Riemannian Geometry

Exam - July 18, 2022

MMAC

Solutions

1.

a) We note that

$$\begin{array}{rcl}
[X,Y] & = & [(\tilde{X},\hat{X}),(\tilde{Y},\hat{Y})] \\
& = & (\tilde{X},0)(\tilde{Y},0)+(0,\hat{X})(\tilde{Y},0)+(\tilde{X},0)(0,\hat{Y})+(0,\hat{X})(0,\hat{Y}) \\
& & -(\tilde{Y},0)(\tilde{X},0)-(0,\hat{Y})(\tilde{X},0)-(\tilde{Y},0)(0,\hat{X})-(0,\hat{Y})(0,\hat{X}) \\
& = & ([\tilde{X},\tilde{Y}],0)+(0,[\hat{X},\hat{Y}]) \\
& = & ([\tilde{X},\tilde{Y}],[\hat{X},\hat{Y}]).
\end{array}$$

because $(0,\hat{X})(\tilde{Y},0)-(\tilde{Y},0)(0,\hat{X})=0$ and $(\tilde{X},0)(0,\hat{Y})-(0,\hat{Y})(\tilde{X},0)=0$. (Indeed, for $f:M\to\mathbb{R}$, we have, for example, that

$$(0,\hat{X})(\tilde{Y},0)f=(\tilde{Y},0)(0,\hat{X})f=\sum_{i,j}\tilde{Y}^i(\tilde{x})\hat{X}^j(\hat{x})\frac{\partial^2 f}{\partial\tilde{x}^i\partial\hat{x}^j}$$

as $\tilde{Y}=\sum_i\tilde{Y}^i(\tilde{x})\partial_{\tilde{x}^i}$ and $\hat{X}=\sum_j\hat{X}^j(\hat{x})\partial_{\hat{x}^j}$, for $\tilde{x}$ and $\hat{x}$ local coordinates on $\tilde{M}$ and $\hat{M}$, respectively). This shows that $\widetilde{[X,Y]}=[\tilde{X},\tilde{Y}]$ and

$\widehat{[X,Y]} = [\hat{X},\hat{Y}]$. Hence, by the Koszul formula, we have

$$\begin{aligned}
2g(\nabla_X Y, Z) &= X\cdot g(Y,Z) + Y\cdot g(X,Z) - Z\cdot g(X,Y) \\
&\quad -g(X,[Y,Z]) - g(Y,[X,Z]) + g(Z,[X,Y]) \\
&= (\tilde{X},\hat{X})\cdot(\tilde{g}(\tilde{Y},\tilde{Z}) + \hat{g}(\hat{Y},\hat{Z})) \\
&\quad +(\tilde{Y},\hat{Y})\cdot(\tilde{g}(\tilde{X},\tilde{Z}) + \hat{g}(\hat{X},\hat{Z})) \\
&\quad -(\tilde{Z},\hat{Z})\cdot(\tilde{g}(\tilde{X},\tilde{Y}) + \hat{g}(\hat{X},\hat{Y})) \\
&\quad -\tilde{g}(\tilde{X},\widetilde{[Y,Z]}) - \hat{g}(\hat{X},\widehat{[Y,Z]}) \\
&\quad -\tilde{g}(\tilde{Y},\widetilde{[X,Z]}) - \hat{g}(\hat{Y},\widehat{[X,Z]}) \\
&\quad +\tilde{g}(\tilde{Z},\widetilde{[X,Y]}) + \hat{g}(\hat{Z},\widehat{[X,Y]}) \\
&= \tilde{X}\cdot\tilde{g}(\tilde{Y},\tilde{Z}) + \hat{X}\cdot\hat{g}(\hat{Y},\hat{Z}) \\
&\quad +\tilde{Y}\cdot\tilde{g}(\tilde{X},\tilde{Z}) + \hat{Y}\cdot\hat{g}(\hat{X},\hat{Z}) \\
&\quad -\tilde{Z}\cdot\tilde{g}(\tilde{X},\tilde{Y}) - \hat{Z}\cdot\hat{g}(\hat{X},\hat{Y}) \\
&\quad -\tilde{g}(\tilde{X},[\tilde{Y},\tilde{Z}]) - \hat{g}(\hat{X},[\hat{Y},\hat{Z}]) \\
&\quad -\tilde{g}(\tilde{Y},[\tilde{X},\tilde{Z}]) - \hat{g}(\hat{Y},[\hat{X},\hat{Z}]) \\
&\quad +\tilde{g}(\tilde{Z},[\tilde{X},\tilde{Y}]) + \hat{g}(\hat{Z},[\hat{X},\hat{Y}]) \\
&= 2\tilde{g}(\tilde{\nabla}_{\tilde{X}}\tilde{Y},\tilde{Z}) + 2\hat{g}(\hat{\nabla}_{\hat{X}}\hat{Y},\hat{Z}).
\end{aligned}$$

This proves that $\widetilde{\nabla_X Y} = \tilde{\nabla}_{\tilde{X}}\tilde{Y}$ and $\widehat{\nabla_X Y} = \hat{\nabla}_{\hat{X}}\hat{Y}$.

b) The Riemann curvature tensor R of M is

$$\begin{aligned}
R(X,Y,Z,W) &= g(\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, W) \\
&= g(\nabla_X(\tilde{\nabla}_{\tilde{Y}}\tilde{Z}, \hat{\nabla}_{\hat{Y}}\hat{Z}), W) \\
&\quad -g(\nabla_Y(\tilde{\nabla}_{\tilde{X}}\tilde{Z}, \hat{\nabla}_{\hat{X}}\hat{Z}), W) \\
&\quad -g(\tilde{\nabla}_{\widetilde{[X,Y]}}\tilde{Z}, \hat{\nabla}_{\widehat{[X,Y]}}\hat{Z}, W) \\
&= g(\tilde{\nabla}_{\tilde{X}}\tilde{\nabla}_{\tilde{Y}}\tilde{Z}, \hat{\nabla}_{\hat{X}}\hat{\nabla}_{\hat{Y}}\hat{Z}, W) \\
&\quad -g(\tilde{\nabla}_{\tilde{Y}}\tilde{\nabla}_{\tilde{X}}\tilde{Z}, \hat{\nabla}_{\hat{Y}}\hat{\nabla}_{\hat{X}}\hat{Z}, W) \\
&\quad -g(\tilde{\nabla}_{[\tilde{X},\tilde{Y}]}\tilde{Z}, \hat{\nabla}_{[\hat{X},\hat{Y}]}\hat{Z}, W) \\
&= \tilde{g}(\tilde{\nabla}_{\tilde{X}}\tilde{\nabla}_{\tilde{Y}}\tilde{Z}, \tilde{W}) + \hat{g}(\hat{\nabla}_{\hat{X}}\hat{\nabla}_{\hat{Y}}\hat{Z}, \hat{W}) \\
&\quad -\tilde{g}(\tilde{\nabla}_{\tilde{Y}}\tilde{\nabla}_{\tilde{X}}\tilde{Z}, \tilde{W}) - \hat{g}(\hat{\nabla}_{\hat{Y}}\hat{\nabla}_{\hat{X}}\hat{Z}, \hat{W}) \\
&\quad -\tilde{g}(\tilde{\nabla}_{[\tilde{X},\tilde{Y}]}\tilde{Z}, \tilde{W}) - \hat{g}(\hat{\nabla}_{[\hat{X},\hat{Y}]}\hat{Z}, \hat{W}) \\
&= \tilde{R}(\tilde{X},\tilde{Y},\tilde{Z},\tilde{W}) + \hat{R}(\hat{X},\hat{Y},\hat{Z},\hat{W}).
\end{aligned}$$

If $X = \tilde{X}$ and $Y = \hat{Y}$, then

$$R(X,Y,X,Y) = \tilde{R}(\tilde{X},0,\tilde{X},0) + \hat{R}(0,\hat{Y},0,\hat{Y}) = 0 + 0 = 0,$$

and so the curvature of the plane Π spanned by X and Y is zero.

2.

a) Since the sphere S^3 has constant sectional curvature equal to 1, we know that

$$R_{ijij} = -1(g_{ii}g_{jj} - g_{ij}g_{ij})$$

and

$$R_{ijkl} = -1(g_{ik}g_{jl} - g_{il}g_{kj}) = -1(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{kj}).$$

In order for us to obtain nonzero components of the curvature tensor:

$$\begin{array}{rcll} i \neq k & \Rightarrow & (l = i \wedge j = k) & R_{ikki} = 1, \\ j \neq l & \Rightarrow & (l = i \wedge j = k) & R_{ljjl} = 1, \\ (i = k \wedge j = l) & \Rightarrow & i \neq j & R_{ijij} = -1. \end{array}$$

The curvature tensor is

$$\begin{aligned} R &= -\sum_{i<j} \omega^i \wedge \omega^j \oplus \omega^i \wedge \omega^j \\ &= -\omega^1 \wedge \omega^2 \oplus \omega^1 \wedge \omega^2 - \omega^1 \wedge \omega^3 \oplus \omega^1 \wedge \omega^3 - \omega^2 \wedge \omega^3 \oplus \omega^2 \wedge \omega^3. \end{aligned}$$

b) We know that

$$K^M(\Pi) - K^{S^3}(\Pi) = \frac{(B(X,X), B(Y,Y)) - \|B(X,Y)\|^2}{\|X\|^2\|Y\|^2 - (X,Y)^2},$$

for X and Y linearly independent in T_pM and spanning $\Pi = \Pi_p$. We take (X, Y) equal to an orthonormal frame formed by principle directions of the embedding of M is S^3. The value of $K^M(\Pi)$ is the Gaussian curvature K of the manifold M. As

$$B(Z,W) = (S_N(Z), W)N, \quad S_N(X) = \lambda_1 X, \quad S_N(Y) = \lambda_2 Y$$

(where $N \in S^3$ is unit and normal to M), we obtain

$$K - 1 = \lambda_1\lambda_2.$$

3.

a) Let X^v and X^w be the left-invariant vector fields corresponding to v and w, respectively, and ϕ_t and ψ_t be the corresponding flows. We claim that $\phi_t \circ \psi_s = \psi_s \circ \phi_t$. This implies that $[X^v, X^w] = 0$. In particular, we have that $[v, w] = [X^v, X^w]_e = 0$.

Recall that the fact that $(L_g)_\star X^v = X^v$ implies that $L_g \circ \phi_t = \phi_t \circ L_g$, and thus

$$g\phi_t(e) = \phi_t(g).$$

Similarly,

$$g\psi_s(e) = \psi_s(g).$$

Note also that $\psi_s(hg) = hg\psi_s(e) = h\psi_s(g)$.
To prove the claim, let $h \in G$. We have

$$\begin{aligned} \phi_t \circ \psi_s(h) &= \phi_t(\psi_s(h)) = \psi_s(h)\phi_t(e) = \phi_t(e)\psi_s(h) \\ &= \psi_s(\phi_t(e)h) = \psi_s(h\phi_t(e)) = \psi_s(\phi_t(h)) \\ &= \psi_s \circ \phi_t(h), \end{aligned}$$

because G is abelian. This means that $\phi_t \circ \psi_s = \psi_s \circ \phi_t$, and so the proof is finished.

b) $[X^v, W^w]$ is left-invariant because, for any $g \in G$, $(L_g)_\star[X^v, W^w] = [(L_g)_\star X^v, (L_g)_\star W^w] = [X^v, W^w]$ and, if $[v, w] = 0$, then $[X^v, W^w]_e = 0$. So, under the hypothesis, $[X^v, W^w]$ is the left-invariant vector field whose value at the identity is zero, which means it is the zero vector field.

c) If

$$\dot{c}(t) = X^v_{c(t)}, \quad c(0) = e,$$

and

$$\dot{\gamma}(t) = X^{sv}_{\gamma(t)}, \quad \gamma(0) = e,$$

then we have the homogeneity property $\gamma(t) = c(st)$, because $(L_{\gamma(t)})_\star(sv) = s(L_{\gamma(t)})_\star(v)$ (and so $X^{sv}_{\gamma(t)} = sX^v_{\gamma(t)}$). In particular, since $\gamma(1) = c(s)$, we have that

$$(d\exp)_0(v) = \frac{d}{ds}\exp(sv)\bigg|_{s=0} = \frac{d}{ds}\gamma(1)\bigg|_{s=0} = \frac{d}{ds}c(s)\bigg|_{s=0} = v.$$

As v is arbitrary, we have shown that $(d\exp)_e = I$. By the Inverse Function Theorem, exp is a diffeomorphism from a neighborhood of zero in T_eG to a neighborhood of e in G.

d) If g and $h \in U$, then there exist v and $w \in \mathfrak{g}$ such that $\exp(v) = g$ and $\exp(w) = h$. Since $[v, w] = 0$, we know that $[X^v, X^w] = 0$, and $\phi_t \circ \psi_s = \psi_s \circ \phi_t$, for all s and t, where ϕ_t and ψ_t are the flows of X^v and X^w, respectively. In particular, taking $t = s = 1$ and evaluating at the identity, we get $\phi_1(h) = \psi_1(g)$. This implies, $h\phi_1(e) = g\psi_1(e)$, or $hg = gh$.

e) If $g \in W^C$, then $gV \subset W^C$. Since V is open, gV is a neighborhood of g (because the map $h \mapsto g^{-1}h$ is continuous, as G is a Lie group, and so the inverse image of the open set V by this map (which is gV) is open). This guarantees that every point in W^C has a neighborhood in W^C. So W^C is open.

f) By the previous item, W is both open and closed. Since it contains the identity (and hence is nonempty) and G is connected, we have that $W = G$. We conclude that G is abelian.

Extra material: Justification of the assertions between items **d)** *and* **e)**.

- We show that for all k, $l \in U^{-1}$, $kl = lk$.
  If k, $l \in U^{-1}$, then k^{-1}, $l^{-1} \in U$, so $k^{-1}l^{-1} = l^{-1}k^{-1}$. This is equivalent to $lk = kl$.
- Let $V := U \cap U^{-1}$. As $V \subset U$, for all g, $h \in V$, we have $gh = hg$. We show that $k \in V$ implies that $k^{-1} \in V$.
  Suppose $k \in V$. Then $k \in U$ and $k \in U^{-1}$. So $k^{-1} \in U^{-1}$ and $k^{-1} \in U$. Thus $k^{-1} \in U \cap U^{-1} = V$.
- We show that for all g $h \in V^2$, we have $gh = hg$. Similarly, for g, $h \in V^n$, we have $gh = hg$, and therefore if g, $h \in W := \cup_{n=1}^{\infty} V^n$, we have $gh = hg$.
  If g, $h \in V^2$, there exist g_1, g_2, h_1 and $h_2 \in V$ such that $g = g_1 g_2$ and $h = h_1 h_2$. Therefore, $gh = g_1 g_2 h_1 h_2 = h_1 h_2 g_1 g_2 = hg$.
- We justify that W is open and that it is invariant under elements of V.
  U^{-1} is open because U is open and, since G is a Lie group, the map that sends an element to its inverse is continuous. V is open because it is the intersection of two open sets. $V^2 = \cup_{g \in V} gV$ is open because it is the union of open sets. Similarly, V^n is open. W is open because it is the union of open sets.
  Suppose $g \in W$. Then there exists n such that $g \in V^n$. So $gV \in V^{n+1} \subset W$. Thus W is invariant under elements of V.
- We justify that W^C, is invariant under V (both under left and right multiplication).
  Suppose, by contradiction, that $g \in W^C$, $h \in V$ and $gh \in W$ or $hg \in W$. Then $g \in Wh^{-1}$ or $g \in h^{-1}W$. Since $h^{-1} \in V$, this contradicts that W is invariant under V (recall that we have commutativity of the group multiplication in W). So $gh \in W^C$ and $hg \in W^C$. This proves that W^C, is invariant under V.