1)

g) Recall that the nonzero Christoffel symbols for the euclidean metric in cylindrical coordinates are $\tilde{\Gamma}^r_{\theta\theta} = -r$ and $\tilde{\Gamma}^\theta_{r\theta} = \tilde{\Gamma}^\theta_{\theta r} = \frac{1}{r}$. Calculate $\tilde{\nabla}_{E_r}n$ and $\tilde{\nabla}_{E_\theta}n$. (1.5)

h) Determine the second fundamental form of $\mathcal{H}$. (1)

i) What are the principal directions and the principal curvatures of $\mathcal{H}$? What is the mean curvature of $\mathcal{H}$? (1)