Geometric Mechanics

2nd Exam - February 6, 2026

MMAC

Duration: 120 minutes

Show your calculations

Consider the model for a (positive) mass m rotating about the origin attached to a spring of natural length l and restitution coefficient κ (both positive), with Lagrangian

$$L = \frac{1}{2}m\left[(v^r)^2 + (l+r)^2(v^\theta)^2\right] - \frac{\kappa}{2}r^2,$$

for $(r,\theta) \in M := \{(r,\theta) \in \mathbb{R} \times S^1 : r > -l\}$.

- **a)** Write the Legendre transformation. Show that the Lagrangian is hyperregular. Determine the Hamiltonian $H : T^*M \to R$. Write Hamilton's equations. (2)
- **b)** Is H completely integrable? Justify. (2)
- **c)** Let $E > 0$ and $(p_\theta)_0 \neq 0$. Consider the level set $L_{(E,(p_\theta)_0)}$ where $H = E$ and the angular momentum, p_θ, is equal to $(p_\theta)_0$. Sketch the intersection of $L_{(E,(p_\theta)_0)}$ with the $(r, \theta_0, p_r, (p_\theta)_0)$ half-plane, where θ_0 is fixed. (3)
- **d)** Give an equation for the radius of the circular orbits. Are they stable or unstable? Why? (2)
- **e)** Classify the equilibrium points according to their stability. Justify. (2)
- **f)** What is the diffeomorphism class of $L_{(E,(p_\theta)_0)}$? Explain. (2)
- **g)** Suppose that the mass on the tip of the spring is now a blade moving in the (r,θ) plane, and that it moves only in the direction of the blade. When the blade makes an angle φ with the direction, θ, of the spring, prove that, for any movement, $\dot{r}\sin\varphi = r\dot{\theta}\cos\varphi$. (2)
- **h)** Suppose that the blade has moment of inertia I about its center, so that the kinetic energy of the blade attached to the spring is equal to (3)

  $$K = \frac{1}{2}m\left[\dot{r}^2 + (l+r)^2\dot{\theta}^2\right] + \frac{1}{2}I\dot{\varphi}^2.$$

  Assuming a perfect reaction force, write the generalized Newton equation

  $$\mu\left(\frac{D\dot{c}}{dt}\right) = (\mathcal{F} + \mathcal{R})(\dot{c})$$

  for the movement of the restricted mass, and a system that determines the motion. Is angular momentum always conserved?
- **i)** Is the constraint holonomic? Justify. (2)