Riemannian Geometry

1st Exam - July 6, 2022

MMAC

Duration: 120 minutes

Show your calculations

1. Consider $M := \{(x, y) \in \mathbb{R}^2 : y > 0\}$ with metric

$$ds^2 = y^2(dx^2 + dy^2)$$

and orientation (∂_x, ∂_y).

a) Using $(\omega^x, \omega^y) = (y\,dx, y\,dy)$ and Cartan's structure equations, calculate ω^x_y, Ω^x_y, and the Gaussian curvature of M. (3)

b) Use the Lagrangian $L = \frac{y^2}{2}(\dot{x}^2 + \dot{y}^2)$ to calculate the equations for the geodesics of M. Use these to write down the Christoffel symbols of M in the coordinates (x, y). (2)

c) Use **b)** to calculate $R(\partial_x, \partial_y)\partial_x$ and use this to obtain again the Gaussian curvature of M. (3)

d) Let $0 < y_0 < y_1$ and $x_0 > 0$. Consider the rectangle (2)

$$R = \{(x, y) \in \mathbb{R}^2 : 0 < x < x_0 \text{ and } y_0 < y < y_1\}.$$

Use **b)** to show that the vertical pieces of the boundary of R are images of geodesics and determine these explicitly.

e) Calculate the geodesic curvature of the horizontal lines $y = y_0$ (transversed from left to right) and $y = y_1$ (transversed from right to left). (2)

f) Check the Gauss-Bonnet Theorem for R (remembering that its boundary has four corners). (2)

g) Let ω be the Riemannian volume form on M and X be the vector field $\frac{1}{y}\partial_y$. Calculate the Lie derivative $L_X\omega$ and the divergence of X. Check the equality of the Divergence Theorem in this particular case, (2)

$$\int_R (\text{div}X)\,\omega = \int_{\partial R} (X, \nu)\,ds,$$

for ν the unit exterior normal to ∂R.

2. Let S be a hypersurface of the Riemannian manifold M. Suppose that S admits global coordinates $(x^1, \ldots, x^n)$. Let S_t be the surface obtained by letting each point p of S flow for time t along the geodesic through p with initial velocity equal to the unit normal to S (chosen smoothly along S).

a) Given $p_0 \in S$, show that there exists a neighborhood V of p_0, such that $(t, x^1, \ldots, x^n)$ define coordinates on V. (2)

b) Show that S_t is orthogonal to the geodesics orthogonal to S. (2)