Mathematical Relativity, Spring 2023/24
Instituto Superior Técnico

`Due March 12`

1. Using the definition of covariant derivative, we showed in class that

$$\nabla\nabla Z(X,Y,\omega) = (\nabla_X \nabla_Y Z)(\omega) - (\nabla_{\nabla_X Y} Z)(\omega). \tag{1}$$

a) Check (1) by calculating both sides in local coordinates.
b) Obtain a formula for

$$\nabla\nabla\nabla W(X,Y,Z).$$

2. Recall that the nonzero Christoffel symbols for the Minkowski metric in spherical coordinates,

$$\eta = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2),$$

are

$$\Gamma^r_{\theta\theta} = -r, \quad \Gamma^r_{\varphi\varphi} = -r\sin^2\theta,$$
$$\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{1}{r}, \quad \Gamma^\theta_{\varphi\varphi} = -\sin\theta\cos\theta,$$
$$\Gamma^\varphi_{r\varphi} = \Gamma^\varphi_{\varphi r} = \frac{1}{r}, \quad \Gamma^\varphi_{\theta\varphi} = \Gamma^\varphi_{\varphi\theta} = \cot\theta.$$

Consider the vector field

$$V = f(r)\partial_r.$$

a) Compute the tensor $\nabla^\mu V^\nu$.
b) We will show in class that

$$\frac{1}{2}(L_V g)_{\mu\nu} = \nabla_{(\mu} V_{\nu)}.$$

Use this equality to compute the deformation tensor $\nabla_{(\mu} V_{\nu)}$. Check your answer using the result of **a)**.

`Due March 21`

3. Draw the Penrose diagram for the Schwarzschild solution with negative mass. Do timelike geodesics hit the naked singularity at $r = 0$?

4. Consider an Oppenheimer-Snyder solution obtained by gluing a FLRW metric

$$-d\tau^2 + a^2(\tau)(d\psi^2 + \psi^2 dl^2_{\mathcal{S}^2})$$

satisfying Friedmann's equations with $k = \Lambda = 0$ and $\alpha > 0$ with a Schwarzschild metric along an hypersurface $\{\psi = \psi_0\}$ of FLRW. Determine the value of a at the center (in terms of α and ψ_0) that corresponds to a light-ray that goes to future timelike infinity i^+.

5. Consider the FLRW metric

$$g = -d\tau^2 + a^2(\tau)\left(\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)\right)$$

and the orthonormal frame

$$\begin{aligned}
\omega^0 &= d\tau, \\
\omega^r &= \frac{a}{\sqrt{1-kr^2}}\, dr, \\
\omega^\theta &= ar\, d\theta, \\
\omega^\varphi &= ar\sin\theta\, d\varphi.
\end{aligned}$$

a) Using Cartan's structure equations, check that

$$\begin{aligned}
\omega^r_0 &= \frac{\dot{a}}{\sqrt{1-kr^2}}\, dr, \\
\omega^\theta_0 &= \dot{a}r\, d\theta, \\
\omega^\varphi_0 &= \dot{a}r\sin\theta\, d\varphi, \\
\omega^\theta_r &= \sqrt{1-kr^2}\, d\theta, \\
\omega^\varphi_r &= \sqrt{1-kr^2}\sin\theta\, d\varphi, \\
\omega^\varphi_\theta &= \cos\theta\, d\varphi.
\end{aligned}$$

Moreover, check that

$$\begin{aligned}
\Omega^r_0 &= \frac{\ddot{a}}{a}\,\omega^0 \wedge \omega^r, \\
\Omega^\theta_0 &= \frac{\ddot{a}}{a}\,\omega^0 \wedge \omega^\theta, \\
\Omega^\varphi_0 &= \frac{\ddot{a}}{a}\,\omega^0 \wedge \omega^\varphi, \\
\Omega^\theta_r &= -\frac{k+\dot{a}^2}{a^2}\,\omega^r \wedge \omega^\theta, \\
\Omega^\varphi_r &= -\frac{k+\dot{a}^2}{a^2}\,\omega^r \wedge \omega^\varphi, \\
\Omega^\varphi_\theta &= -\frac{k+\dot{a}^2}{a^2}\,\omega^\theta \wedge \omega^\varphi.
\end{aligned}$$

Finally, check that

$$\begin{aligned}
R_{00} &= -\frac{3\ddot{a}}{a}, \\
R_{rr} &= R_{\theta\theta} = R_{\varphi\varphi} = 2\frac{k+\dot{a}^2}{a^2} + \frac{\ddot{a}}{a},
\end{aligned}$$

and

$$R = 6\left(\frac{k+\dot{a}^2}{a^2} + \frac{\ddot{a}}{a}\right).$$

b) Using Einstein's equation

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu},$$

with $T = \rho\, d\tau \otimes d\tau$, check that

$$\frac{d}{d\tau}\left(\frac{a\dot{a}^2}{2} + \frac{ka}{2} - \frac{\Lambda}{6}a^3\right) = 0$$

and obtain Friedmann's equations.

`Due March 28`

6. Consider the Riemannian or Lorentzian metric

$$g = dt^2 + h_{ij}(t,x)dx^i dx^j.$$

Show that

a) The Christoffel symbols are

$$\Gamma^0_{ij} = -K_{ij}, \qquad \Gamma^i_{jk} = \bar{\Gamma}^i_{jk}, \qquad \Gamma^i_{0j} = K^i{}_j,$$

where $\bar{\Gamma}^i_{jk}$ are the Christoffel symbols of h and $K(t)$ is the second fundamental form of the hypersurface $t =$ constant.

b) The components of the Riemann tensor are

$$\begin{aligned} R_{0i0}{}^j &= -\frac{\partial}{\partial t}K^j{}_i - K_{il}K^{lj}, \\ R_{ij0}{}^l &= -\bar{\nabla}_i K^l{}_j + \bar{\nabla}_j K^l{}_i, \\ R_{ijl}{}^m &= \bar{R}_{ijl}{}^m - K_{il}K^m{}_j + K_{jl}K^m{}_i, \end{aligned}$$

where $\bar{\nabla}$ is the Levi-Civita connection of h and $\bar{R}_{ijl}{}^m$ are the components of the Riemann tensor of h.

c) The components of the Ricci tensor are

$$\begin{aligned} R_{00} &= -\frac{\partial}{\partial t}K^i{}_i - K_{ij}K^{ij}, \\ R_{0i} &= -\bar{\nabla}_i K^j{}_j + \bar{\nabla}_j K^j{}_i, \\ R_{ij} &= \bar{R}_{ij} - \frac{\partial}{\partial t}K_{ij} + 2K_{il}K^l{}_j - K^l{}_l K_{ij}, \end{aligned}$$

where $\bar{R}_{ij}$ are the components of the Ricci tensor of h.

d) The time derivative of the inverse of h is

$$\frac{\partial h^{ij}}{\partial t} = -2K^{ij}.$$

e) The scalar curvature is

$$R = \bar{R} - 2\frac{\partial}{\partial t}K^i{}_i - (K^i{}_i)^2 - K_{ij}K^{ij}, \tag{2}$$

where $\bar{R}$ is the scalar curvature of h.

f) The component G_{00} of the Einstein tensor is

$$G_{00} = \frac{1}{2}\left(-\bar{R} + (K^i{}_i)^2 - K_{ij}K^{ij}\right). \tag{3}$$

7. Let (M, g) be the quotient of the 2-dimensional Minkowski spacetime by the group of isometries generated by the map $(t, x) \mapsto (t + 1, x + 1)$. Show directly that (M, g) is not stably causal, i.e. it is not possible to define a global time function.

Due April 26

8. Consider $(\mathbb{R}^3, -dt^2 + dx^2 + dy^2)$ and the congruence of timelike geodesics with velocity

$$X = \frac{t\partial_t + x\partial_x + y\partial_y}{\sqrt{t^2 - x^2 - y^2}}.$$

Consider the orthonormal frame

$$\mathcal{F} = \left(X, \frac{(x^2 + y^2)\partial_t + tx\partial_x + ty\partial_y}{\sqrt{x^2 + y^2}\sqrt{t^2 - x^2 - y^2}}, \frac{-y\partial_x + x\partial_y}{\sqrt{x^2 + y^2}} \right).$$

This frame is parallel along the geodesics.

- **a)** Calculate the second fundamental form of X in the frame $(\partial_t, \partial_x, \partial_y)$.
- **b)** Calculate the second fundamental form of X in the frame $\mathcal{F}$.
- **c)** Calculate the expansion θ.
- **d)** Verify Raychaudhuri's equation.
- **e)** Let Y be a deviation vector orthogonal to X and let τ be arc length along a geodesic. What is the relation between $\dot{Y}$ and Y?

9. Consider $(\mathbb{R}^3, -dt^2 + dx^2 + dy^2)$ and the congruence of timelike geodesics through the x-axis with velocity

$$X = \frac{t\partial_t + y\partial_y}{\sqrt{t^2 - y^2}}.$$

Consider the orthonormal frame $\mathcal{F}$, given by

$$\left(X, \partial_x, \frac{y\partial_t + t\partial_y}{\sqrt{t^2 - y^2}} \right).$$

- **a)** Write the second fundamental form $B_{\mu\nu}$ of X in the frame $\mathcal{F}$.
- **b)** Without actually calculating $\nabla_X \frac{y\partial_t + t\partial_y}{\sqrt{t^2-y^2}}$, justify that $\frac{y\partial_t + t\partial_y}{\sqrt{t^2-y^2}}$ is parallel along each integral curve of X.
- **c)** Write the spatial metric $h_{\mu\nu}$. Calculate the expansion, deformation and vorticity, and use these to decompose the second fundamental form.
- **d)** Verify the Raychaudhuri equation.
- **e)** Define an appropriate fundamental solution A of the Jacobi equation to encode the evolution of a general deviation vector orthogonal to X. Calculate the fundamental solution and check that $B = \dot{A}A^{-1}$.

Due May 14

10. Consider $\mathbb{R}^3$ with the Minkowski metric written in polar coordinates as

$$g = -dt^2 + dr^2 + r^2\, d\theta^2.$$

Let $f : \mathbb{R} \to \mathbb{R}$ be periodic with period 2π and

$$X = f(\theta)(\partial_t + \partial_r).$$

Consider the frame

$$\mathcal{V} = \left(\partial_t, X, \frac{1}{r}\partial_\theta\right).$$

- **a)** Verify that X is null geodesic.
- **b)** Compute the second fundamental form $B^\mu{}_\nu$ in the coordinates corresponding to $\mathcal{V}$ by calculating $\nabla_{\partial_t} X$ and $\nabla_{\frac{1}{r}\partial_\theta} X$.
- **c)** Determine the integral curves (t, r, θ) of X through $(0, 1, \theta_0)$ in terms of the affine parameter u. Express r in terms of t.
- **d)** The vector field $Y = \partial_\theta$ is a Jacobi field $\left(\partial_\theta = \partial_{\theta_0} - u\frac{f'(\theta_0)}{f(\theta_0)}\partial_u\right)$. Note however that Y does not commute with X. Correct the equation $\nabla_X Y^\mu = B^\mu{}_\nu Y^\nu$ to take this into account and verify the corrected equation directly.
- **e)** Write the expression for the metric g in the frame $\mathcal{V}$. Compute the covector $X_\flat$.
- **f)** Compute the second fundamental form $B_{\mu\nu}$ in the coordinates corresponding to $\mathcal{V}$ by calculating $\nabla_{\partial_t} X_\flat$ and $\nabla_{\frac{1}{r}\partial_\theta} X_\flat$. To check your answer, verify that $B_{\mu\nu} = g_{\mu\gamma} B^\gamma{}_\nu$.
- **g)** Compute the second fundamental form $B_{\mu\nu}$ in the coordinates corresponding to $(\partial_t, \partial_r, \partial_\theta)$ by using

$$B_{\mu\nu} = \frac{1}{2}\left(L_X g\right)_{\mu\nu} - \frac{1}{2}\left(dX_\flat\right)_{\mu\nu}.$$

  Check that your answer agrees with the one obtained in **f)**.

11. Consider the metric

$$g = -dt^2 + a^2(t)(d\psi^2 + f^2(\psi)(d\theta^2 + \sin^2\theta d\varphi^2))$$

defined on M. When $M = \mathbb{R} \times S^3$, $f(\psi) = \sin\psi$; when $M = \mathbb{R} \times \mathbb{R}^3$, $f(\psi) = \psi$; when $M = \mathbb{R} \times H^3$, $f(\psi) = \sinh\psi$.

a) Using the geodesic Lagrangian obtained from the metric, compute the Christoffel symbols in the coordinates $(t, \psi, \theta, \varphi)$.

b) Verify that the null vector field

$$X = \frac{1}{a}\partial_t + \frac{1}{a^2}\partial_\psi$$

is geodesic.

c) Compute

$$\nabla_{\partial_\theta} X \qquad \text{and} \qquad \nabla_{\partial_\varphi} X.$$

Use these to obtain the expansion of the congruence of null geodesics tangent to X.

d) Let Σ be the sphere $t = t_0$ and $\psi = \psi_0$. Denote by ξ_r the flow of X. The 2-dimensional surface $\xi_r(\Sigma)$ is parametrized by

$$(\theta, \varphi) \mapsto (t(r), \psi(r), \theta, \varphi).$$

Here $t(r)$ is determined by

$$r = \int_{t_0}^{t} a(s)\, ds$$

and $\psi(r)$ satisfies

$$\psi = \psi_0 + \int_0^r \frac{1}{a^2(t(s))}\, ds.$$

Define

$$\tilde{A}(t, \psi) = a^2(t) f^2(\psi)$$

and

$$A(r) = \tilde{A}(t(r), \psi(r)).$$

Note that

$$\frac{dA}{dr} = X \cdot \tilde{A}.$$

Use the area element of $\xi_r(\Sigma)$, which is

$$A(r) \sin\theta \, d\theta \wedge d\varphi,$$

to confirm the result you obtained above for the expansion of the congruence of null geodesics.