Riemannian Geometry, Fall 2016/17
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The Exam, given on February 3, 2017, consists of part of Problem 1 and of Problem 2.

1. To each $n \times n$ matrix $A = [a_{ij}]$ we may associate the vector field in $\mathbb{R}^n$

$$X^A = (AX)^T \frac{\partial}{\partial x} = \sum_{i,j=1}^{n} x^i a_{ji} \frac{\partial}{\partial x^j},$$

where $\left\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\right\}$ is the canonical basis of $\mathbb{R}^n$.

a) Knowing that $[X^A, X^B] = X^C$, express C in terms of A and B.

b) Suppose G is a Lie group, which is a subgroup of $GL(n)$, with Lie algebra $\mathfrak{g}$, and $A \in \mathfrak{g}$. What is the relation between the exponential of At and the flow $F(\,\cdot\,, t)$ of X^A at time t?

c) Consider $A \in \mathfrak{g}$. What is the value of left invariant vector field Y^A on $G \subset GL(n)$ corresponding to A at the matrix Y?

d) Consider the case where $n = 2$, and define A and B to be

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

The matrices A and B do not commute but $[B, A] = A$. Determine X^A, X^B and $[X^A, X^B]$. Check that your answer is according to the one you gave in **a)**.

e) Determine the real numbers $s = s(\beta)$ and $t = t(\beta)$ such that

$$e^{\beta B} e^{\alpha A} = e^{s\alpha A + \beta B}, \qquad e^{\alpha A} e^{\beta B} = e^{t\alpha A + \beta B}.$$

Suggestion: Compute both sides of the previous equalities.

f) There is a very simple relation between t and s. Explain it.

g) Show that

$$G = \{M \in GL(2) : M = e^{\alpha A + \beta B}, \text{ with } \alpha, \beta \in \mathbb{R}\}.$$

is a subgroup of $GL(2)$. (In fact, it is a Lie group.)

h) Show that the Lie algebra of G is spanned by A and B. Suggestion: you may want to use the definition of the exponential of a matrix.

i) Let

$$(g_1, g_2) \cong \begin{bmatrix} g_2 & g_1 \\ 0 & 1 \end{bmatrix} = e^{\alpha A+\beta B} = g \in G.$$

Show that the volume form

$$\omega = \frac{dx \wedge dy}{y^2},$$

defined on $\mathbb{R} \times \mathbb{R}^+$, is invariant under the pull-back by L_g. Note: If $g = (g_1, g_2)$, then $L_{(g_1,g_2)}(a, b) = (g_2 a + g_1, g_2 b) = (x, y)$.

j) Define $\eta_0 = \frac{dx}{y}$. Check that $d\eta_0 = \omega$. Let $R > 0$. Knowing that you can apply Stokes' Theorem to the region

$$S := \{(x, y) \in \mathbb{R} \times \mathbb{R}^+ : x \in]-R, R[\text{ and } x^2 + y^2 > R^2\},$$

use it to calculate the area of S.

k) Interpret the value you obtained for the area heuristically using the Gauss-Bonnet Theorem.

l) Characterize the forms η such that $d\eta = \omega$.

2. Consider the cylinder $M = \mathbb{R} \times S^1$ with metric

$$ds^2 = d\gamma^2 + \cosh^2 \gamma \, d\theta^2,$$

and orthonormal frame

$$(E_\gamma, E_\theta) = \left(\frac{\partial}{\partial \gamma}, \frac{1}{\cosh \gamma} \frac{\partial}{\partial \theta} \right).$$

a) Show that M has constant curvature equal to -1.

b) Consider the closed curve $c(\theta) = (\gamma_0, \theta)$, and the vector field

$$X(\theta) := a(\theta)(E_\gamma)_{c(\theta)} + b(\theta)(E_\theta)_{c(\theta)},$$

defined for $\theta \in [0, 2\pi[$, with $a(0) = 1$ and $b(0) = 0$. Knowing that it is parallel along c, determine X using connection forms.

c) Let $Y = \lim_{\theta \to 2\pi} X(\theta)$. Compute Y using the result of **b)**. What is the angle between Y and $X(0)$? Confirm your answer by calculating the integral of the geodesic curvature of c. For what values of $\gamma_0 \geq 0$ are $X(0)$ and Y parallel with the same direction?

d) Let $(\gamma_0)_n$ and $(\gamma_0)_{n+1}$ be two consecutive values of $\gamma_0 \geq 0$ as in your answer to **c)**. Use the Gauss-Bonnet Theorem to calculate the area of the portion of M where $(\gamma_0)_n \leq \gamma \leq (\gamma_0)_{n+1}$.

e) Let f be a smooth function of M. Recall that the gradient of f is the vector field X such that, for all $Y \in \mathcal{X}(M)$,

$$(\nabla f, Y) = df(Y).$$

Deduce a formula for the gradient of a vector field in a general system of coordinates where the metric is g_{ij}. Particularize to the case of the coordinates (γ, θ) above.

f) Let ω be a volume form on a Riemannian manifold. Recall that, by definition, the divergence of $X \in \mathcal{X}(M)$ is the function $\operatorname{div} X$ such that

$$L_X \omega = (\operatorname{div} X)\, \omega.$$

Using the formula about the Lie derivative of the tensor product and the fact that the Lie derivative commutes with the exterior derivative, show that

$$\operatorname{div} X = \frac{1}{\sqrt{\det g}} \partial_i \left(\sqrt{\det g}\, X^i \right).$$

Particularize to the case of the coordinates (γ, θ) above.

g) Write down the expression for the Laplacian of f in the coordinates (γ, θ).

1. Solution.

a)

$$\begin{aligned} [X^A, X^B] &= \left(x^i a_{ji} \frac{\partial}{\partial x^j} \right) \left(x^k b_{lk} \frac{\partial}{\partial x^l} \right) - \left(x^i b_{ji} \frac{\partial}{\partial x^j} \right) \left(x^k a_{lk} \frac{\partial}{\partial x^l} \right) \\ &= x^i (a_{ji} b_{lj} - b_{ji} a_{lj}) \frac{\partial}{\partial x^l} \end{aligned}$$

Let $D = AB$ and $E = BA$. Then

$$d_{li} = a_{lj} b_{ji}, \qquad e_{li} = b_{lj} a_{ji}.$$

So,

$$[X^A, X^B] = x^i (e_{li} - d_{li}) \frac{\partial}{\partial x^l} = -x^i (d - e)_{li} \frac{\partial}{\partial x^l}.$$

We conclude that $C = -[A, B]$.

b)

$$e^{At} = [F((1,0,\dots,0),t)\ \dots\ F((0,0,\dots,1),t)].$$

Indeed, $Y(t) = e^{At}$ is the solution of

$$\begin{cases} \dot{Y} = AY, \\ Y(0) = I. \end{cases}$$

On the other hand, if $Y = [Y^{ij}]$ and

$$X^j := \begin{bmatrix} Y^{1j} \\ \dots \\ Y^{nj} \end{bmatrix},$$

then

$$\begin{cases} \dot{X}^j = AX^j, \\ X^j(0) = \begin{bmatrix} 0 \\ \dots \\ 0 \\ 1 \\ 0 \\ \dots \\ 0 \end{bmatrix}, \end{cases}$$

where the one is in position j. Therefore,

$$Y(t) = [X^1(t)\ \dots\ X^n(t)].$$

c) $(Y^A)_Y = YA$.

d)

$$X^A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ 0 \end{bmatrix} = y\frac{\partial}{\partial x},$$

$$X^B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix} = x\frac{\partial}{\partial x}.$$

So,

$$\begin{aligned} [X^A, X^B] &= \left(y\frac{\partial}{\partial x}\right)\left(x\frac{\partial}{\partial x}\right) - \left(x\frac{\partial}{\partial x}\right)\left(y\frac{\partial}{\partial x}\right) \\ &= \left(y\frac{\partial}{\partial x}\right) = X^A. \end{aligned}$$

This is consistent with the result of **a)** because $C = A = -[A, B]$.

e)

$$e^{\alpha A} = \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}, \qquad e^{\beta B} = \begin{bmatrix} e^\beta & 0 \\ 0 & 1 \end{bmatrix}.$$

Let

$$C = s\alpha A + \beta B = \begin{bmatrix} \beta & s\alpha \\ 0 & 0 \end{bmatrix}.$$

Then
$$C^2 = \begin{bmatrix} \beta^2 & \beta s\alpha \\ 0 & 0 \end{bmatrix}, \quad C^3 = \begin{bmatrix} \beta^3 & \beta^2 s\alpha \\ 0 & 0 \end{bmatrix}, \quad C^n = \begin{bmatrix} \beta^n & \beta^{n-1} s\alpha \\ 0 & 0 \end{bmatrix}.$$
Thus,
$$\begin{aligned} e^C &= \sum_{n=0}^{\infty} \frac{C^n}{n!} = \begin{bmatrix} \sum_{n=0}^{\infty} \frac{\beta^n}{n!} & s\alpha \sum_{n=1}^{\infty} \frac{\beta^{n-1}}{n!} \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} e^\beta & s\alpha\frac{e^\beta - 1}{\beta} \\ 0 & 1 \end{bmatrix}. \end{aligned}$$
On the other hand,
$$e^{\beta B} e^{\alpha A} = \begin{bmatrix} e^\beta & \alpha e^\beta \\ 0 & 1 \end{bmatrix}, \qquad e^{\alpha A} e^{\beta B} = \begin{bmatrix} e^\beta & \alpha \\ 0 & 1 \end{bmatrix}.$$
We conclude that the equality $e^{\beta B} e^{\alpha A} = e^{s\alpha A + \beta B}$ is true for
$$s\frac{e^\beta - 1}{\beta} = e^\beta \ \Leftrightarrow \ s = \frac{\beta}{1 - e^{-\beta}},$$
whereas $e^{\alpha A} e^{\beta B} = e^{t\alpha A + \beta B}$ is true for
$$t\frac{e^\beta - 1}{\beta} = 1 \ \Leftrightarrow \ t = \frac{\beta}{e^\beta - 1}.$$

f) From the expressions above for t and s, we see that $t(-\beta) = s(\beta)$. This is immediate from
$$(e^{\beta B} e^{\alpha A})^{-1} = (e^{s(\beta)\alpha A + \beta B})^{-1} \ \Leftrightarrow \ e^{-\alpha A} e^{-\beta B} = e^{s(\beta)(-\alpha)A + (-\beta)B}$$

g) We remark that
$$e^{\beta B} e^{\alpha A} = e^{s\alpha A + \beta B} = e^{\frac{s}{t}\alpha A} e^{\beta B},$$
and
$$\frac{s}{t} = e^\beta.$$
The inverse of $e^{\alpha A + \beta B}$ is $e^{-(\alpha A + \beta B)}$. So, to check that G is a group, we just have to check that the product of two elements of G is in G. This is a consequence of the following computation:
$$\begin{aligned} e^{\alpha A + \beta B} e^{\gamma A + \delta B} &= e^{\beta B} e^{\frac{\alpha}{s(\beta)} A} e^{\delta B} e^{\frac{\gamma}{s(\delta)} A} \\ &= e^{\beta B} e^{\delta B} e^{\frac{e^{-\delta}\alpha}{s(\beta)} A} e^{\frac{\gamma}{s(\delta)} A} \\ &= e^{(\beta+\delta)B} e^{\left(\frac{e^{-\delta}\alpha}{s(\beta)} + \frac{\gamma}{s(\delta)}\right) A} \\ &= e^{\left[s(\beta+\delta)\left(\frac{e^{-\delta}\alpha}{s(\beta)} + \frac{\gamma}{s(\delta)}\right)\right] A + (\beta+\delta)B}. \end{aligned}$$
This shows that G is a subgroup of $GL(n)$.

h) For these matrices A and B, the formula for $e^{\alpha A+\beta B}$ obtained above shows that this exponential is equal to the identity if and only if $\alpha = \beta = 0$. So, to obtain the tangent space to G at the identity, we just have to consider curves $\tau \mapsto c(\tau) := e^{\alpha(\tau)A+\beta(\tau)B}$ in G, with $\alpha(0) = \beta(0) = 0$, and compute $\dot{c}(0)$. But even if the map $(\alpha, \beta) \to e^{\alpha A+\beta B}$ were not injective, the vector field tangent to this flow would still be well defined, because it is a left invariant vector field, and so these curves would suffice to calculate $\mathfrak{g}$. We note that

$$c(\tau) = \sum_{n=0}^{\infty} \frac{(\alpha(\tau)A + \beta(\tau)B)^n}{n!}.$$

So clearly,

$$\dot{c}(0) = \dot{\alpha}(0)A + \dot{\beta}(0)B.$$

This shows that $\mathfrak{g}$ is spanned by A and B. If we wanted to compute $\dot{c}(\tau)$, for τ different from 0, then we should write

$$c(\tau) := e^{\frac{\alpha(\tau)}{t(\beta(\tau))}A} e^{\beta(\tau)B}.$$

Thus, the derivative of c is

$$\dot{c}(\tau) \quad = \quad e^{\frac{\alpha(\tau)}{t(\beta(\tau))}A} \left(\frac{d}{d\tau} \frac{\alpha(\tau)}{t(\beta(\tau))} \right) A e^{\beta(\tau)B} + e^{\frac{\alpha(\tau)}{t(\beta(\tau))}A} e^{\beta(\tau)B} \dot{\beta}(\tau)B,$$

and

$$\frac{d}{d\tau} \frac{\alpha(\tau)}{t(\beta(\tau))} = \frac{\dot{\alpha}(\tau)t(\beta(\tau)) - \alpha(\tau)\dot{t}(\beta(\tau))\dot{\beta}(\tau)}{[t(\beta(\tau))]^2}$$

i)

$$dx = g_2\, da, \qquad dy = g_2\, da.$$

The computation

$$\frac{dx \wedge dy}{y^2} = g_2^2 \frac{da \wedge db}{(g_2 b)^2} = \frac{da \wedge db}{b^2}$$

shows that this volume form is invariant under the pull-back by L_g.

j) Clearly $d\eta_0 = \omega$. Using Stokes' Theorem, the area of S is

$$\begin{aligned} \int_S \omega \quad &= \quad \int_{\partial S} \eta_0 \quad = \quad \int_{\substack{x^2+y^2=R^2 \\ y>0}} \frac{dx}{y} \\ &= \int_{-\pi}^{0} \frac{d(R\cos\theta)}{(-R\sin\theta)} \\ &= \pi. \end{aligned}$$

k) According to the Gauss-Bonnet Theorem,

$$\int_S K + \int_{\partial S} k_g = 2\pi\chi = 2\pi,$$

as the Euler characteristic of a triangle is 1. Now the boundary of S is formed by geodesics, curves whose geodesic curvature is equal to 0. We know that the integral $\int_c k_g$ measures $\Delta\theta$, the change in angle of the vector $\dot{c}$ with respect to a parallel vector field along c. At the 'vertices' $(-R, 0)$, $(0, R)$ and $(0, \infty)$, $\Delta\theta$ is equal to π. Taking into account that $K \equiv -1$, we obtain $-A + 3\pi = 2\pi$, or $A = \pi$. This argument could be made rigorous by applying the Gauss-Bonnet Theorem to the region $S_\epsilon := S \cap \left\{(x, y) \in \mathbb{R} \times \mathbb{R}^+ : \epsilon < y < \frac{1}{\epsilon}\right\}$, for $\epsilon > 0$, and then letting $\epsilon \to 0$.

l) If $d\eta = \omega$, then $\eta = \eta_0 + \xi$ where ξ is a one form on $\mathbb{R} \times \mathbb{R}^+$ satisfying $d\xi = 0$. Writing ξ as $\xi = f\,dx + g\,dy$, we have $-f_y + g_x = 0$. This means that the vector field (f, g) is a gradient: $(f, g) = \nabla\phi = (\phi_x, \phi_y)$. So $\xi = \phi_x\,dx + \phi_y\,dy = d\phi$.

2. Solution.

a) The dual frame is $(\omega^\gamma, \omega^\theta)$, where

$$\omega^\gamma = d\gamma, \qquad \omega^\theta = \cosh\gamma\,d\theta.$$

From Cartan's structure equations,

$$\begin{aligned} d\omega^\theta &= -\omega^\theta_\gamma \wedge \omega^\gamma = -\omega^\theta_\gamma \wedge d\gamma \\ &= \sinh\gamma\,d\gamma \wedge d\theta. \end{aligned}$$

This implies

$$\omega^\theta_\gamma = \sinh\gamma\,d\theta.$$

The curvature form is

$$\begin{aligned} \Omega^\theta_\gamma &= d\omega^\theta_\gamma = \cosh\gamma\,d\gamma \wedge d\theta = R_{\gamma\theta\gamma}{}^\theta \omega^\gamma \wedge \omega^\theta \\ &= R_{\gamma\theta\gamma\theta}\cosh\gamma\,d\gamma \wedge d\theta. \end{aligned}$$

This shows that $R_{\gamma\theta\gamma\theta}$. The manifold M has curvature

$$K = -R_{\gamma\theta\gamma\theta} = -1.$$

b) Clearly $\dot{c} = \frac{\partial}{\partial\theta}$ and so

$$\nabla_{\dot{c}}X = 0 \ \Leftrightarrow\ \frac{1}{\cosh\gamma}\nabla_{\frac{\partial}{\partial\theta}}X = 0 \ \Leftrightarrow\ \nabla_{E_\theta}X = 0.$$

This equation is equivalent to

$$\begin{aligned}
0 &= \nabla_{E_\theta}(aE_\gamma + bE_\theta)\\
&= (E_\theta\cdot a)E_\gamma + a(\nabla_{E_\theta}E_\gamma, E_\gamma)E_\gamma + a(\nabla_{E_\theta}E_\gamma, E_\theta)E_\theta\\
&\quad +(E_\theta\cdot b)E_\theta + b(\nabla_{E_\theta}E_\theta, E_\gamma)E_\gamma + b(\nabla_{E_\theta}E_\theta, E_\theta)E_\theta\\
&= \frac{1}{\cosh\gamma}\dot{a}E_\gamma + a\omega_\gamma^\gamma(E_\theta)E_\gamma + a\omega_\gamma^\theta(E_\theta)E_\theta\\
&\quad +\frac{1}{\cosh\gamma}\dot{b}E_\theta + b\omega_\theta^\gamma(E_\theta)E_\gamma + b\omega_\theta^\theta(E_\theta)E_\theta\\
&= \frac{1}{\cosh\gamma}\dot{a}E_\gamma + 0 + a\tanh\gamma E_\theta\\
&\quad +\frac{1}{\cosh\gamma}\dot{b}E_\theta - b\tanh\gamma E_\gamma + 0.
\end{aligned}$$

We deduce that

$$\begin{cases} \dot{a} = \sinh\gamma\, b,\\ \dot{b} = -\sinh\gamma\, a,\end{cases}$$

or

$$\begin{cases} \ddot{a} = -\sinh^2\gamma\, a,\\ \ddot{b} = -\sinh^2\gamma\, b.\end{cases}$$

Our initial conditions are $a(0) = 1$, $\dot{a}(0) = 0$, and $b(0) = 0$, $\dot{b}(0) = -\sinh\gamma$. Therefore,

$$\begin{cases} a(\theta) = \cos(\theta\sinh\gamma),\\ b(\theta) = -\sin(\theta\sinh\gamma).\end{cases}$$

In conclusion,

$$X(\theta) = \cos(\theta\sinh\gamma_0)E_\gamma - \sin(\theta\sinh\gamma_0)E_\theta.$$

c)

$$Y = \cos(2\pi\sinh\gamma_0)E_\gamma - \sin(2\pi\sinh\gamma_0)E_\theta.$$

The angle between Y and $X(0)$ is $-2\pi\sinh\gamma_0$. Since E_θ is tangent to c and has unit length, and $(E_\theta, -E_\gamma)$ has positive orientation, the geodesic curvature of c is

$$k_g = -\omega_\theta^\gamma(E_\theta) = \sinh\gamma_0\, d\theta\left(\frac{1}{\cosh\gamma_0}\frac{\partial}{\partial\theta}\right) = \tanh\gamma_0.$$

The integral of the geodesic curvature over c is

$$\int_c k_g\,ds = \int_0^{2\pi} \tanh\gamma_0\,\cosh\gamma_0\,d\theta = 2\pi\sinh\gamma_0.$$

This is the angle by which $\dot c$ rotates with respect to X when we go once around the curve c. The angle between Y and $X(0) = \dot c(0)$ is the opposite angle. $X(0)$ and Y parallel with the same direction if $2\pi\sinh\gamma_0 = 2\pi k$, with $k \in \mathbb{N}_0$ (because $\gamma_0 \geq 0$). Now

$$\begin{aligned}\sinh\gamma_0 = k \ &\Leftrightarrow\ e^{2\gamma_0} - 2ke^{\gamma_0} - 1 = 0 \ \Leftrightarrow\ e^{\gamma_0} = k + \sqrt{k^2+1}\\ &\Leftrightarrow\ \gamma_0 = \ln(k + \sqrt{k^2+1}).\end{aligned}$$

d) Using the Gauss-Bonnet Theorem,

$$\int_{(\gamma_0)_n \leq \gamma \leq (\gamma_0)_{n+1}} K + \int_{\gamma=(\gamma_0)_{n+1}} k_g - \int_{\gamma=(\gamma_0)_{n+1}} k_g = 2\pi\chi = 0,$$

as the Euler characteristic of a 'slice' of a cylinder is 0. But $K \equiv -1$ and, according to the definition of $(\gamma_0)_n$, we have $\int_{\gamma=(\gamma_0)_n} k_g = 2\pi n$. Therefore, the value of the area of the portion of M under consideration is

$$\int_{(\gamma_0)_n \leq \gamma \leq (\gamma_0)_{n+1}} 1 = 2\pi.$$

e) From the definition of the gradient,

$$(\nabla f, Y) = df(Y) \ \Leftrightarrow\ (\nabla f)^i g_{ij} Y^j = Y^j \frac{\partial f}{\partial x^j}.$$

Since this equality is valid for all Y, we must have

$$(\nabla f)^i g_{ij} = \frac{\partial f}{\partial x^j}.$$

Multiplying both sides by g^{jk} and summing over j, it follows

$$(\nabla f)^i \delta_i^k = g^{jk} \frac{\partial f}{\partial x^j}.$$

Thus

$$(\nabla f)^k = g^{kj} \frac{\partial f}{\partial x^j}.$$

For the above metric, we have

$$\nabla f = \partial_\gamma f \frac{\partial}{\partial\gamma} + \frac{1}{\cosh^2\gamma} \partial_\theta f \frac{\partial}{\partial\theta}.$$

f) The volume form on a Riemannian manifold is

$$\omega = \sqrt{\det g}\, dx^1 \wedge \ldots \wedge dx^n.$$

Using the formula about the Lie derivative of the tensor product and the fact that the Lie derivative commutes with the exterior derivative,

$$\begin{aligned} L_X\omega &= X \cdot \sqrt{\det g}\, dx^1 \wedge \ldots \wedge dx^n \\ &\quad + \sqrt{\det g}\, d(L_X x^1) \wedge \ldots \wedge dx^n + \ldots \\ &\quad + \sqrt{\det g}\, dx^1 \wedge \ldots \wedge d(L_X x^n) \\ &= X^i \frac{\partial \sqrt{\det g}}{\partial x^i}\, dx^1 \wedge \ldots \wedge dx^n \\ &\quad + \sqrt{\det g} \frac{\partial X^1}{\partial x^1}\, dx^1 \wedge \ldots \wedge dx^n + \ldots \\ &\quad + \sqrt{\det g} \frac{\partial X^n}{\partial x^n}\, dx^1 \wedge \ldots \wedge dx^n \\ &= \partial_i(\sqrt{\det g}\, X^i)\, dx^1 \wedge \ldots \wedge dx^n \\ &= \frac{1}{\sqrt{\det g}} \partial_i(\sqrt{\det g}\, X^i)\, \omega \\ &= \operatorname{div} X\, \omega. \end{aligned}$$

We have used the fact that

$$d(L_X x^i) = d(X \cdot x^i) = dX^i = \frac{\partial X^i}{\partial x_j}\, dx^j.$$

Therefore,

$$\operatorname{div} X = \frac{1}{\sqrt{\det g}} \partial_i(\sqrt{\det g}\, X^i).$$

For X a vector field on M, in the above coordinates,

$$\begin{aligned} \operatorname{div} X &= \frac{1}{\cosh\gamma} \partial_\gamma(\cosh\gamma\, X^\gamma) + \frac{1}{\cosh\gamma} \partial_\theta(\cosh\gamma\, X^\theta) \\ &= \partial_\gamma X^\gamma + \tanh\gamma X^\gamma + \partial_\theta X^\theta. \end{aligned}$$

g)

$$\Delta f = \frac{1}{\cosh\gamma} \partial_\gamma(\cosh\gamma\, \partial_\gamma f) + \frac{1}{\cosh^2\gamma} \partial_\theta \partial_\theta f.$$