Riemannian Geometry

Exam - July 6, 2022

MMAC

Solutions

1.

a) Using Cartan's structure equations, we have

$$\begin{aligned} d\omega^x &= dy \wedge dx = \omega^y \wedge \omega^x_y = y\, dy \wedge \frac{1}{y}\, dx \\ d\omega^x &= 0 = \omega^x \wedge \omega^y_x = y\, dx \wedge \left(-\frac{1}{y}\, dx\right), \end{aligned}$$

and so

$$\omega^x_y = \frac{1}{y}\, dx.$$

This implies

$$\Omega^x_y = d\omega^x_y = \frac{1}{y^2}\, dx \wedge dy = \frac{1}{y^4}\, \omega^x \wedge \omega^y.$$

Since

$$\Omega^x_y = R_{xyy}{}^x\, \omega^x \wedge \omega^y$$

and we are working on an orthonormal frame, we have

$$R_{xyyx} = R_{xyy}{}^x = \frac{1}{y^4}.$$

The Gaussian curvature of M is equal to

$$K = R_{xyyx} = \frac{1}{y^4}.$$

b) Since the equations of the geodesics are

$$\begin{aligned} \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0, \\ \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} = 0, \end{aligned}$$

we obtain

$$\begin{aligned}\frac{d}{dt}(y^2\dot{x}) &= 0, \\ \frac{d}{dt}(y^2\dot{y}) - y(\dot{x}^2 + \dot{y}^2) &= 0,\end{aligned}$$

which is equivalent to

$$\ddot{x} + \frac{2}{y}\dot{x}\dot{y} = 0, \tag{1}$$

$$\ddot{y} - \frac{1}{y}\dot{x}^2 + \frac{1}{y}\dot{y}^2 = 0. \tag{2}$$

We read out that the the nonzero Christoffel symbols are

$$\Gamma^x_{xy} = \frac{1}{y}, \quad \Gamma^x_{yx} = \frac{1}{y}, \quad \Gamma^y_{xx} = -\frac{1}{y}, \quad \Gamma^y_{yy} = \frac{1}{y}.$$

c)

$$\begin{aligned}R(\partial_x, \partial_y)\partial_x &= \nabla_{\partial_x}\nabla_{\partial_y}\partial_x - \nabla_{\partial_y}\nabla_{\partial_x}\partial_x - \nabla_{[\partial_x,\partial_y]}\partial_x \\ &= \nabla_{\partial_x}(\Gamma^x_{yx}\partial_x + \Gamma^y_{yx}\partial_y) - \nabla_{\partial_y}(\Gamma^x_{xx}\partial_x + \Gamma^y_{xx}\partial_y) \\ &= \nabla_{\partial_x}\left(\frac{1}{y}\,\partial_x\right) - \nabla_{\partial_y}\left(-\frac{1}{y}\,\partial_y\right) \\ &= -\frac{1}{y}\frac{1}{y}\partial_y - \frac{1}{y^2}\partial_y + \frac{1}{y}\frac{1}{y}\partial_y \\ &= -\frac{1}{y^2}\partial_y.\end{aligned}$$

It follows that

$$(R(\partial_x, \partial_y)\partial_x, \partial_y) = \left(-\frac{1}{y^2}\partial_y, \partial_y\right) = -1$$

and

$$K = -\frac{R(\partial_x, \partial_y, \partial_x, \partial_y)}{\|\partial_x\|^2\|\partial_y\|^2 - (\partial_x, \partial_y)^2} = \frac{1}{y^4}.$$

d) For $x(\,\cdot\,) = c$ equation (1) is satisfied and equation (2) becomes

$$\ddot{y} + \frac{\dot{y}^2}{y} = 0.$$

This may be rewritten as

$$\frac{\ddot{y}}{\dot{y}} + \frac{\dot{y}}{y} = 0,$$

and integrated to

$$\ln|y\dot{y}| = \ln|\dot{y}| + \ln|y| = \tilde{\alpha},$$

for some constant $\tilde{\alpha}$. So

$$y\dot{y} = \frac{\alpha}{2},$$

for some nonzero constant α. In fact, α may be zero (we divided the original equation by $\dot{y}$). Thus we have

$$y = \sqrt{\alpha t + \beta},$$

for some constants α and β.

e) The geodesic curvature of the horizontal lines $y = y_0$ (transversed from left to right) is

$$k_g = \omega_x^y\left(\frac{\partial_x}{y}\right) = -\frac{1}{y}\,dx\left(\frac{\partial_x}{y}\right) = -\frac{1}{y^2} = -\frac{1}{y_0^2}.$$

Thus, the geodesic curvature of the horizontal lines $y = y_1$ (transversed from right to left) is

$$k_g = \frac{1}{y_1^2}.$$

f) According to the Gauss-Bonnet Theorem

$$\int_R K\omega^x \wedge \omega^y + \int_{\partial R} k_g\,ds + 4 \times \frac{\pi}{2} = 2\pi \times 1$$

because the tangent to the boundary of R turns $\frac{\pi}{2}$ at each corner (check this) and the Euler characteristic of a rectangle is equal to 1. Therefore, we must verify that

$$\int_0^{x_0}\int_{y_0}^{y_1} \frac{1}{y^4}y^2\,dx \wedge dy + \int_0^{x_0}\left(-\frac{1}{y_0^2}\right)y_0\,dx + \int_0^{x_0}\frac{1}{y_1^2}y_1\,dx = 0.$$

This is true.

g) The Riemannian volume form on M is

$$\omega = \omega^x \wedge \omega^y = y^2\, dx \wedge dy.$$

Lie derivative $L_X\omega$ is

$$L_{\frac{1}{y}\partial_y}(y^2\, dx \wedge dy) = 2\, dx \wedge dy + y^2\, dx \wedge d\frac{1}{y} = dx \wedge dy = \frac{1}{y^2}\omega.$$

Hence, the divergence of X is equal to $\frac{1}{y^2}$. The unit exterior normals to ∂R are $\pm\frac{\partial_x}{y}$ and $\pm\frac{\partial_y}{y}$. We obtain $(X, \nu) = 0$ or $(X, \nu) = \pm 1$. As $ds = y\, dy$ and $ds = y\, dx$, it follows that

$$\int_0^{x_0}\int_{y_0}^{y_1} dx \wedge dy = \int_R (\mathrm{div} X)\, \omega$$
$$= \int_{\partial R} (X, \nu)\, ds = \int_0^{x_0} 1 \times y_1\, dx + \int_0^{x_0} (-1) \times y_0\, dx.$$

Both sides are equal to $x_0(y_1 - y_0)$.

2.

a) Denote by F the map from (a subset of) $\mathbb{R} \times S$ to M such that $F(t, x_1, \ldots, x_n)$ is the point on M that one reaches by starting at the point of coordinates $(x_1, \ldots, x_n)$ on S and flowing during time t along the geodesic through that point with initial velocity equal to the unit normal to S. Since, by hypothesis, $(x^1, \ldots, x^n)$ are coordinates on S, the vectors $\partial_{x_1}, \ldots, \partial_{x_n}$ are linearly independent. As ∂_t is orthogonal to S, the set $\{\partial_t, \partial_{x_1}, \ldots, \partial_{x_n}\}$ is linearly independent. Therefore the derivative of the map F is invertible at any point of S. Since F is smooth, by the Inverse Function Theorem, given $p_0 \in S$, there exists a neighborhood V of p_0, such that $(t, x^1, \ldots, x^n)$ define coordinates on V.

b) On S we have that $(\partial_t, \partial_x) \equiv 0$ and

$$\partial_t \cdot (\partial_t, \partial_x) = (\partial_t, \nabla_{\partial_t}\partial_x) = (\partial_t, \nabla_{\partial_x}\partial_t) = \frac{1}{2}\partial_x \cdot (\partial_t, \partial_t) = \frac{1}{2}\partial_x \cdot 1 = 0,$$

since $\nabla_{\partial_t}\partial_t = 0$ and $[\partial_t, \partial_x] = 0$. We conclude that $(\partial_t, \partial_x) \equiv 0$ for all t (where this is defined).