Geometric Mechanics, Fall 2025/26
Instituto Superior Técnico

`Due October 6`

1. Let $(M, \langle \cdot, \cdot \rangle, \mathcal{F})$ be a mechanical system. Show that the Newton equation defines a flow on TM, generated by the vector field $X \in \mathcal{X}(TM)$ whose local expression is

$$X = v^i \frac{\partial}{\partial x^i} + \left(\sum_{j=1}^{n} g^{ij}(x) F_j(x, v) - \sum_{j,k=1}^{n} \Gamma^i_{jk}(x) v^j v^k \right) \frac{\partial}{\partial v^i},$$

where $(x^1, \ldots, x^n)$ are local coordinates on M, $(x^1, \ldots, x^n, v^1, \ldots, v^n)$ are the local coordinates induced on TM, and

$$\mathcal{F} = \sum_{i=1}^{n} F_i(x, v) dx^i$$

on these coordinates. What are the fixed points of this flow?

2. The harmonic oscillator (in appropriate units) is the conservative mechanical system $(\mathbb{R}, dx \otimes dx, -dU)$, where $U : \mathbb{R} \to \mathbb{R}$ is given by

$$U(x) := \frac{1}{2} \omega^2 x^2.$$

a) Write the equation of motion and its general solution.
b) Friction can be included in this model by considering the external force

$$\mathcal{F}\left(u \frac{d}{dx} \right) = -dU - 2ku \, dx$$

(where $k > 0$ is a constant). Write the equation of motion of this new mechanical system and its general solution.
c) Generalize **a)** to the n-dimensional harmonic oscillator, whose potential energy $U : \mathbb{R}^n \to \mathbb{R}$ is given by

$$U(x^1, \ldots, x^n) := \frac{1}{2} \omega^2 \left(\left(x^1\right)^2 + \ldots + \left(x^n\right)^2 \right).$$

3. Use spherical coordinates to write the equations of motion for the spherical pendulum of length l, i.e. a particle of mass $m > 0$ moving in $\mathbb{R}^3$ subject to a constant gravitational acceleration g and with the holonomic constraint

$$N = \left\{ (x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = l^2 \right\}.$$

Which parallels of N are possible trajectories of the particle?

4. Write the equations of motion for a particle moving on a frictionless surface of revolution with equation $z = f(r)$, $r = \sqrt{x^2 + y^2}$, under a constant gravitational acceleration g.

`Due November 10`

5. Let $S : \mathbb{R} \to SO(3)$ describe the orientation of a rigid body. In terms of the Euler angles, show that the angular velocity Ω and ω are given by

$$\Omega = (\dot{\theta}\cos\psi + \dot{\varphi}\sin\theta\sin\psi)e_1 + (-\dot{\theta}\sin\psi + \dot{\varphi}\sin\theta\cos\psi)e_2 + (\dot{\psi} + \dot{\varphi}\cos\theta)e_3,$$

$$\omega = (\dot{\theta}\cos\varphi + \dot{\psi}\sin\theta\sin\varphi)e_1 + (\dot{\theta}\sin\varphi - \dot{\psi}\sin\theta\cos\varphi)e_2 + (\dot{\varphi} + \dot{\psi}\cos\theta)e_3.$$

You may use Mathematica to help with the computations.

6. (*Precession of the equinoxes*) Due to its rotation, the Earth is not a perfect sphere, but an oblate ellipsoid; therefore its moments of inertia are not quite equal, satisfying approximately

$$I_1 = I_2 \neq I_3;$$
$$\frac{I_3 - I_1}{I_1} \simeq \frac{1}{306}.$$

As a consequence, the combined gravitational attraction of the Moon and the Sun disturbs the Earth's rotation motion. This perturbation can be approximately modeled by the potential energy $U : SO(3) \to \mathbb{R}$ given in the Euler angles (θ, φ, ψ) by

$$U = -\frac{\Omega^2}{2}(I_3 - I_1)\cos^2\theta,$$

where $\frac{2\pi}{\Omega} \simeq 168$ days.

a) Write the equations of motion and determine the equilibrium points.
b) Show that there exist solutions such that θ, $\dot{\varphi}$ and $\dot{\psi}$ are constant, which in the limit $|\dot{\varphi}| \ll |\dot{\psi}|$ (as is the case with the Earth) satisfy

$$\dot{\varphi} \simeq -\frac{\Omega^2(I_3 - I_1)\cos\theta}{I_3\dot{\psi}}.$$

Given that for the Earth $\theta \simeq 23°$, determine the approximate value of the period of $\varphi(t)$.

7. Consider a sphere of radius R and mass M rolling without slipping on a plane. We solved questions **a), b), c)** in class, please solve **d)** and **e)**:

a) Show that the condition of rolling without slipping is

$$\dot{x} = R\omega^y, \qquad \dot{y} = -R\omega^x,$$

where (x, y) are the Cartesian coordinates of the contact point on the plane and ω is the angular velocity of the sphere.

b) Show that if the sphere's mass is symmetrically distributed then its kinetic energy is

$$K = \frac{M}{2}\left(\dot{x}^2 + \dot{y}^2\right) + \frac{I}{2}\langle \omega, \omega \rangle,$$

where I is the sphere's moment of inertia and $\langle \cdot, \cdot \rangle$ is the Euclidean inner product.

c) Using ω as coordinates on the fibers of $TSO(3)$, show that

$$\frac{D\dot{c}}{dt} = \ddot{x}\frac{\partial}{\partial x} + \ddot{y}\frac{\partial}{\partial y} + \dot{\omega}.$$

(**Hint:** Recall that the integral curves of left-invariant vector fields on a Lie group with a bi-invariant metric are geodesics).

d) Since we are identifying the fibers of $TSO(3)$ with $\mathbb{R}^3$, we can use the Euclidean inner product to also identify the fibers of $T^*SO(3)$ with $\mathbb{R}^3$. Show that under this identification the non-holonomic constraint yielding the condition of rolling without slipping is the distribution determined by the kernels of the 1-forms

$$\theta^x := dx - R\, e_y, \qquad \theta^y := dy + R\, e_x$$

(where $\{e_x, e_y, e_z\}$ is the canonical basis of $\mathbb{R}^3$). Is this distribution integrable? (**Hint:** Show that any two points of $\mathbb{R}^2 \times SO(3)$ can be connected by a piecewise smooth curve compatible with the distribution).

e) Show that the sphere moves along straight lines with constant speed and constant angular velocity orthogonal to its motion.

8. Consider an Euler top with moment of inertia tensor I, and let $S : \mathbb{R} \to SO(3)$ be a motion with constant angular momentum $p \neq 0$ and constant kinetic energy $K > 0$. Recall that if we set $\dot{S} = SA$ and $\Omega = \Omega(A)$, we have the relations

$$p = SP, \quad P = I\Omega \quad K = \frac{1}{2}(P, \Omega), \quad \dot{P} = P \times \Omega.$$

Let $n = \frac{p}{\|p\|}$, and let e, $f \in \mathbb{R}^3$ be unit vectors such that (e, f, n) is a right-handed orthonormal frame. Show that e and f are rotating about n with angular velocity $\frac{2K}{\|p\|}$, so that

$$\dot{e} = \frac{2K}{\|p\|} f, \qquad \dot{f} = -\frac{2K}{\|p\|} e,$$

then the vectors E, F defined by

$$e = SE, \qquad f = SF,$$

regarded as tangent vectors to the sphere $\{\|P\| = \|p\|\}$ at P, are parallel transported along the curve described by the moving point P. (This gives a picture of the Euler top motion provided one knows the motion of P: if $R(t)$ is the rotation about n by $\frac{2Kt}{\|p\|}$ and $T(t)$ is the rotation that effects the parallel transport from $P(0)$ to $P(t)$, then $S(t) = R(t)S(0)T^{-1}(t)$.)