Geometric Mechanics

2nd Exam - February 6, 2026

MMAC

Solutions

a) The Legendre transformation is

$$\begin{aligned} p_r &= m v^r, \\ p_\theta &= m(l+v)^2 v^\theta. \end{aligned}$$

The Lagrangian is hyper-regular because the Legendre transformation $(r, \theta, v^r, v^\theta) \to (r, \theta, p_r, p_\theta)$ is invertible. The Hamiltonian is

$$H = p_r v^r + p_\theta v^\theta - L = \frac{1}{2m}\left((p_r)^2 + \frac{(p_\theta)^2}{(l+r)^2}\right) + \frac{\kappa}{2} r^2.$$

Hamilton's equations are

$$\begin{aligned} \dot{r} &= \frac{p_r}{m}, \\ \dot{\theta} &= \frac{p_\theta}{m(l+r)^2}, \\ \dot{p}_r &= \frac{(p_\theta)^2}{m(l+r)^3} - \kappa r, \\ \dot{p}_\theta &= 0. \end{aligned}$$

b) The function p_θ is a first integral of H. Moreover, the vector fields

$$X_H = \frac{p_r}{m}\frac{\partial}{\partial r} + \frac{p_\theta}{m(l+r)^2}\frac{\partial}{\partial \theta} + \left(\frac{(p_\theta)^2}{m(l+r)^3} - \kappa r\right)\frac{\partial}{\partial p_r}$$

and

$$X_{p_\theta} = \frac{\partial}{\partial \theta}$$

are independent in

$$U := T^*M \setminus \left\{(r, \theta, p_r, p_\theta) \in T^*M : p_r = 0 \text{ and } \frac{(p_\theta)^2}{m(l+r)^3} = \kappa r\right\}.$$

The set U is open and is dense in T^*M, because

$$\mathbb{R}^4 \setminus \{(r, \theta, p_r, p_\theta) \in \mathbb{R}^4 : p_r = 0\} \subset U$$

is dense in $\mathbb{R}^4$. Therefore, H completely integrable.

c) The function $V : (-l, +\infty) \to \mathbb{R}$, defined by

$$V(r) = \frac{1}{2m}\frac{((p_\theta)_0)^2}{(l+r)^2} + \frac{\kappa}{2}r^2,$$

has first derivative

$$V'(r) = -\frac{1}{m}\frac{((p_\theta)_0)^2}{(l+r)^3} + \kappa r$$

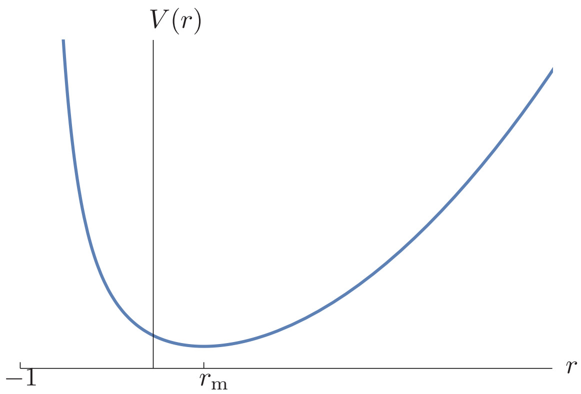
and second derivative

$$V''(r) = \frac{3}{m}\frac{((p_\theta)_0)^2}{(l+r)^4} + \kappa.$$

So, it is convex, and goes to $+\infty$ when $r \searrow -l$ and $r \nearrow +\infty$. It has a strict global minimum at $r_{\rm m}$, where

$$(l + r_{\rm m})^3 r_{\rm m} = \frac{((p_\theta)_0)^2}{\kappa m}. \tag{1}$$

In particular, r_m is positive (unless $(p_\theta)_0 = 0$).



Since

$$H = \frac{(p_r)^2}{2m} + V(r),$$

we may consider three cases:

(i) If $E < V(r_{\rm m})$, then $L_{(E,(p_\theta)_0)}$ is empty.

(ii) If $E = V(r_{\rm m})$, then $L_{(E,(p_\theta)_0)}$ is equal to

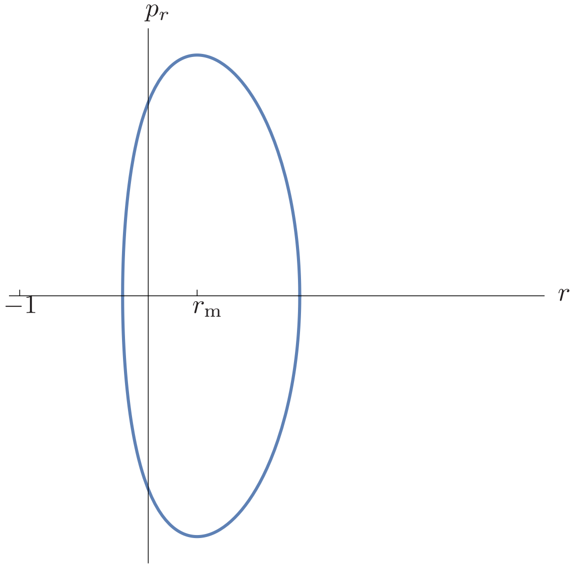
$$\{(r, \theta, p_r, p_\theta) \in T^*M : r = r_{\rm m},\ \theta \in S^1,\ p_r = 0,\ p_\theta = (p_\theta)_0\}.$$

(iii) If $E > V(r_\mathrm{m})$, then $L_{(E,(p_\theta)_0)}$ is equal to

$$\Big\{(r,\theta,p_r,p_\theta)\in T^*M : r\in[r_0,r^0],\ \theta\in S^1,\\ p_r = \pm 2m\sqrt{E-V(r)},\ p_\theta = (p_\theta)_0\Big\}.$$

Here $r_0 < r_m < r^0$ and

$$V(r_0) = V(r^0) = E.$$



Sketch of the intersection of $L_{(E,(p_\theta)_0)}$ with the $(r,\theta_0,p_r,(p_\theta)_0)$ half-plane.

Note that the Implicit Function Theorem guarantees that the curve depicted in the figure is smooth when it crosses the r-axis because $V'(r_0)$ and $V'(r^0)$ are different from zero. Indeed, the functions $p_r \mapsto r(p_r)$ satisfy

$$0 = \frac{p_r}{m} + V'(r(p_r))r'(p_r).$$

At $p_r = 0$, $r(p_r) = r_0$ or $r(p_r) = r^0$ and $r'(p_r) = 0$.

d) A circular orbit (which depends on $(p_\theta)_0$) satisfies

$$\begin{aligned} r &= r_\mathrm{m},\\ \theta &= \frac{(p_\theta)_0}{m(l+r_\mathrm{m})^2}t + \theta_0,\\ p_r &= 0,\\ p_\theta &= (p_\theta)_0. \end{aligned}$$

The value r_{m} satisfies (1). Note that the potential V depends on p_θ, $V = V_{p_\theta}$. The orbit is stable because orbits such that r, p_r and p_θ start close to $r_{\mathrm{m}} = r_{\mathrm{m}}((p_\theta)_0)$, 0 and $(p_\theta)_0$, respectively, have energy E close to $V_{(p_\theta)_0}(r_m((p_\theta)_0))$. Since the function V_{p_θ} is strictly convex and V_{p_θ} is close to $V_{(p_\theta)_0}$ (for, say, $-\frac{l}{2} \leq r \leq L$, for any fixed $L > 0$), the two values $r_0 = r_0(E, p_\theta)$ and $r^0 = r^0(E, p_\theta)$ will be close to $r_{\mathrm{m}}((p_\theta)_0)$; since $r_0 \leq r \leq r^0$, the orbit will be close to the circular orbit.

e) The equilibrium points are $(r, \theta, p_r, p_\theta) = (0, \theta_0, 0, 0)$, where $\theta_0 \in S^1$. They are unstable because a solution starting at $(0, \theta_0, 0, \varepsilon)$, $\varepsilon > 0$, has $\dot{\theta} \geq \frac{\varepsilon}{4ml^2}$ while $-l < r \leq l$, and so satisfies $\theta \geq \theta_0 + \frac{\varepsilon}{4ml^2}t$ while $-l < r \leq l$. So, either r attains the value l, or θ goes to $+\infty$. This proves instability of the equilibrium point.

f) In case (i) $L_{(E,(p_\theta)_0)}$ is empty. In case (ii) $L_{(E,(p_\theta)_0)}$ is diffeomorphic to S^1. In case (iii) $L_{(E,(p_\theta)_0)}$ is diffeomorphic to T^2 because the curve in the previous figure is diffeomorphic to S^1 and θ belongs to S^1. These assertions also hold for $(p_\theta)_0 = 0$ (what changes in this case is the shape of $V = V_0 = \frac{\kappa}{2}r^2$, and the shape of $L_{(E,0)}$ (it becomes the ellipse $E = \frac{(p_r)^2}{2m} + \frac{\kappa}{2}r^2$))

g) The position of the blade is

$$x = re_r,$$

and its velocity is

$$\dot{x} = \dot{r}\, e_r + r\dot{\theta}\, e_\theta.$$

Since the blade moves in a direction that makes an angle φ with the spring, it moves in the direction of

$$\cos\varphi\, e_r + \sin\varphi\, e_\theta.$$

So,

$$\dot{x} = \lambda(\cos\varphi\, e_r + \sin\varphi\, e_\theta),$$

for some real λ. This implies that

$$\dot{r} = \lambda\cos\varphi \qquad \text{and} \qquad r\dot{\theta} = \lambda\sin\varphi.$$

These equalities imply that

$$\dot{r}\sin\varphi = r\dot{\theta}\cos\varphi.$$

h) Newton's equation,

$$\mu\left(\frac{D\dot{c}}{dt}\right) = (\mathcal{F} + \mathcal{R})(\dot{c}),$$

reads

$$\begin{aligned}
\frac{d}{dt}\frac{\partial K}{\partial \dot r} - \frac{\partial K}{\partial r} &= (\mathcal{F}+\mathcal{R})\left(\frac{\partial}{\partial r}\right), \\
\frac{d}{dt}\frac{\partial K}{\partial \dot\theta} - \frac{\partial K}{\partial \theta} &= (\mathcal{F}+\mathcal{R})\left(\frac{\partial}{\partial \theta}\right), \\
\frac{d}{dt}\frac{\partial K}{\partial \dot\varphi} - \frac{\partial K}{\partial \varphi} &= (\mathcal{F}+\mathcal{R})\left(\frac{\partial}{\partial \varphi}\right).
\end{aligned}$$

Since the constraint is defined by the one form

$$\omega = \sin\varphi\, dr - r\cos\varphi\, d\theta,$$

$\mathcal{R} = \lambda\omega$, for some real function $\lambda = \lambda(\dot c)$. On the other hand,

$$\mathcal{F} = -dU = -\kappa r\, dr,$$

as $U = \frac{\kappa}{2}r^2$. We obtain

$$\begin{aligned}
m\ddot r - m(l+r)\dot\theta^2 &= -\kappa r + \lambda\sin\varphi, && (2)\\
\frac{d}{dt}\left(m(l+r)^2\dot\theta\right) &= -\lambda r\cos\varphi, && (3)\\
I\ddot\varphi = 0. & && (4)
\end{aligned}$$

These three equations together with the restriction

$$\dot r\sin\varphi = r\dot\theta\cos\varphi$$

determine the motion. Angular momentum does not have to be conserved. But, in some situations it is conserved. For example, suppose $\varphi \equiv \frac{\pi}{2}$. Then (the restriction equation shows that) r has to be constant. By (3), $\dot\theta$ has to be constant. Equation (2) is satisfied with $\lambda = \kappa r - m(l+r)\dot\theta^2$.

i) We have

$$d\omega = \cos\varphi\, d\varphi\wedge dr - \cos\varphi\, dr\wedge d\theta + r\sin\varphi\, d\varphi\wedge d\theta.$$

Thus

$$\begin{aligned}
d\omega\wedge\omega &= -r\cos^2\varphi, d\varphi\wedge dr\wedge d\theta + r\sin^2\varphi\, d\varphi\wedge d\theta\wedge dr \\
&= r\, d\varphi\wedge d\theta\wedge dr.
\end{aligned}$$

The constraint is non-holonomic because $d\omega\wedge\omega$ is not identically equal to zero.