Geometric Mechanics

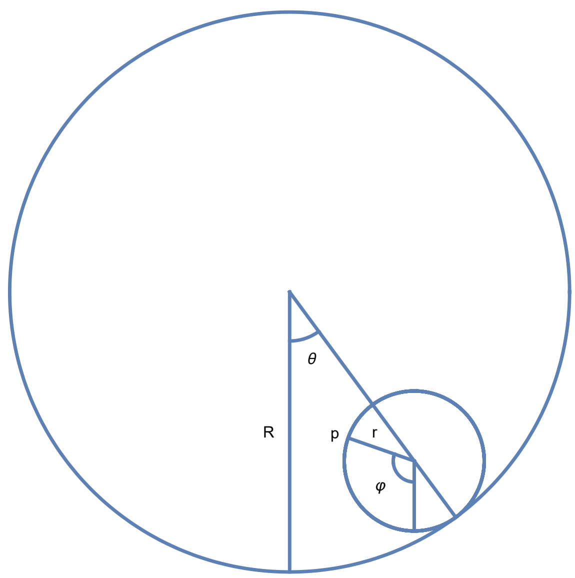
1^{st} Exam - January 27, 2025

MMAC

Duration: 120 minutes

Show your calculations

1. Consider a disc of radius r which rolls without slipping inside a circle of radius R centered at the origin. Let θ be the angle between the segment that connects the origin to the center of the disc and the x-axis, which is a vertical axis pointing down. Consider a point p on the boundary of the disc, so that when $\theta = 0$ the point p is at position $(R, 0)$. Let $-\varphi$ be the angle between the segment that connects the center of the disc to p and the direction of the x-axis. Notice that, because of the minus sign, when θ increases, φ also increases.



a) Check that the relation between θ and φ is (1)

$$\varphi = \frac{R-r}{r}\theta.$$

b) Assuming that the disc has total mass m, which is radially distributed, and that the disc has moment of inertia I, check that its kinetic energy is given by (1)

$$K = \frac{m}{2}(R-r)^2\dot{\theta}^2 + \frac{I}{2}\dot{\varphi}^2.$$

c) Write a 1-form, ω, whose kernel gives the constraint. Is the constraint holonomic? (2)

d) Assume a conservative external force, $\mathcal{F}$ corresponding to a potential energy equal to $U = -mgx$. Assuming a perfect reaction force $\mathcal{R}$, write the system that arises from the Newton equation (2)

$$\mu\left(\frac{D\dot{c}}{dt}\right) = (\mathcal{F} + \mathcal{R})(\dot{c}).$$

e) Obtain an ordinary differential equation for θ. (2)

2. Consider the hyperboloid $\mathcal{H}$, parameterized by

$$g(u,\varphi) = (\cosh u \cos\varphi, \cosh u \sin\varphi, \sinh u), \ (u,\varphi) \in (-\infty, +\infty) \times (0, 2\pi).$$

a) Check that the metric induced by the euclidean metric on $\mathcal{H}$ is (2)

$$ds^2 = \cosh(2u)\, du^2 + \cosh^2 u\, d\varphi^2.$$

b) Consider a particle of mass one, moving on $\mathcal{H}$ under the action of a perfect reaction force and the gravitational potential energy (2)

$$U = gz,$$

where $g \geq 0$. Write the equations of motion. Which parallels $\{z = \text{constant}\}$ are images of motion? Characterize these motions.

c) Compute the Legendre transformation. Show that the Lagrangian is hyper-regular. Determine the Hamiltonian $H : T^*\mathcal{H} \to \mathbb{R}$. Prove that it is completely integrable. (2)

d) Consider $g = 0$. Fix l and k positive, with $l^2 = 2k$. Give a rough sketch of the intersection of the level set (2)

$$L_{(k,l)} = \{\beta \in T^*\mathcal{H} : H(\beta) = k, p_\varphi(\beta) = l\}$$

with the plane $(u, \varphi_0, p_u, (p_\varphi)_0)$, for fixed φ_0 and $(p_\varphi)_0 = l$. Suggestion: write the hyperbolic functions in terms of $\sinh u$; note that $\frac{1+2x}{1+x}$ is a strictly increasing function in $\mathbb{R}^+$, going from 1 at $x = 0$ to 2 at $x = +\infty$.

e) Are the orbits whose images are the circle $\{z = 0\}$ stable? (2)

f) The Lie group S^1 acts on $\mathcal{H}$ through (2)

$$e^{i\theta} \cdot (u,\varphi) = (u, \varphi + \theta),$$

and θ and φ should be understood mod 2π. Calculate the infinitesimal action of S^1 on $\mathcal{H}$. Determine the momentum map of the action. Relate to Noether's theorem.