Riemannian Geometry

February 3, 2017

LMAC and MMA

1st Test	–	Question 1	–	90 minutes
2nd Test	–	Question 2	–	90 minutes
Exam	–	Both questions	–	3 hours

Show your calculations

1. To each $n \times n$ matrix $A = [a_{ij}]$ we may associate the vector field in $\mathbb{R}^n$

$$X^A = (AX)^T \frac{\partial}{\partial x} = \sum_{i,j=1}^{n} x^i a_{ji} \frac{\partial}{\partial x^j},$$

where $\left\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\right\}$ is the canonical basis of $\mathbb{R}^n$.

a) Knowing that $[X^A, X^B] = X^C$, express C in terms of A and B. (1.5)

b) Consider the case where $n = 2$, and define A and B to be (1)

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

The matrices A and B do not commute but $[B, A] = A$. Determine X^A, X^B and $[X^A, X^B]$. Check that your answer is according to the one you gave in **a)**.

c) Determine the real numbers $s = s(\beta)$ and $t = t(\beta)$ such that (2)

$$e^{\beta B} e^{\alpha A} = e^{s\alpha A + \beta B}, \qquad e^{\alpha A} e^{\beta B} = e^{t\alpha A + \beta B}.$$

Suggestion: Compute both sides of the previous equalities.

d) Show that (1.5)

$$G = \{M \in GL(2) : M = e^{\alpha A + \beta B}, \text{ with } \alpha, \beta \in \mathbb{R}\}.$$

is a subgroup of $GL(2)$. (In fact, it is a Lie group.)

e) Show that the Lie algebra of G is spanned by A and B. Suggestion: you may want to use the definition of the exponential of a matrix. (1)

f) Let (1.5)

$$(g_1, g_2) \cong \begin{bmatrix} g_2 & g_1 \\ 0 & 1 \end{bmatrix} = e^{\alpha A + \beta B} = g \in G.$$

Show that the volume form

$$\omega = \frac{dx \wedge dy}{y^2},$$

defined on $\mathbb{R} \times \mathbb{R}^+$, is invariant under the pull-back by L_g. Note: If $g = (g_1, g_2)$, then $L_{(g_1,g_2)}(a, b) = (g_2 a + g_1, g_2 b) = (x, y)$.

g) Define $\eta = \frac{dx}{y}$. Check that $d\eta = \omega$. Let $R > 0$. Knowing that you can apply Stokes' Theorem to the region (1.5)

$$S := \{(x, y) \in \mathbb{R} \times \mathbb{R}^+ : x \in]-R, R[\text{ and } x^2 + y^2 > R^2\},$$

use it to calculate the area of S.

2. Consider the cylinder $M = \mathbb{R} \times S^1$ with metric

$$ds^2 = d\gamma^2 + \cosh^2 \gamma \, d\theta^2,$$

and orthonormal frame

$$(E_\gamma, E_\theta) = \left(\frac{\partial}{\partial \gamma}, \frac{1}{\cosh \gamma} \frac{\partial}{\partial \theta} \right).$$

a) Show that M has constant curvature equal to -1. (2)

b) Consider the closed curve $c(\theta) = (\gamma_0, \theta)$, and the vector field (2)

$$X(\theta) := a(\theta)(E_\gamma)_{c(\theta)} + b(\theta)(E_\theta)_{c(\theta)},$$

defined for $\theta \in [0, 2\pi[$, with $a(0) = 1$ and $b(0) = 0$. Knowing that it is parallel along c, determine X using connection forms.

c) Let $Y = \lim_{\theta \to 2\pi} X(\theta)$. Compute Y using the result of **b)**. What is the angle between Y and $X(0)$? Confirm your answer by calculating the integral of the geodesic curvature of c. For what values of $\gamma_0 \geq 0$ are $X(0)$ and Y parallel with the same direction? (1)

d) Let $(\gamma_0)_n$ and $(\gamma_0)_{n+1}$ be two consecutive values of $\gamma_0 \geq 0$ as in your answer to **c)**. Use the Gauss-Bonnet Theorem to calculate the area of the portion of M where $(\gamma_0)_n \leq \gamma \leq (\gamma_0)_{n+1}$. (1.5)

e) Let f be a smooth function of M. Recall that the gradient of f is the vector field X such that, for all $Y \in \mathcal{X}(M)$, (1)

$$(\nabla f, Y) = df(Y).$$

Deduce a formula for the gradient of a vector field in a general system of coordinates where the metric is g_{ij}. Particularize to the case of the coordinates (γ, θ) above.

f) Let ω be a volume form on a Riemannian manifold. Recall that, by definition, the divergence of $X \in \mathcal{X}(M)$ is the function $\operatorname{div} X$ such that (1.5)

$$L_X \omega = (\operatorname{div} X)\, \omega.$$

Using the formula about the Lie derivative of the tensor product and the fact that the Lie derivative commutes with the exterior derivative, show that

$$\operatorname{div} X = \frac{1}{\sqrt{\det g}} \partial_i \left(\sqrt{\det g}\, X^i \right).$$

Particularize to the case of the coordinates (γ, θ) above.

g) Write down the expression for the Laplacian of f in the coordinates (γ, θ). (1)