Geometric Mechanics

2[nd] Exam - February 7, 2025
MMAC

Solutions

1.

a) Let

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

be the matrix that rotates (x, y) by θ radians counterclockwise around the origin, i.e.

$$R(\theta) \begin{bmatrix} x \\ y \end{bmatrix}$$

is the vector which will be at (x, y) after a rotation by θ radians clockwise around the origin. The vector

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} R(\theta) \begin{bmatrix} x \\ y \end{bmatrix}$$

is orthogonal to the hyperbolas $x^2 - y^2 = c$ at $R(\theta) \begin{bmatrix} x \\ y \end{bmatrix}$. After a rotation by θ radians clockwise around the origin, these hyperbolas have normal vector

$$R(-\theta) \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} R(\theta) \begin{bmatrix} x \\ y \end{bmatrix}$$

at $\begin{bmatrix} x \\ y \end{bmatrix}$. Since

$$R(-\theta) \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} R(\theta) = \begin{bmatrix} \cos(2\theta) & -\sin(2\theta) \\ -\sin(2\theta) & -\cos(2\theta) \end{bmatrix},$$

the velocities of the particles are in the kernel of

$$\omega = (x\cos(2\theta) - y\sin(2\theta))\, dx - (x\sin(2\theta) + y\cos(2\theta))\, dy.$$

b) The distribution Σ is the kernel of ω. As

$$\begin{aligned} d\omega &= -2x\sin(2\theta)\,d\theta\wedge dx - 2y\cos(2\theta)\,d\theta\wedge dx \\ &\quad -\sin(2\theta)\,dy\wedge dx \\ &\quad -2x\cos(2\theta)\,d\theta\wedge dy + 2y\sin(2\theta)\,d\theta\wedge dy \\ &\quad -\sin(2\theta)\,dx\wedge dy \\ &= -2(x\sin(2\theta) + y\cos(2\theta))\,d\theta\wedge dx \\ &\quad +2(-x\cos(2\theta) + y\sin(2\theta))\,d\theta\wedge dy, \end{aligned}$$

we have

$$\begin{aligned} d\omega\wedge\omega &= 2(x\sin(2\theta) + y\cos(2\theta))^2\,d\theta\wedge dx\wedge dy \\ &\quad +2(x\cos(2\theta) - y\sin(2\theta))^2\,d\theta\wedge dx\wedge dy \\ &= 2(x^2+y^2)\,d\theta\wedge dx\wedge dy. \end{aligned}$$

The distribution is not integrable because $d\omega\wedge\omega$ is not zero.

c) The motion of the particle and the plane are determined by Newton's equation

$$\mu\left(\frac{D\dot c}{dt}\right) = \mathcal{R}(\dot c),$$

where $\mathcal{R}(\dot c) = \lambda\omega(\dot c)$, for some real $\lambda = \lambda(x, y, \theta, \dot x, \dot y, \dot c)$. Indeed, if c is admissible, then

$$\left(\mu^{-1}\mathcal{R}, \dot c\right) = \mathcal{R}(\dot c) = \lambda\omega(\dot c) = 0.$$

In our coordinates, Newton's and the constraint equation are written as

$$\begin{cases} m\ddot x &= \lambda(x\cos(2\theta) - y\sin(2\theta)), \\ m\ddot y &= -\lambda(x\sin(2\theta) + y\cos(2\theta)), \\ I\ddot\theta &= 0, \\ 0 &= (x\cos(2\theta) - y\sin(2\theta))\dot x - (x\sin(2\theta) + y\cos(2\theta))\dot y. \end{cases}$$

d) If θ is constant, we may integrate the constraint equation to obtain

$$\frac{x^2}{2}\cos(2\theta) - xy\sin(2\theta) - \frac{y^2}{2}\cos(2\theta) = \text{constant}.$$

So, the assertion follows from

$$\begin{aligned} &(x\cos\theta - y\sin\theta)^2 - (x\sin\theta + y\cos\theta)^2 \\ &\qquad = x^2\cos(2\theta) - 2xy\sin(2\theta) - y^2\cos(2\theta). \end{aligned}$$

e) This is immediate as

$$\begin{aligned}(\cos t\cos(2t)+\sin t\sin(2t)) &= \cos t,\\ (\cos t\sin(2t)-\sin t\cos(2t)) &= \sin t,\end{aligned}$$

so that the constraint equation becomes

$$\cos t(-\sin t)+\sin t\cos t=0.$$

The value of λ is equal to $-m$.

2.

a) The length of the arch o catenary between $(0,0)$ and $(s, 1-\cosh s)$ is

$$\int_0^s \sqrt{1+\sinh^2\tau}\, d\tau = \sinh s.$$

The unit tangent vector to the catenary at $(s, 1-\cosh s)$ is

$$\frac{(1, -\sinh s)}{\cosh s}.$$

Thus, the angle, φ, between the bar and the s-axis satisfies

$$\tan\varphi = -\sinh s, \qquad \text{or} \qquad \varphi = -\arctan\sinh s.$$

The position o the center of mass of the bar is at

$$(s, 1-\cosh s) - \frac{(1, -\sinh s)}{\cosh s}\sinh s = (s-\tanh s, 1-\operatorname{sech} s).$$

b) The velocity of the center of mass of the bar is

$$(\dot{x}(s), \dot{y}(s)) = (1-\operatorname{sech}^2 s, \operatorname{sech} s\tanh s) = (\tanh^2 s, \operatorname{sech} s\tanh s).$$

So,

$$\left\|\dot{x}^2+\dot{y}^2\right\|(s) = \tanh^2 s.$$

Moreover, we have

$$\dot{\varphi}(s) = -\frac{\cosh s}{1+\sinh^2 s} = \operatorname{sech} s.$$

Finally, the center of mass of the bar satisfies $y = (1-\operatorname{sech} s)$.

c) The equation for the motion of the bar is

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{s}} - \frac{\partial L}{\partial s} = 0,$$

or

$$\ddot{s} + \operatorname{sech} s \tanh s = 0.$$

d) The Legendre transformation is

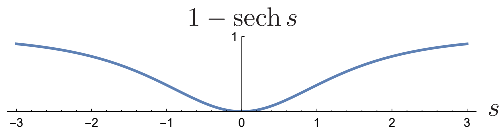
$$p = \frac{\partial L}{\partial \dot{s}} = \dot{s}.$$

It is invertible, $\dot{s} = p$, so the Lagrangian is hyper-regular. The Hamiltonian is

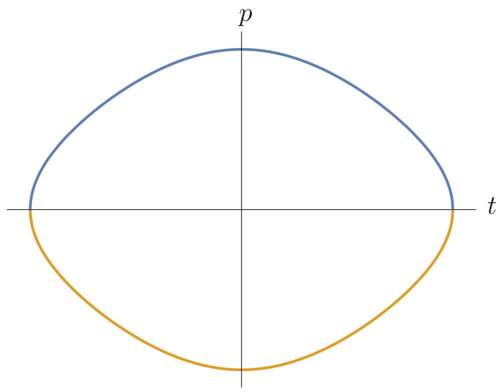
$$H = p\dot{s} - L = \frac{p^2}{2} + 1 - \operatorname{sech} s.$$

It is completely integrable because H itself is an integral, whose differential only vanishes at the origin.

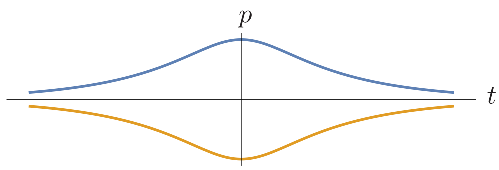
e) We sketch the graph of $s \mapsto 1 - \operatorname{sech} s$.



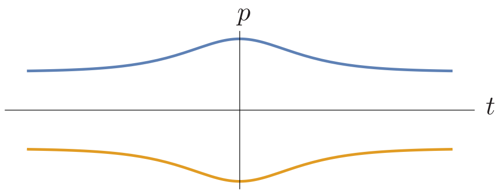
For $l < 0$, the l level set of H is empty. The 0 level set of H is the origin. For $0 < l < 1$, the l level set of H has the following sketch in the (s, p) plane:



The 1 level set of H has the following sketch in the (s, p) plane:



For $l > 1$, the l level set of H has the following sketch in the (s,p) plane:



The level sets of H are compact for levels $l \in (-\infty, 1)$. The flow of H progresses along its level sets. For p positive, s increases, and for p negative, s decreases. The origin corresponds to a stable equilibrium point.

f) The symplectic form is $\omega = dp \wedge ds$. The Hamiltonian gradient of H is

$$X_H = \frac{\partial H}{\partial p}\frac{\partial}{\partial s} - \frac{\partial H}{\partial s}\frac{\partial}{\partial p} = p\frac{\partial}{\partial s} - \operatorname{sech} s \tanh s \frac{\partial}{\partial p}.$$

The Poisson bracket is

$$\{H, G\} = \frac{\partial H}{\partial p}\frac{\partial G}{\partial s} - \frac{\partial H}{\partial s}\frac{\partial G}{\partial p}.$$

The Poisson bivector is

$$B = \frac{\partial}{\partial p} \otimes \frac{\partial}{\partial s} - \frac{\partial}{\partial s} \otimes \frac{\partial}{\partial p}.$$