Riemannian Geometry

2^{nd} Exam - July 18, 2022
MMAC

Duration: 120 minutes
Show your calculations

1. Let $(\tilde{M}, \tilde{g})$ and $(\hat{M}, \hat{g})$ be Riemannian manifolds and consider the product manifold $\tilde{M} \times \hat{M}$ with the product metric g,

$$g(X,Y) = \tilde{g}(\tilde{X}, \tilde{Y}) + \hat{g}(\hat{X}, \hat{Y}),$$

where $X = (\tilde{X}, \hat{X})$ and $Y = (\tilde{Y}, \hat{Y})$, for $\tilde{X}$, $\tilde{Y} \in \mathcal{X}(\tilde{M})$ and $\hat{X}$, $\hat{Y} \in \mathcal{X}(\hat{M})$.

a) Show that the Levi-Civita connection ∇ on $\tilde{M} \times \hat{M}$ satisfies (3)

$$\nabla_{(\tilde{X},\hat{X})}(\tilde{Y}, \hat{Y}) = \left(\tilde{\nabla}_{\tilde{X}}\tilde{Y}, \hat{\nabla}_{\hat{X}}\hat{Y}\right),$$

where $\tilde{\nabla}$ and $\hat{\nabla}$ are the Levi-Civita connections on $\tilde{M}$ and $\hat{M}$.

b) Calculate the Riemann curvature tensor R of M in terms of the Riemann curvature tensor $\tilde{R}$ and $\hat{R}$ of $\tilde{M}$ and $\hat{M}$. Does there necessarily exist a plane Π whose sectional curvature is zero? (2)

2. As you know, the sphere S^3 has constant sectional curvature equal to 1.

a) In some local orthonormal frame (E_1, E_2, E_3) for S^3 with dual frame $(\omega^1, \omega^2, \omega^3)$: What is R_{ijij}? What is R_{ijkl} for $i \neq k$? What is R_{ijkl} for $j \neq l$? Write the curvature tensor of S^3. (2)

b) Suppose that M is a 2-dimensional manifold which can be embedded in S^3. Prove that the product of the principal curvatures of M does not depend on the embedding and relate this product to the Gaussian curvature K of M. (3)

3.

a) Suppose that G is an abelian (or commutative) group. Let v, $w \in \mathfrak{g}$. Show that $[v, w] = 0$ (i.e. that the Lie algebra is abelian). Suggestion: use a property of the flow of left invariant vector fields. (3)

b) Justify that $[v, w] = 0$ implies that $[X^v, X^w] = 0$. (2)

c) Justify that exp is a diffeomorphism from a neighborhood of zero in T_eG to an open neighborhood U of e in G. (2)

d) Show that if $\mathfrak{g}$ is abelian, then, for all g and $h \in U$, $gh = hg$. (1)

Under the hypothesis that $\mathfrak{g}$ is abelian and for U as in **c)**, one can easily show that:

- Let $V := U \cap U^{-1}$. For all g, $h \in V$, we have $gh = hg$.
- For all g, $h \in W := \cup_{n=1}^{\infty} V^n$, we have $gh = hg$.
- Both V and W are open, and both W and W^C (the complement of W) are invariant under elements of V (both under left and right multiplication).

e) Justify that W^C is open. (1)

f) Suppose that G is connected and that its Lie algebra is abelian. What may you conclude? (1)