3 de Novembro

Tuesday, November 3, 2020 9:58 AM

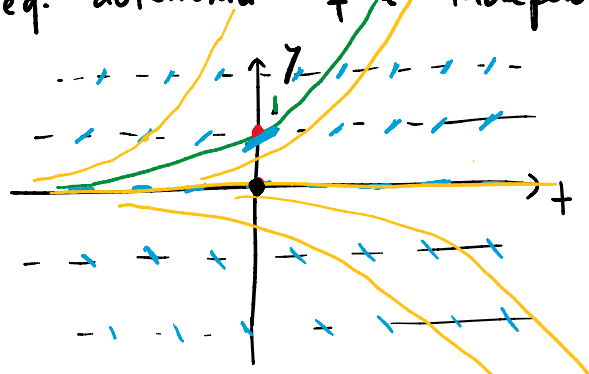
Equações diferenciais ordinárias escalares de 1ª ordem

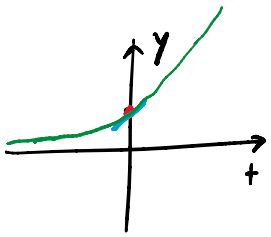
$$\begin{cases} y' = f(t,y) \\ y(t_0) = y_0 \end{cases} \qquad \begin{cases} y'(t) = f(t, y(t)) \\ y(t_0) = y_0 \end{cases}$$

PVI (problema de valor inicial)

Exemplo 1 eq. autónoma f é independente de t

$$\begin{cases} y' = y \\ y(0) = 1 \end{cases}$$





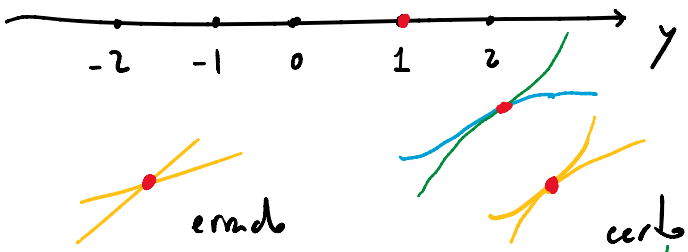
$y = y(0)e^t$

campo de direções da equação diferencial

Questões:

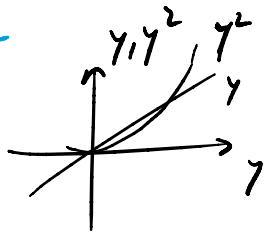
- Existência de soluções?
- Unicidade de solução
- Prolongamento de soluções ou intervalo máximo de existência
- Cálculo das soluções

Lei de Malthus

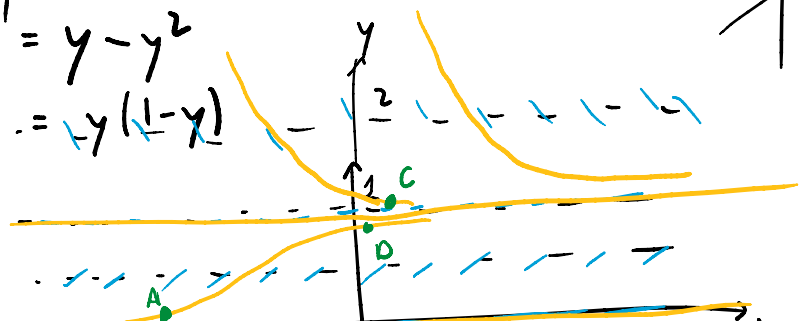


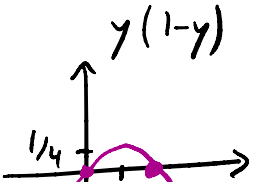
zero é equilíbrio instável

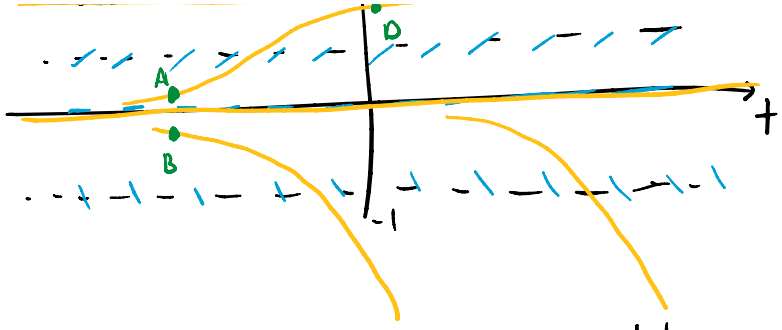
Exemplo 2 Lei logística

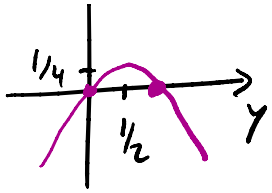


$$y' = y - y^2$$
$$= y(1-y)$$









um é um equilíbrio estável
zero " " " instável

A equação $y' = f(t,y)$ atribui a cada ponto do plano (mais precisamente do domínio de f) um declive. As soluções são as funções cujos gráficos têm os declives especificados!

4 de Novembro

Wednesday, November 4, 2020 11:59 AM

Resolução de eq. separáveis

Eq. separável é aquela que se pode escrever na forma

$g(y)y' = h(t)$ Exemplos $y' = y$ $\frac{1}{y}y' = 1$

$y' = y(1-y)$ $\frac{1}{y(1-y)}y' = 1$

Supomos que g e h são funções contínuas.

Existem G, H tais que $G' = g$ e $H' = h$.

$g(y)y' = h(t)$ $\frac{d}{dy}(G(y))\frac{dy}{dt} = \frac{dH}{dt}$

$\frac{d}{dt}(G(y)) = \frac{dH}{dt}$ $G(y) = H(t) + c$

$G(y(t)) = H(t) + c$

$t = t_0 \Rightarrow$ $G(y(t_0)) = H(t_0) + c$

$G(y_0) = H(t_0) + c$

$$\boxed{G(y) = H(t) + G(y_0) - H(t_0)}$$

Alternativa

$$\int_{t_0}^{t} \frac{d}{ds}G(y(s))\,ds = \int_{t_0}^{t} \frac{d}{ds}H(s)\,ds$$

$$G(y(t)) - G(y(t_0)) = H(t) - H(t_0)$$

<u>Exemplo</u>

$y' = a(t)y$ $\frac{1}{y}y' = a(t)$

$$\frac{d}{dy}\ln|y|\frac{dy}{dt} = \frac{d}{dt}\int a(t)\,dt$$

$$\frac{d}{dt}\left(\ln|y(t)|\right) = \frac{d}{dt}\int a(t)\,dt$$

$$\ln|y(t)| = \int a(t)\,dt + c_1 \qquad c_1 \in \mathbb{R}$$

$$|y(t)| = e^{\ln|y(t)|} = e^{\int a(t)dt + c_1} = e^{c_1} e^{\int a(t)dt}$$

$$= c_2 e^{\int a(t)dt} \qquad c_2 \in \mathbb{R}^+$$

$$y(t) = \pm c_2 e^{\int a(t)dt} \qquad \pm c_2 = c_3 \in \mathbb{R}\setminus\{0\}$$

$$y(t) = c_3 e^{\int a(t)dt} \qquad c_3 \in \mathbb{R}$$

$$y'(t) = a(t)y \iff y(t) = c e^{\int a(t)dt} \qquad c \in \mathbb{R}$$

$$\begin{cases} y' = a(t)y \\ y(t_0) = y_0 \end{cases} \iff y(t) = y_0 e^{\int_{t_0}^{t} a(s)ds}$$

verificação

$$y' = y_0 e^{\int_{t_0}^{t} a(s)ds} \frac{d}{dt}\int_{t_0}^{t} a(s)ds$$

$$= y\, a(t)$$

$$y(t_0) = y_0 e^{\int_{t_0}^{t_0} a(s)ds} = y_0 e^0 = y_0$$

$$\begin{cases} y' = f(t,y) \\ y(t_0) = y_0 \end{cases} \qquad f \in C^1$$

existe uma e uma só solução definida numa vizinhança de t_0

$$y' = y(1-y) \qquad \boxed{\frac{1}{y(1-y)} y' = 1}$$

$$\left(\frac{A}{y} + \frac{B}{y-1}\right) y' = 1 \qquad \left(\frac{1}{y} - \frac{1}{y-1}\right) y' = 1$$

$$\frac{d}{dy}\left(\ln|y| - \ln|y-1|\right)\frac{dy}{dt} = 1$$

$$\frac{1}{y} + \frac{1}{1-y}$$

$$\frac{d}{dy} \ln\left|\frac{y}{y-1}\right| \frac{dy}{dt} = 1$$

$$\frac{1}{y} + \frac{1}{1-y}$$

$$= \frac{1-y}{y(1-y)} + \frac{y}{y(1-y)}$$

$$= \frac{1}{y(1-y)}$$

$$\frac{d}{dt}\left(\ln\left|\frac{y}{y-1}\right|\right) = 1$$

$$\ln\left|\frac{y}{y-1}\right| = t + c_1 \qquad c_1 \in \mathbb{R}$$

$$\left|\frac{y}{y-1}\right| = e^{t+c_1} = e^{c_1} e^{t} = c_2 e^{t} \qquad c_2 \in \mathbb{R}^+$$

$$\frac{y}{y-1} = \pm c_2 e^{t} = c_3 e^{t}$$

$$\pm c_2 = c_3 \in \mathbb{R}\setminus\{0\}$$

$$c_3 = c \in \mathbb{R}$$

$$y = (y-1)\, c e^{t} = y c e^{t} - c e^{t}$$

$$y(1 - c e^{t}) = -c e^{t}$$

$$y = -\frac{c e^{t}}{1 - c e^{t}} = \boxed{\frac{c e^{t}}{c e^{t} - 1}} = \frac{e^{t}}{e^{t} - 1/c} = \boxed{\frac{e^{t}}{e^{t} - \tilde{c}}}$$

$c \in \mathbb{R}$

$\tilde{c} \in \mathbb{R}$

$$\tilde{c} = 1/c \in \mathbb{R}\setminus\{0\}$$

6 de Novembro

Friday, November 6, 2020 12:01 PM

Equações lineares

Forma canónica $y' = a(t)y + b(t)$

Obs. Se $b(t) \equiv 0$ a equação diz-se homogénea

$$y' = a(t)y \iff y = ce^{\int a(t)dt}$$

$$\iff e^{-\int a(t)dt} y = c$$

Resolução das eq. lineares.

Faz-se em 5 passos:

1) Escreve-se na forma canónica

$$y' - a(t)y = b(t) \qquad L(y) = y' - a(t)y$$

2) Calcula-se o factor integrante

$$\mu(t) = e^{-\int a(t)dt}$$

3) Multiplicam-se ambos os membros da e.d. pelo factor integrante $\mu(t)$

$$e^{-\int a(t)dt} y' - e^{-\int a(t)dt} a(t)y = e^{-\int a(t)dt} b(t)$$

4) Reconhece-se no 1º membro a derivada do produto do factor integrante por y.

$$\frac{d}{dt}\left(e^{-\int a(t)dt} y\right) = e^{-\int a(t)dt} b(t)$$

5) Primitivam-se ambos os membros em ordem a t

5) Primitivam-se ambos os membros em ordem a t

$$e^{-\int a(t)dt} y = \int e^{-\int a(t)dt} b(t)\, dt + C$$

<u>Exemplo</u>

$$y' = 2ty + te^{t^2}$$

1) $y' - 2ty = te^{t^2}$

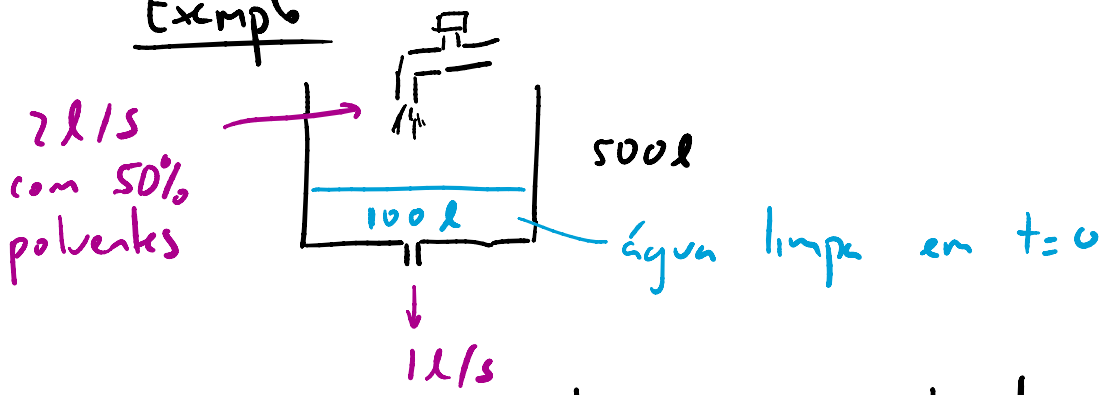
2) $\mu(t) = e^{\int -2t\,dt} = e^{-t^2}$

3) $e^{-t^2} y' - 2te^{-t^2} y = te^{t^2} e^{-t^2} = t$

4) $\frac{d}{dt}\left(e^{-t^2} y\right) = t$

5) $e^{-t^2} y = \frac{t^2}{2} + C \iff y = \frac{t^2}{2} e^{t^2} + Ce^{t^2}$

<u>Exemplo</u>



Qual é a concentração de poluentes no reservatório quando transbordar?

$Q \to$ a quantidade de poluentes no reservatório

$$\frac{dQ}{dt} = 1 - \frac{Q}{100+t} \qquad \frac{dQ}{dt} + \frac{Q}{100+t} = 1$$

$$100 + t = s$$

1) $\frac{dQ}{ds} + \frac{1}{s} Q = 1$

$\int \frac{1}{s} ds \qquad \ln|s| = |s|$

$$\frac{dQ}{ds} \quad s$$

2) $\mu(s) = e^{\int \frac{1}{s} ds} = e^{\ln|s|} = |s|$ $\qquad \mu(s) = s$

3) $s \frac{dQ}{ds} + Q = s$

4) $\frac{d}{ds}(sQ) = s$

$$\frac{Q}{s} = \frac{1}{2} + \frac{c}{s^2}$$

$$\to \frac{1}{2}$$

5) $sQ = \frac{s^2}{2} + c \qquad Q = \frac{s}{2} + \frac{c}{s}$

$t = 0 \Rightarrow s = 100 \Rightarrow Q = 0 \Rightarrow$

$$0 = \frac{100}{2} + \frac{c}{100} \Leftrightarrow c = -5000$$

$$Q = \frac{s}{2} - \frac{5000}{s}$$

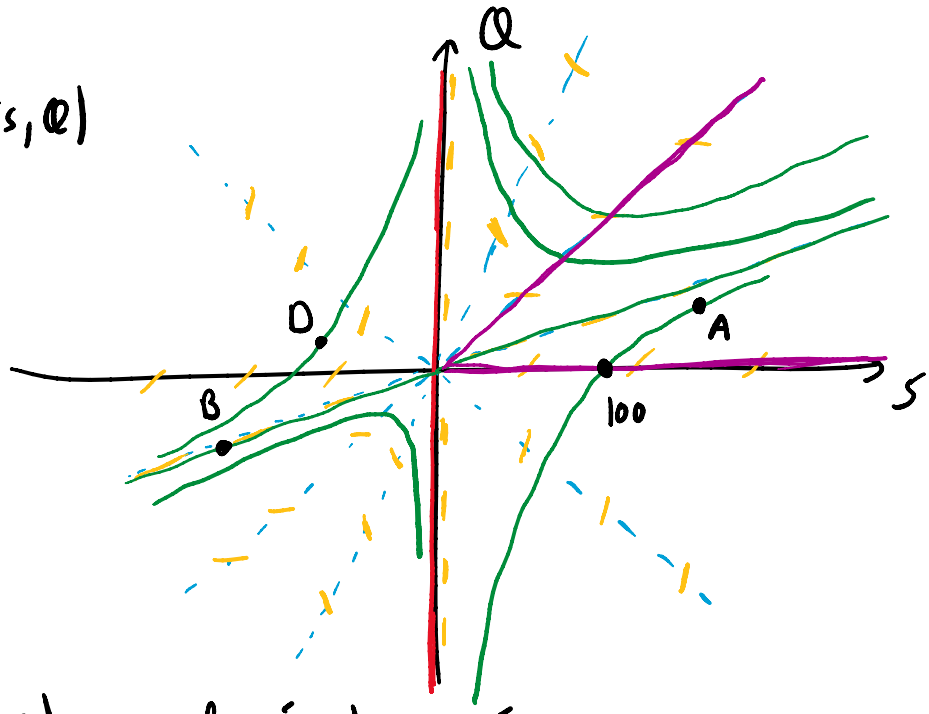
$s = 500 \Rightarrow Q = \frac{500}{2} - \frac{5000}{500} = 250 - 10 = 240$

concentracão $= \frac{Q}{s} = \frac{240}{500} = 48\%$

9 de Novembro

Monday, November 9, 2020 12:00 PM

$$\frac{dQ}{ds} = 1 - \frac{Q}{s} = f(s,Q)$$



$$f(s,Q) = f(\alpha s, \alpha Q) \qquad f \text{ é homogénea}$$
$$= f\left(1, \frac{Q}{s}\right) \qquad \left(\alpha = \frac{1}{s}\right)$$

$$\frac{Q}{s} = m \iff Q = ms \qquad Q' = 1 - m$$

$$\exists_m \; m = Q' = 1 - m \iff 2m = 1 \iff m = 1/2$$

$$\frac{dQ}{ds} = 1 - \frac{Q}{s} = \frac{s - Q}{s} \qquad \frac{ds}{dQ} = \frac{s}{s - Q}$$

$$\boxed{\frac{dy}{dt} = \frac{y}{y - t}}$$

$$(y - t)\frac{dy}{dt} = y \qquad \boxed{y + (t - y)y' = 0}$$

É da forma $M + Ny' = 0$ com $M = y$, $N = t - y$

$N = t-y$

Neste caso o campo $(M,N) = (y, t-y)$ é fechado: $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t} = 1$. Como está definido num conj. s.c. é um gradiente. A eq. dif diz-se exacta.

Cálculo do potencial $\nabla\phi = \left(\frac{\partial\phi}{\partial t}, \frac{\partial\phi}{\partial y}\right) = (M,N)$

$$= (y, t-y)$$

$$\begin{cases} \frac{\partial\phi}{\partial t} = y \\ \frac{\partial\phi}{\partial y} = t-y \end{cases} \Rightarrow \phi = ty - \frac{y^2}{2} + \cancel{cte}$$

$$M + Ny' = 0 \Leftrightarrow \frac{\partial\phi}{\partial t} + \frac{\partial\phi}{\partial y}\frac{dy}{dt} = 0$$

$$\Leftrightarrow \frac{d}{dt}\left[\phi(t, y(t))\right] = 0$$

$$\Leftrightarrow \phi(t, y(t)) = cte$$

$$\Leftrightarrow ty - \frac{y^2}{2} = c \Leftrightarrow y^2 - 2ty + 2c = 0$$

$$\Leftrightarrow y = t \pm \sqrt{t^2 - 2c}$$

$$\Leftrightarrow t = \frac{y}{2} + \frac{c}{y}$$

$y' = f(t,y)$

$$-f(t,y) + y' = 0$$

$$M + Ny' = 0$$

$$(M,N) = (-f, 1)$$

é exacta se $-\frac{\partial f}{\partial y} = \frac{\partial 1}{\partial t} = 0$

é exacta se $-\frac{\partial f}{\partial y} = \frac{\partial 1}{\partial t} = 0$

$$1 + \left(\frac{t-y}{y}\right) y' = 0$$

Não é exacta

$$\frac{\partial M}{\partial y} = \frac{\partial 1}{\partial y} = 0$$

$$\frac{\partial N}{\partial t} = \frac{\partial}{\partial t}\left(\frac{t-y}{y}\right) = \frac{1}{y}$$

A classificação da eq. como exacta depende da forma como está escrita.

10 de Novembro

Tuesday, November 10, 2020 9:58 AM

$$1 + \frac{t-y}{y} y' = 0 \qquad \mu = \mu(y)$$

$$M + N y' = 0$$

$$\mu(y) + \mu(y) \frac{t-y}{y} y' = 0 \quad (*)$$

Para que esta eq. seja exacta, devemos ter

$$\frac{\partial \mu(y)}{\partial y} = \frac{\partial}{\partial t}\left(\mu(y) \frac{t-y}{y}\right) \Leftrightarrow \mu'(y) = \frac{\mu(y)}{y}$$

$$\Leftrightarrow \mu(y) = c\, e^{\int \frac{1}{y} dy}$$

$$\Leftrightarrow \mu(y) = c\, e^{\ln|y|}$$

$$\Leftrightarrow \mu(y) = c\,|y|$$

Vou escolher $\mu(y) = y$

Substituindo $\mu(y) = y$ na eq. $(*)$ obtém-se

$$y + (t-y) y' = 0.$$

$$1 + \frac{t-y}{y} y' = 0$$

Tentativa de determinação de um factor integrante $\mu = \mu(t)$

$$\mu(t) + \mu(t) \frac{t-y}{y} y' = 0$$

$$\frac{\partial}{\partial y} \mu(t) = \frac{\partial}{\partial t}\left(\mu(t) \frac{t-y}{y}\right) \Leftrightarrow 0 = \mu'(t) \frac{t-y}{y} + \frac{\mu(t)}{y}$$

$$\frac{\partial}{\partial y}\mu(t) = \frac{\partial}{\partial t}\left(\mu(t)\frac{t-y}{y}\right) \Leftrightarrow 0 = \mu'(t)\frac{t-y}{y} + \frac{\mu(t)}{y}$$

$$\Leftrightarrow \quad 0 = \mu'(t)(t-y) + \mu(t)$$

$$\Leftrightarrow \quad \mu'(t)(y-t) = \mu(t) \quad \Leftrightarrow \quad \frac{\mu'(t)}{\mu(t)} = \frac{1}{y-t}$$

impossível

$$(2y^2+2y) + (3ty+2t)\,y' = 0$$

Procura de um factor integrante da forma

$$\mu = \mu(ty)$$

$$\mu(ty)(2y^2+2y) + \mu(ty)(3ty+2t)y' = 0$$

Para que a eq. seja exacta, devemos ter

$$\frac{\partial}{\partial y}\left[\mu(ty)(2y^2+2y)\right] = \frac{\partial}{\partial t}\left[\mu(ty)(3ty+2t)\right]$$

$$\mu'(ty)\,t\,(2y^2+2y) + \mu(ty)(4y+2)$$
$$= \mu'(ty)\,y\,(3ty+2t) + \mu(ty)(3y+2)$$

$$\Leftrightarrow \quad \mu'(ty)\left(2ty^2+2ty-3ty^2-2ty\right)$$
$$= \mu(ty)\left(3y+2-4y-2\right)$$

$$\Leftrightarrow \quad \mu'(ty)\left(-ty^2\right) = \mu(ty)(-y)$$

$$\Leftrightarrow \quad \mu'(ty)(ty) = \mu(ty) \qquad s = ty$$

$$\Leftrightarrow \quad \mu'(s)\, s = \mu(s) \quad \Leftrightarrow \quad \mu'(s) = \frac{1}{s}\mu(s)$$

$$\Leftrightarrow \quad \mu(s) = c\, e^{\int \frac{1}{s} ds} = c\, e^{\ln|s|}$$

$$= c|s|$$

$$\mu(s) = s$$

A eq. admite o factor integrante $\mu(ty) = ty$

Classes de eq. que conseguimos resolver explicitamente:

- separáveis
- lineares
- exactas
- redutíveis a exactas

Qd não conseguimos resolver uma eq. df. por uma das 4 técnicas acima, por vezes tentamos fazer uma mudança de variável.

Exemplo $\quad y' = g\left(\frac{y}{t}\right) \qquad Q' = 1 - \frac{Q}{s}$

$$y' = 1 - \frac{y}{t}$$

Façamos a mudança de variável $v = \frac{y}{t}$

Façamos a mudança de variável $v = \frac{y}{t}$

$$y' = g(v) \qquad \frac{dy}{dt} = g(v)$$

$$y = tv \Rightarrow y' = v + tv'$$

$$v + tv' = g(v) \qquad h(v)v' = l(t)$$

$$tv' = g(v) - v$$

$$\frac{1}{g(v)-v} v' = \frac{1}{t}$$

11 de Novembro

Wednesday, November 11, 2020 11:59 AM

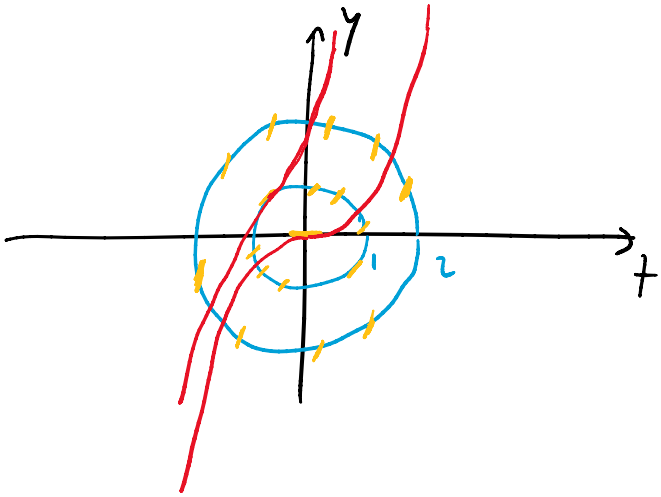
f dif em "a" $\Rightarrow$ f é cont em "a"

$$\lim_{z\to a}\left[f(z)-f(a)\right] = \lim_{z\to a}\left(\frac{f(z)-f(a)}{z-a}(z-a)\right)$$

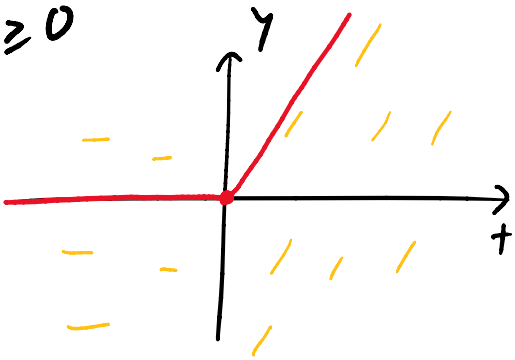
$$= f'(a)\cdot 0 = 0$$

PVI $\begin{cases} y' = f(t,y) \\ y(t_0) = y_0 \end{cases}$

$\begin{cases} y' = t^2+y^2 \\ y(0)=0 \end{cases}$



Exemplo $y' = \begin{cases} 0 & \text{se } t<0 \\ 1 & \text{se } t\geq 0 \end{cases}$



Def de solução: É uma função de classe C^1 definida num intervalo, cujo gráfico pertence ao domínio de f, e que satisfaz a eq. dif.

$\begin{cases} y'(s) = f(s,y(s)) \\ y(t_0) = y_0 \end{cases}$ Supomos que f é contínua

contínua

$$\begin{cases} y'(t) = f(t, y(t)) \\ y(t_0) = y_0 \end{cases}$$

$$\Leftrightarrow \int_{t_0}^{t} y'(s)\,ds = \int_{t_0}^{t} f(s, y(s))\,ds$$

$$\Leftrightarrow y(t) - y(t_0) = \int_{t_0}^{t} f(s, y(s))\,ds$$

$$\Leftrightarrow y(t) = y_0 + \int_{t_0}^{t} f(s, y(s))\,ds$$

O PVI e a eq. integral são equivalentes.

Aparte: $x = h(x)$ $\qquad h: \mathbb{R} \to \mathbb{R}$

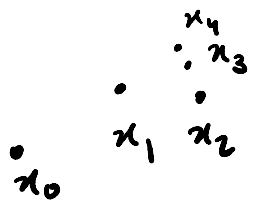
Pretendo determinar um ponto fixo da função h.

$$\xrightarrow{x} \boxed{h} \longrightarrow h(x)$$

$$y(t) \longrightarrow \boxed{L} \longrightarrow z(t) = y_0 + \int_{t_0}^{t} f(s, y(s))\,ds$$

$$\exists_{L<1} \ \forall_{x,y} \quad |h(x) - h(y)| \leq L|x-y|$$

h diz-se uma contração



$$x_{n+1} = h(x_n)$$

Construímos uma sucessão de funções (designadas por iteradas de Picard) da seguinte forma

$$y_0(t) \equiv y_0$$

$$y_1(t) = y_0 + \int_{t_0}^{t} f(s, y_0(s))\,ds$$

$$y_2(t) = y_0 + \int_{t_0}^{t} f(s, y_1(s))\,ds$$

$$\cdots\cdots$$

$$y_{n+1}(t) = y_0 + \int_{t_0}^{t} f(s, y_n(s))\,ds$$

$$\cdots\cdots$$

$$y_n(t) \longrightarrow y(t)$$

↳ será a solução da eq. integral e portanto do PVI

<u>Exemplo</u>

$$\begin{cases} y' = y \\ y(0) = 1 \end{cases}$$

$$y'(s) = y(s) \qquad \int_0^t y'(s)\,ds = \int_0^t y(s)\,ds$$

$$y(t) - y(0) = \int_0^t y(s)\,ds$$

$$y(t) = 1 + \int_0^t y(s)\,ds$$

$$y_0(t) \equiv 1 \qquad y_0 = y(t_0)$$

$$y_1(t) = 1 + \int_0^t y_0(s)\,ds = 1 + \int_0^t 1\,ds = 1 + t$$

$$y_2(t) = 1 + \int_0^t y_1(s)\,ds = 1 + \int_0^t (1+s)\,ds$$

$$= 1 + t + \frac{t^2}{2}$$

$$y_3(t) = 1 + \int_0^t y_2(s)\,ds = 1 + \int_0^t \left(1 + s + \frac{s^2}{2}\right) ds$$

$$y_3(t) = 1 + \int_0^t y_2(s)\,ds = 1 + \int_0^t \left(1 + s + \frac{s^2}{2}\right) ds$$

$$= 1 + t + \frac{t^2}{2} + \frac{t^3}{3!}$$

$$y_n(t) = \sum_{k=0}^{n} \frac{t^k}{k!}$$

$$y_{n+1}(t) = 1 + \int_0^t y_n(s)\,ds = 1 + \int_0^t \sum_{k=0}^{n} \frac{s^k}{k!}\,ds$$

$$= \sum_{k=0}^{n+1} \frac{t^k}{k!}$$

. . . .

$$y_n(t) \xrightarrow[n \to +\infty]{} \sum_{k=0}^{+\infty} \frac{t^k}{k!} = e^t$$

$$f(x) = y \iff x = x + y - f(x) \iff x = g(x)$$

$$\mathcal{L}\{y(t)\} = y_0 + \int_{t_0}^{t} f(s, y(s))\,ds$$

13 de Novembro

Friday, November 13, 2020 12:02 PM

Uma distância em S é uma função $d: S \times S \to \mathbb{R}_0^+$ que satisfaz

$\forall_{x,y \in S}$ $d(x,y) \geq 0$ anulando-se sse $x=y$

$d(x,y) = d(y,x)$

$d(x,y) \leq d(x,z) + d(z,y)$

Exemplos $\mathbb{R}$ $d(x,y) = |x-y|$

$\mathbb{R}^n$ $d(x,y) = \|x-y\|$

$\mathbb{C}$ $d(z,w) = |z-w|$

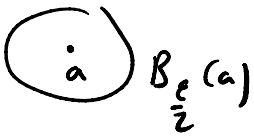
A um par (S,d) chama-se espaço métrico.

Nos espaços métricos define-se a noção de sucessão de Cauchy:

(x_n) é de Cauchy sse

$$\forall_{\varepsilon>0} \exists_p \quad n,m > p \Rightarrow d(x_n, x_m) < \varepsilon.$$

Nota: Qq sucessão convergente é de Cauchy



Os espaços métricos em que todas as sucessões de Cauchy convergem dizem-se completos.

Exemplo $\mathbb{R}$ é um espaço métrico completo

$\mathbb{Q}$ não é um " " " .

Teorema de Ponto Fixo de Banach.

Seja S um espaço métrico completo e $f: S \to S$ uma contracção

(i.e. $\exists_{L<1} \forall_{x,y} \; d(f(x), f(y)) \leq L\, d(x,y)$).

(i.e. $\exists_{L<1} \ \forall_{x,y} \ d(f(x),f(y)) \leq L d(x,y)$).

Então f tem um e um só ponto fixo

<u>Prova</u>: $x_0 \in S$

$x_1 = f(x_0)$

$x_2 = f(x_1)$

....

$x_{n+1} = f(x_n)$

....

Vamos provar que (x_n) é uma sucessão de Cauchy. Seguir-se-á $x_n \to x$. Isto implica

$$\lim x_{n+1} = \lim f(x_n) \overset{?}{=} f(\lim x_n)$$

$$\begin{array}{ccc} \| & & \| \\ x & & f(x) \end{array}$$

sim pq f é contínua

Prova da unicidade do ponto fixo:

Sejam $x = f(x)$ e $y = f(y)$

$$d(x,y) = d(f(x),f(y)) \leq L\, d(x,y)$$

$\Rightarrow d(x,y) = 0$ caso contrário obtém-se $1 \leq L$ contradição.

Falta mostrar que (x_n) é de Cauchy.

$$d(x_2,x_1) = d(f(x_1),f(x_0)) \leq L\, d(x_1,x_0)$$

$$d(x_3,x_2) = d(f(x_2),f(x_1)) \leq L\, d(x_2,x_1) \leq L^2 d(x_1,x_0)$$

$$d(x_4,x_3) = d(f(x_3),f(x_2)) \leq L\, d(x_3,x_2) \leq L^3 d(x_1,x_0)$$

...

$$d(x_{n+1},x_n) \leq L^n d(x_1,x_0)$$

...

Suponhamos que $n > m$

Suponhamos que $n > m$

$$d(x_n, x_m) \leq d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + \cdots + d(x_{m+2}, x_{m+1}) + d(x_{m+1}, x_m)$$

$$\leq L^{n-1} d(x_1, x_0) + L^{n-2} d(x_1, x_0) + \cdots + L^{m+1} d(x_1, x_0) + L^m d(x_1, x_0)$$

$$= d(x_1, x_0)\left(L^m + L^{m+1} + \cdots + L^{n-1}\right)$$

$$\leq d(x_1, x_0)\left(L^m + L^{m+1} + \cdots\right)$$

$$= d(x_1, x_0) \frac{L^m}{1-L} \xrightarrow{m \to +\infty} 0$$

Seja $\varepsilon > 0$. $\exists p \ \forall n > m > p \quad d(x_n, x_m) < \varepsilon$

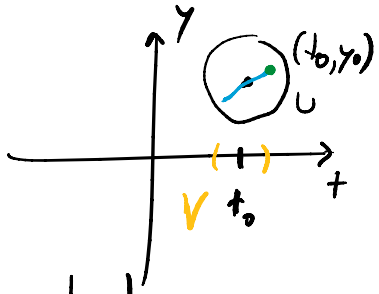
Está provado que (x_n) é uma sucessão de Cauchy

16 de Novembro
Monday, November 16, 2020 12:01 PM

Teorema de Picard-Lindelof.

PVI $\begin{cases} y' = f(t,y) \\ y(t_0) = y_0 \end{cases}$

Suponhamos que f é contínua, diferenciável em ordem a y e com $\frac{\partial f}{\partial y}$ contínua numa vizinhança do ponto (t_0,y_0). Então existe uma vizinhança V de t_0 tal que o PVI tem uma solução definida em V. Esta solução é única.



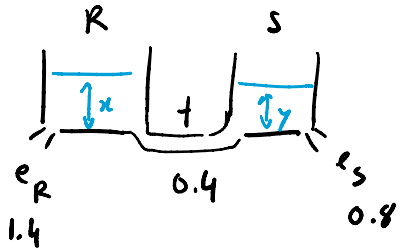
Este é um teorema de existência local.

Sistemas de EDOs lineares de coeficientes constantes homogéneos

$\begin{cases} \dot{X} = AX \\ X(0) = X_0 \end{cases}$ X vector coluna $n \times 1$, $A \in M_{n\times n}$

$\dot{x} = ax \Leftrightarrow x(t) = ce^{at}$

Exemplo



$X = \begin{bmatrix} x \\ y \end{bmatrix}$

$$\begin{cases} \dot{x} = -1.4x + 0.4(y-x) = -1.8x + 0.4y = -\frac{9}{5}x + \frac{2}{5}y \\ \dot{y} = -0.8y + 0.4(x-y) = 0.4x - 1.2y = \frac{2}{5}x - \frac{6}{5}y \end{cases}$$

$$\begin{cases} \dot{\begin{bmatrix} x \\ y \end{bmatrix}} = \frac{1}{5}\begin{bmatrix} -9 & 2 \\ 2 & -6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \\ \begin{bmatrix} x \\ y \end{bmatrix}(0) = \begin{bmatrix} 10 \\ 10 \end{bmatrix} \end{cases}$$

$\begin{cases} \dot{x} = ax \\ x(0) = x_0 \end{cases} \Leftrightarrow x(t) = x_0 e^{at}$

$$\begin{cases} \dot{X} = AX \\ X(0) = X_0 \end{cases} \Leftrightarrow X(t) = e^{At}X_0$$

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots$$

$\{ X(0) = X_0$ $\qquad 2! \quad 3!$

Suponhamos que A tem n vectores próprios linearmente independentes. Seja $S = [v_1 \dots v_n]$ e

$\Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$. Então $A = S \Lambda S^{-1}$

$$\underbrace{\begin{bmatrix} \\ \\ \end{bmatrix}}_{X} = S \underbrace{\begin{bmatrix} \\ \\ \end{bmatrix}}_{Y} = [v_1 \dots v_n] \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = y_1 v_1 + \dots + y_n v_n$$

$\uparrow$ escrito na base canónica de $\mathbb{R}^n$

$\hookrightarrow$ Y é o vector X escrito na base $(v_1 \dots v_n)$

$$AX = \underbrace{S \underbrace{\Lambda \underbrace{S^{-1} X}_{Y}}_{\downarrow}}_{S \Lambda S^{-1} X}$$

$X = SY \iff Y = S^{-1} X$

$$\Lambda Y = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} \lambda_1 y_1 \\ \vdots \\ \lambda_n y_n \end{pmatrix}$$

A imagem de X escrita na base dos vectores próprios

$\hookrightarrow$ A imagem de X escrita na base canónica de $\mathbb{R}^n$

$$A = S \Lambda S^{-1}$$

$$A^2 = S \Lambda S^{-1} S \Lambda S^{-1} = S \Lambda I \Lambda S^{-1} = S \Lambda^2 S^{-1}$$

$$A^3 = A^2 A = S \Lambda^2 S^{-1} S \Lambda S^{-1} = S \Lambda^2 I \Lambda S^{-1} = S \Lambda^3 S^{-1}$$

$\dots$

$$A^k = S \Lambda^n S^{-1} \checkmark$$

$$A^{k+1} = A^n A = S \Lambda^k S^{-1} S \Lambda S^{-1} = S \Lambda^{k+1} S^{-1}$$

$\dots$

$$\Lambda^k = \begin{bmatrix} \lambda_1^k & & 0 \\ & \ddots & \\ 0 & & \lambda_n^k \end{bmatrix}$$

$$e^{At} = I + S \Lambda S^{-1} t + S \Lambda^2 S^{-1} \frac{t^2}{2!} + S \Lambda^3 S^{-1} \frac{t^3}{3!} + \dots$$

$$= S\left(I + \Lambda t + \frac{\Lambda^2 t^2}{2!} + \cdots + \frac{\Lambda^k t^k}{k!} + \cdots\right) S^{-1}$$

$$= S e^{\Lambda t} S^{-1}$$

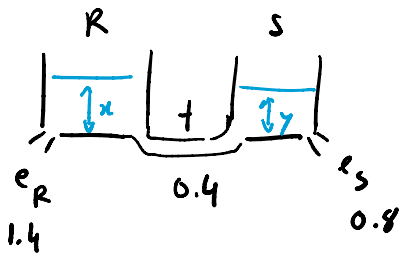
$$e^{\Lambda t} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \lambda_1 t & 0 \\ 0 & \lambda_2 t \end{bmatrix} + \begin{bmatrix} \frac{\lambda_1^2 t^2}{2!} & 0 \\ 0 & \frac{\lambda_2^2 t^2}{2!} \end{bmatrix} + \begin{bmatrix} \frac{\lambda_1^3 t^3}{3!} & 0 \\ 0 & \frac{\lambda_2^3 t^3}{3!} \end{bmatrix} + \cdots$$

$$= \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}$$

17 de Novembro

Tuesday, November 17, 2020 9:56 AM

$$\begin{cases} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -9 & 2 \\ 2 & -6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \\ \begin{bmatrix} x \\ y \end{bmatrix}(0) = \begin{bmatrix} 10 \\ 10 \end{bmatrix} \end{cases}$$



$$\begin{cases} \dot{X} = AX \\ X(0) = X_0 \end{cases} \iff X(t) = e^{At}X_0$$

$$\begin{cases} \dot{X} = AX \\ X(0) = X_0 \end{cases} \iff X(t) = e^{At}X_0 \qquad e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots$$

Caso A diagonalizável

$$A = S\Lambda S^{-1} \Rightarrow e^{At} = Se^{\Lambda t}S^{-1} \qquad e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & e^{\lambda_2 t} & \\ 0 & & \ddots \; e^{\lambda_n t} \end{bmatrix}$$

$$X(0) = e^{A0}X_0 = IX_0 = X_0$$

$$\frac{d}{dt}e^{At} = 0 + A + A^2t + \frac{A^3t^2}{2!} + \cdots \frac{A^n t^{n-1}}{(n-1)!} + \cdots$$

$$= A\left(I + At + \frac{A^2t^2}{2!} + \cdots + \frac{A^{n-1}t^{n-1}}{(n-1)!} + \cdots\right)$$

$$= Ae^{At} = e^{At}A$$

$$\dot{X}(t) = \left(e^{At}X_0\right)' = \left(e^{At}\right)'X_0 = Ae^{At}X_0 = AX$$

Nota: $e^{At}e^{As} = e^{A(t+s)}$

Fazendo $s=-t$ concluímos que $e^{At}e^{-At} = I$, ou seja, $(e^{At})^{-1} = e^{-At}$

$$\left(e^{At}e^{A(\tau-t)}\right)' = e^{At}Ae^{A(\tau-t)} + e^{At}(-A)e^{A(\tau-t)} = 0$$

$$e^{At}e^{A(\tau-t)} = cte = e^{A0}e^{A\tau} = Ie^{A\tau} = e^{A\tau}$$

Prova da unicidade de soluções. Seja X uma solução do PVI.

$$\left(e^{-At}X(t)\right)' \qquad e^{-At}(-A)X(t) + e^{-At}AX(t) = 0$$

Prova da unicidade de solução. Seja X uma solução do PVI.

$$\left(e^{-At}X(t)\right)' = e^{-At}(-A)X(t) + e^{-At}AX(t) = 0$$

$$e^{-At}X(t) = cte = e^{-A0}X(0) = IX_0 = X_0$$

$$\therefore \quad e^{-At}X(t) = X_0 \quad \Rightarrow \quad X(t) = e^{At}X_0$$

Aplicação ao exemplo acima:

$$X(t) = e^{At}X_0 = Se^{\Lambda t}S^{-1}X_0$$

$$A = \frac{1}{5}\begin{bmatrix} -9 & 2 \\ 2 & -6 \end{bmatrix}$$

$$Av = \lambda v \iff (A-\lambda I)v = 0$$

$$\det(A-\lambda I) = p(\lambda) = \lambda^2 - (\operatorname{tr} A)\lambda + \det A = (\lambda-\lambda_1)(\lambda-\lambda_2)$$

$$\begin{vmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{vmatrix} = \lambda^2 - (a_{11}+a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21})$$

$$p(\lambda) = \lambda^2 + 3\lambda + 2 = (\lambda+1)(\lambda+2)$$

$$\lambda_1 = -1 \qquad (A-\lambda I)v = 0 \iff \frac{1}{5}\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} v = 0$$

$$v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \lambda_2 = -2 \qquad v_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$S = [v_1 \ v_2] = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \qquad \Lambda = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$$

$$S = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow S^{-1} = \frac{1}{\det S}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$S^{-1} = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$$

$$X(t) = Se^{\Lambda t}S^{-1}X_0 = [v_1 \ v_2]\begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix}\frac{1}{5}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

$$= [v_1 \ v_2]\begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix}\frac{1}{5}\begin{bmatrix} x_0 + 2y_0 \\ -2x_0 + y_0 \end{bmatrix}$$

$$= \frac{1}{}[v_1 \ v_2]\begin{bmatrix} (x_0+2y_0)e^{-t} \\ \ \ 2t \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} (x_0+2y_0)e^{-t} \\ (-2x_0+y_0)e^{-2t} \end{bmatrix}$$

$$= \frac{1}{5}(x_0+2y_0)e^{-t}v_1 + \frac{1}{5}(-2x_0+y_0)e^{-2t}v_2$$

$$v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad v_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$= \alpha e^{\lambda_1 t}v_1 + \beta e^{\lambda_2 t}v_2$$

Seja v um vector próprio de A.
Procuremos soluções de $\dot{X} = AX$ da forma $X(t) = c(t)v$:

$$\dot{X} = \underline{\dot{c}v} = AX = Acv = cAv = \underline{c\lambda v}$$

$$\Rightarrow \quad \dot{c} = \lambda c \iff c = \alpha e^{\lambda t}$$

$$\Rightarrow X = \alpha e^{\lambda t}v$$

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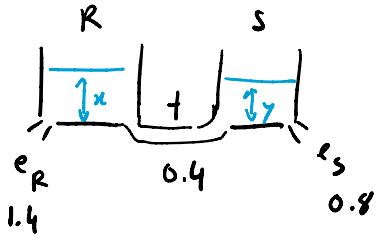
Wednesday, November 18, 2020 12:02 PM

$$\frac{1}{5}(x_0+2y_0)e^{-t}v_1 + \frac{1}{5}(-2x_0+y_0)e^{-2t}v_2 = X(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

$$v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad v_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$\begin{cases} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -9 & 2 \\ 2 & -6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \\ \begin{pmatrix} x \\ y \end{pmatrix}(0) = \begin{bmatrix} 10 \\ 10 \end{bmatrix} \end{cases}$$



$$\begin{cases} \dot{X} = AX \\ X(0) = X_0 \end{cases} \iff X(t) = e^{At}X_0$$

$$\begin{cases} \dot{X} = AX \\ X(0) = X_0 \end{cases} \iff X(t) = e^{At}X_0$$

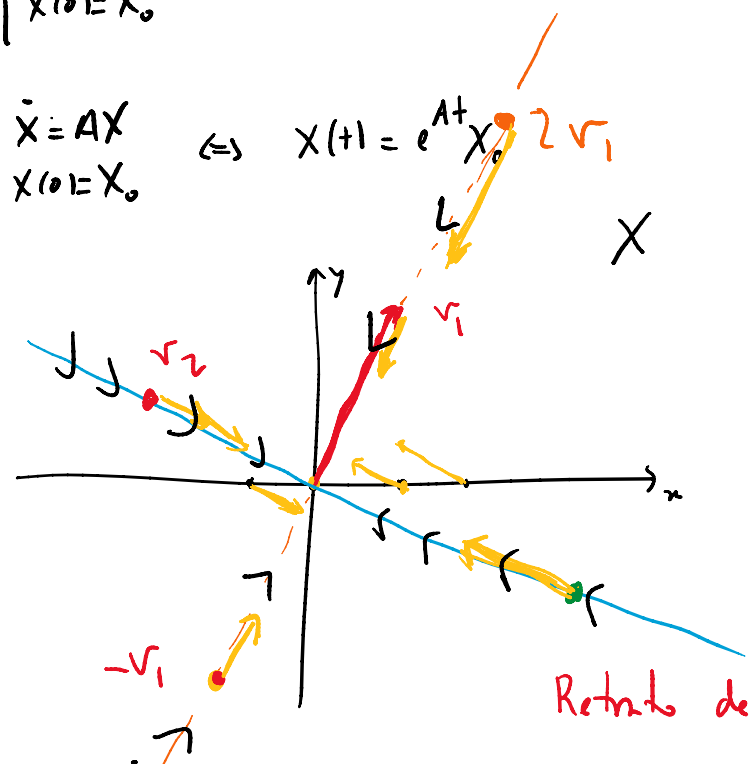
$$X(t) = S e^{\Lambda t} \underbrace{S^{-1}X_0}_{\begin{bmatrix} \alpha \\ \beta \end{bmatrix}}$$

$$= \alpha e^{\lambda_1 t} v_1 + \beta e^{\lambda_2 t} v_2$$

$$\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \alpha v_1 + \beta v_2$$

$$= (v_1 \; v_2)\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = S\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

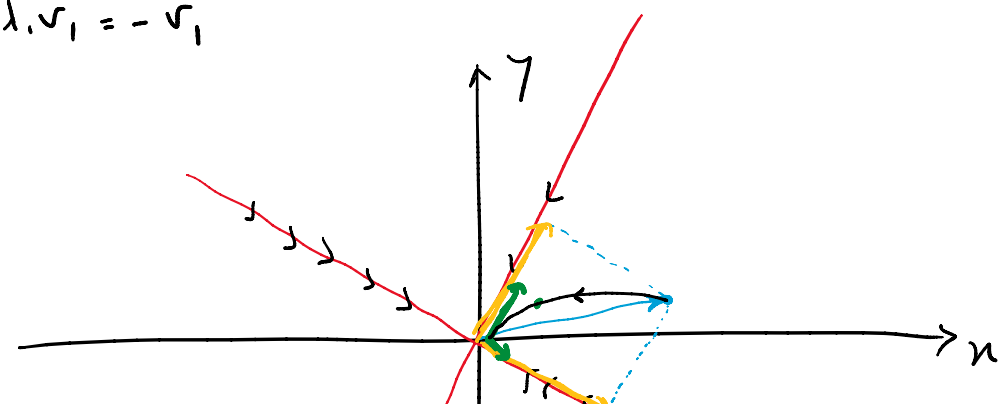
$$\Rightarrow \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = S^{-1}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

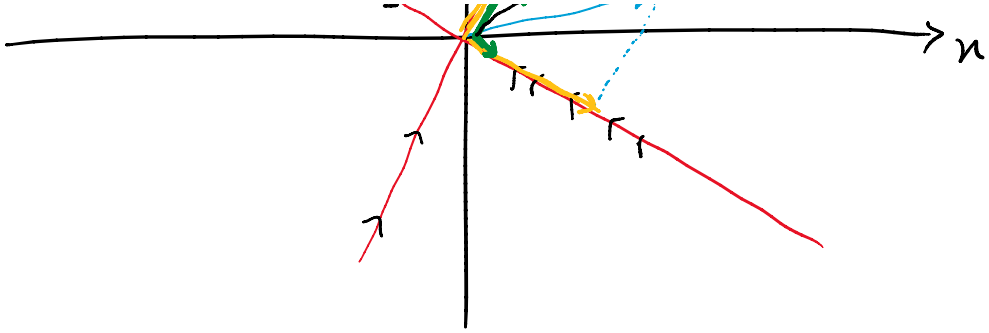


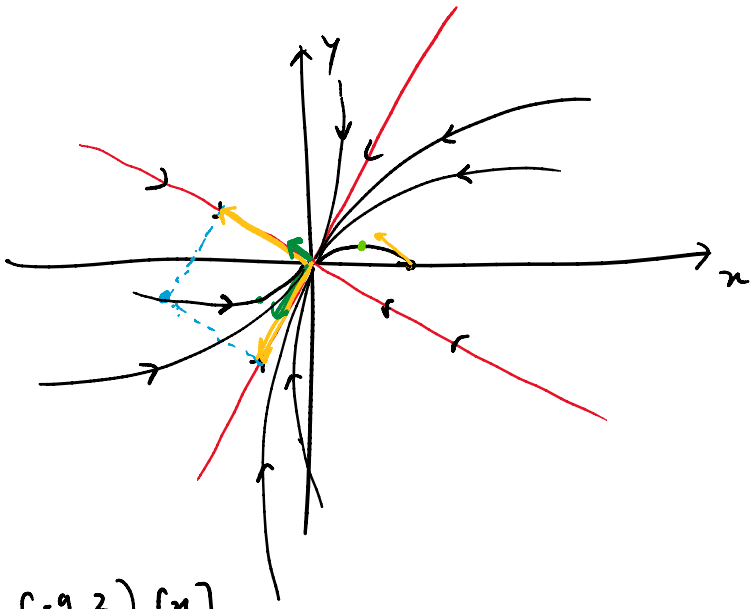
Retrato de fase

$$\dot{X} = AX = A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{5}\begin{bmatrix} -9 \\ 2 \end{bmatrix}$$

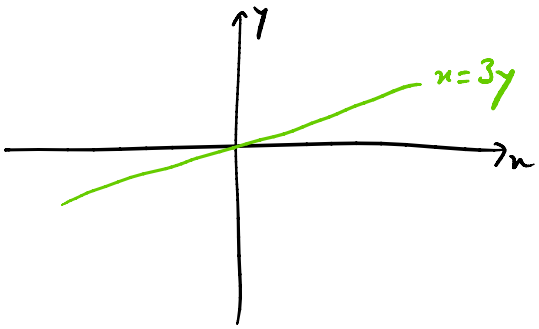
$$\dot{X} = Av_1 = \lambda_1 v_1 = -v_1$$







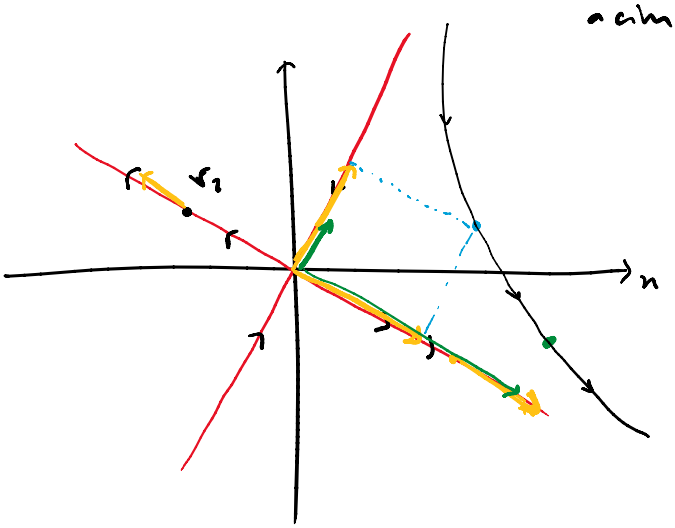
$$\dot{X} = \begin{bmatrix} * \\ 0 \end{bmatrix} = \frac{1}{5}\begin{pmatrix} -9 & 2 \\ 2 & -6 \end{pmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \iff 2x - 6y = 0 \iff x = 3y$$

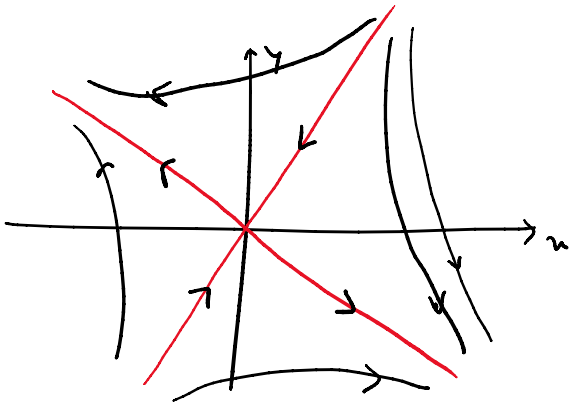


Outro exemplo

$$X(t) = \alpha e^{-t} v_1 + \beta e^{2t} v_2$$

v_1 e v_2 como acima



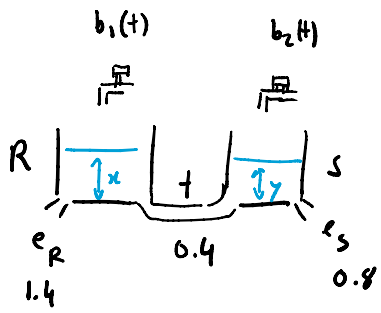

y
x

20 de Novembro

Friday, November 20, 2020 11:59 AM

$$B(t) = \begin{bmatrix} b_1(t) \\ b_2(t) \end{bmatrix}$$

$$\begin{cases} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -9 & 2 \\ 2 & -6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} b_1(t) \\ b_2(t) \end{bmatrix} \\ \begin{bmatrix} x \\ y \end{bmatrix}(0) = \begin{bmatrix} 10 \\ 10 \end{bmatrix} \end{cases}$$



(i) $\dot{X} - AX = B(t)$

(ii) $M(t) = e^{-At}$

(iii) $e^{-At}\dot{X} - e^{-At}AX = e^{-At}B(t)$

(iv) $\frac{d}{dt}\left(e^{-At}X\right) = e^{-At}B(t)$

(v) $\frac{d}{ds}\left(e^{-As}X(s)\right) = e^{-As}B(s)$

$$\int_{t_0}^{t} \frac{d}{ds} e^{-As}X(s)\,ds = \int_{t_0}^{t} e^{-As}B(s)\,ds$$

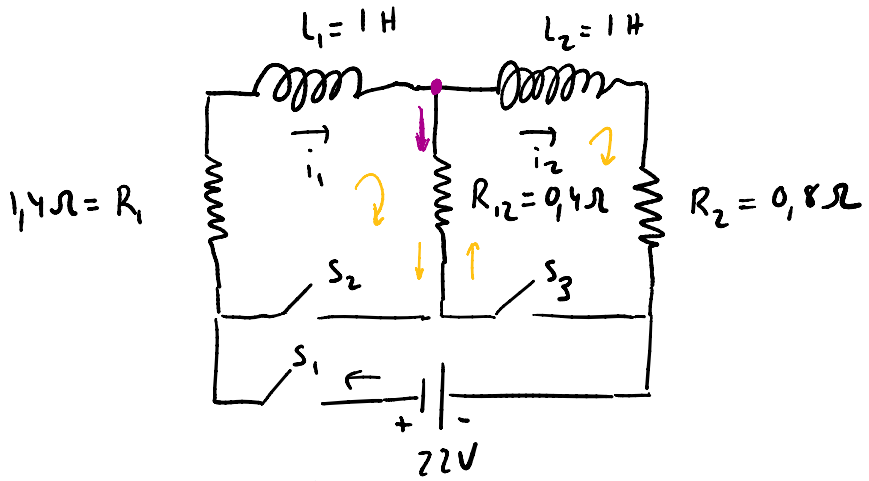
$$e^{-As}X(s)\Big|_{t_0}^{t} = \int_{t_0}^{t} e^{-As}B(s)\,ds$$

$$e^{-At}X(t) - e^{-At_0}X(t_0) = \int_{t_0}^{t} e^{-As}B(s)\,ds$$

$$X(t) = e^{At}e^{-At_0}X_0 + e^{At}\int_{t_0}^{t} e^{-As}B(s)\,ds$$

$$\boxed{X(t) = e^{A(t-t_0)}X_0 + \int_{t_0}^{t} e^{A(t-s)}B(s)\,ds}$$

Fórmula de variação das constantes



$t < 0$ S_1 fechado, S_2 e S_3 abertos 10 A

$t \geq 0$ S_1 aberto, S_2 e S_3 fechados

$$\begin{cases} L_1 \dfrac{di_1}{dt} + R_1 i_1 + R_{12}(i_1 - i_2) = 0 \\ L_2 \dfrac{di_2}{dt} + R_2 i_2 + R_{12}(i_2 - i_1) = 0 \end{cases}$$

$$\begin{cases} i_1' + 1.4 i_1 + 0.4(i_1 - i_2) = 0 \\ i_2' + 0.8 i_2 + 0.4(i_2 - i_1) = 0 \end{cases} \qquad \begin{cases} i_1' = -1.8 i_1 + 0.4 i_2 \\ i_2' = 0.4 i_1 - 1.2 i_2 \end{cases}$$

$$\begin{bmatrix} i_1 \\ i_2 \end{bmatrix}' = \frac{1}{5} \begin{bmatrix} -9 & 2 \\ 2 & -6 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$$

Eq. do Pêndulo Simples

$$\ddot{\theta} = -\frac{g}{l} \sin\theta$$

$$\omega^2 = \frac{g}{l}$$

$$\ddot{\theta} = -\omega^2 \sin\theta$$

$\sin\theta \approx \theta$ para θ pequeno

A esta aproximação chamamos pêndulo linear

$$\boxed{\ddot{\theta} = -\omega^2 \theta}$$

$x = \theta$
$y = \dot{\theta}$

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}' = \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} \dot{\theta} \\ -\omega^2 \theta \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$X = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$$

$$\dot{X} = AX \qquad A = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix} \qquad \ddot{\theta} + \omega^2 \theta = 0$$

$$p(\lambda) = \lambda^2 - \operatorname{tr} A \lambda + \det A = \lambda^2 + \omega^2 = 0$$

$$\ddot{\theta} = -b\theta - a\dot{\theta}$$

$$\ddot{\theta} + a\dot{\theta} + b\theta = 0$$

matriz companheira

$$\begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$$

$$p(\lambda) = \lambda^2 + a\lambda + b = 0$$

$$p(\lambda) = \lambda^2 + a\lambda + b = 0$$

$p(\lambda) = 0 \iff \lambda = \pm i\omega$

$\lambda_1 = i\omega$ $\qquad Av = \lambda v_1 \iff (A - \lambda I)v_1 = 0$

$$\iff \begin{pmatrix} -i\omega & 1 \\ -\omega^2 & -i\omega \end{pmatrix} v_1 = 0$$

$$\Leftarrow \quad v_1 = \begin{bmatrix} 1 \\ i\omega \end{bmatrix}$$

$-i\omega\alpha + \beta = 0$

$\beta = i\omega\alpha$

$v_1 = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{bmatrix} \alpha \\ i\omega\alpha \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ i\omega \end{bmatrix}$

$\lambda_2 = -i\omega$ $\qquad Av = \lambda v \xRightarrow{A \text{ real}} A\bar{v} = \bar{\lambda}\bar{v}$

$$v_2 = \begin{bmatrix} 1 \\ -i\omega \end{bmatrix}$$

$$S = \begin{bmatrix} 1 & 1 \\ i\omega & -i\omega \end{bmatrix} \qquad \Lambda = \begin{bmatrix} i\omega & 0 \\ 0 & -i\omega \end{bmatrix}$$

$$A = S\Lambda S^{-1}$$

$$\Downarrow$$

$$e^{At} = S e^{\Lambda t} S^{-1}$$

$$S^{-1} = \frac{1}{-2i\omega} \begin{bmatrix} -i\omega & -1 \\ -i\omega & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & \frac{-i}{\omega} \\ 1 & i/\omega \end{bmatrix}$$

$$X(t) = e^{At} X_0$$

$$X_{n+1} = AX_n \qquad X_{n+1} - X_n = (A - I)X_n$$

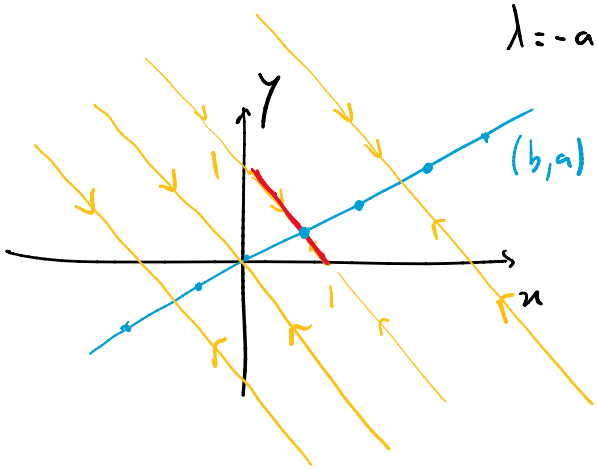
$$\frac{dX}{dt} = BX$$

$$A = \begin{bmatrix} 1-a & b \\ a & 1-b \end{bmatrix} \qquad A - I = B = \begin{bmatrix} -a & b \\ a & -b \end{bmatrix}$$

$$A = \begin{bmatrix} 1-a & b \\ a & 1-b \end{bmatrix} \qquad A - I = B = \begin{bmatrix} -a & b \\ a & -b \end{bmatrix}$$

$$\lambda = 0 \qquad v = \begin{pmatrix} b \\ a \end{pmatrix}$$

$$\lambda = -a-b \qquad v = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$



$x + y = 1$

23 de Novembro

Monday, November 23, 2020 11:56 AM

$$\boxed{\ddot{\theta} = -\omega^2 \theta}$$

$x = \theta$

$y = \dot{\theta}$

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}' = \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} \dot{\theta} \\ -\omega^2 \theta \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$X = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$$

$$\dot{X} = AX \qquad A = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix} \qquad \ddot{\theta} + \omega^2 \theta = 0$$

$$S = \begin{bmatrix} 1 & 1 \\ i\omega & -i\omega \end{bmatrix} \qquad \Lambda = \begin{bmatrix} i\omega & 0 \\ 0 & -i\omega \end{bmatrix}$$

$$A = S \Lambda S^{-1}$$

$$\Downarrow$$

$$e^{At} = S e^{\Lambda t} S^{-1}$$

$$S^{-1} = \frac{1}{-2i\omega} \begin{bmatrix} -i\omega & -1 \\ -i\omega & 1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1 & -\frac{i}{\omega} \\ 1 & i/\omega \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} 1 & 1 \\ i\omega & -i\omega \end{bmatrix} \begin{bmatrix} e^{i\omega t} & 0 \\ 0 & e^{-i\omega t} \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & -\frac{i}{\omega} \\ 1 & i/\omega \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1 & 1 \\ i\omega & -i\omega \end{bmatrix} \begin{bmatrix} e^{i\omega t} & -\frac{i}{\omega} e^{i\omega t} \\ e^{-i\omega t} & \frac{i}{\omega} e^{-i\omega t} \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} e^{i\omega t} + e^{-i\omega t} & -\frac{i}{\omega} e^{i\omega t} + \frac{i}{\omega} e^{-i\omega t} \\ i\omega e^{i\omega t} - i\omega e^{-i\omega t} & e^{i\omega t} + e^{-i\omega t} \end{bmatrix}$$

$$= \begin{pmatrix} \cos(\omega t) & \frac{1}{\omega} \sin(\omega t) \\ -\omega \sin(\omega t) & \cos(\omega t) \end{pmatrix}$$

$$\begin{bmatrix} -\omega\sin(\omega t) & \cos(\omega t) \end{bmatrix}$$

$$\begin{pmatrix} \theta \\ \dot\theta \end{pmatrix}(t) = \begin{bmatrix} \cos(\omega t) & \frac{1}{\omega}\sin(\omega t) \\ -\omega\sin(\omega t) & \cos(\omega t) \end{bmatrix} \begin{bmatrix} \theta_0 \\ \dot\theta_0 \end{bmatrix}$$

$$\theta(t) = \theta_0\cos(\omega t) + \frac{\dot\theta_0}{\omega}\sin(\omega t)$$

Notar: $\alpha\cos(\omega t) + \beta\sin(\omega t)$

$\alpha + i\beta = Re^{i\varphi}$

$e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$

$$= Re\{(\alpha - i\beta)(\cos(\omega t) + i\sin(\omega t))\}$$

$$= Re\{Re^{-i\varphi}e^{i\omega t}\} = Re\{Re^{i(\omega t-\varphi)}\}$$

$$= Re\{R\cos(\omega t-\varphi) + iR\sin(\omega t-\varphi)\}$$

$$= R\cos(\omega t-\varphi)$$

$$\theta(t) = R\cos(\omega t-\varphi)$$

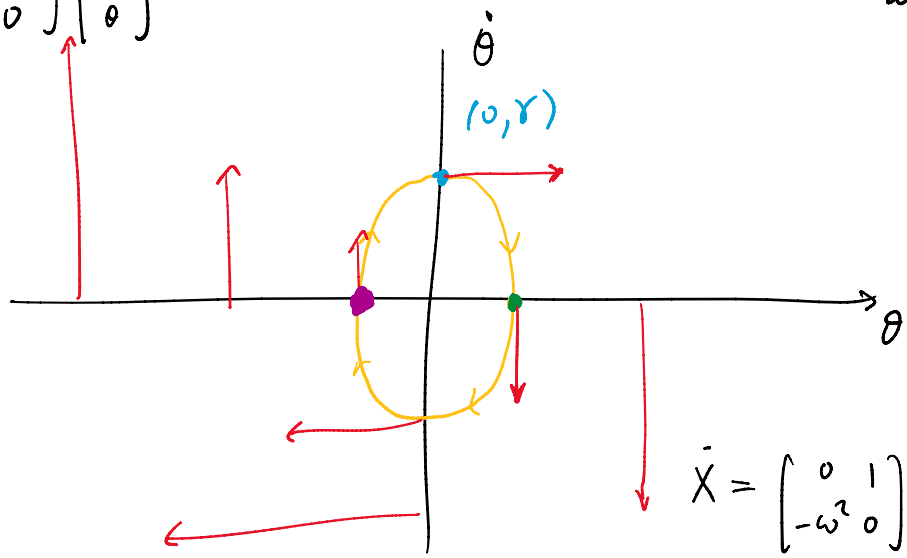
$$Re^{i\varphi} = \theta_0 + \frac{\dot\theta_0}{\omega}i = \alpha + i\beta$$

$$\dot\theta(t) = -R\omega\sin(\omega t-\varphi)$$

$$R = \sqrt{\theta_0^2 + \frac{\dot\theta_0^2}{\omega^2}}$$

$$\varphi = \arg\left(\theta_0 + i\frac{\dot\theta_0}{\omega}\right)$$

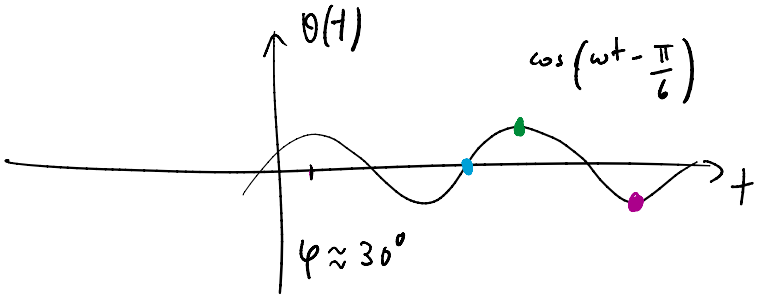
$$\begin{pmatrix} \theta \\ \dot\theta \end{pmatrix}' = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix}\begin{pmatrix} \theta \\ \dot\theta \end{pmatrix}$$

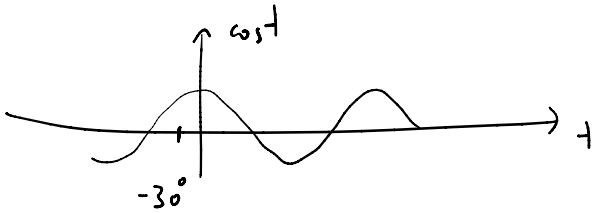


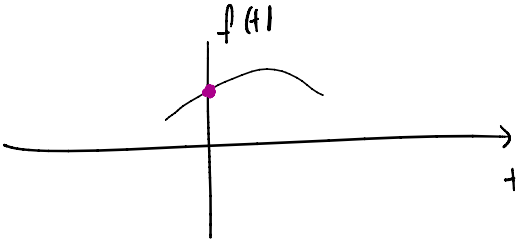
$$\dot X = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix}\begin{pmatrix} \alpha \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -\alpha\omega^2 \end{pmatrix}$$

$$X(t) = \begin{bmatrix} R\cos(\omega t-\varphi) \\ -R\omega\sin(\omega t-\varphi) \end{bmatrix}$$

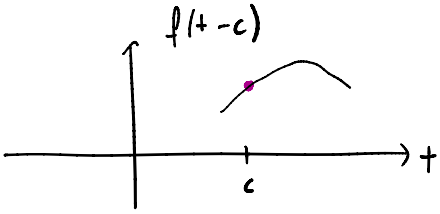
$$\dot X = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix}\begin{pmatrix} 0 \\ \gamma \end{pmatrix} = \begin{pmatrix} \gamma \\ 0 \end{pmatrix}$$







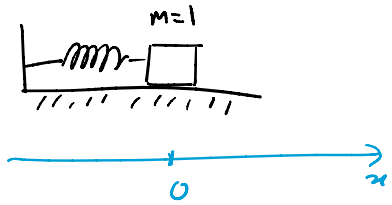
$c > 0$



24 de Novembro

Tuesday, November 24, 2020 9:58 AM

Mola criticamente amortecida



$$\ddot{x} = -a^2 x - 2a\dot{x} \qquad a>0$$

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a^2 & -2a \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$$

$$p(\lambda) = \lambda^2 - \operatorname{tr} A\,\lambda + \det A = \lambda^2 + 2a\lambda + a^2 = (\lambda+a)^2$$

$\lambda = -a$ $\qquad A = S \begin{bmatrix} -a & 0 \\ 0 & -a \end{bmatrix} S^{-1} = S(-a)I S^{-1} = -aI$

$$Av = -av \iff (A+aI)v = 0 \iff \begin{bmatrix} a & 1 \\ -a^2 & -a \end{bmatrix} v = 0 \Leftarrow v = \begin{bmatrix} 1 \\ -a \end{bmatrix}$$

$$A = SJS^{-1} \qquad J = \begin{bmatrix} -a & 1 \\ 0 & -a \end{bmatrix}$$

$$\Updownarrow$$

$$AS = SJ \iff A[v_1 \ v_2] = [v_1 \ v_2] \begin{bmatrix} -a & 1 \\ 0 & -a \end{bmatrix}$$

$$\iff (Av_1 \ \ Av_2) = [-av_1 \ \ v_1 - av_2]$$

$$\iff \begin{cases} Av_1 = -av_1 \\ Av_2 = v_1 - av_2 \end{cases} \iff \begin{cases} (A+aI)v_1 = 0 \\ (A+aI)v_2 = v_1 \end{cases}$$

$$\begin{cases} (A-\lambda I)v_1 = 0 \\ (A-\lambda I)v_2 = v_1 \end{cases}$$

$$\begin{aligned} (A-\lambda I)^2 v_2 &= (A-\lambda I)(A-\lambda I)v_2 \\ &= (A-\lambda I)v_1 \\ &= 0 \end{aligned}$$

Nota: $A_{3\times 3}$ $\quad \lambda_0$ é valor próprio com multiplicidade algébrica 3 e multiplicidade geométrica 1. A forma canónica de Jordan de A é

$$J = \begin{bmatrix} \lambda_0 & 1 & 0 \\ 0 & \lambda_0 & 1 \\ 0 & 0 & \lambda_0 \end{bmatrix}$$

λ mul. algébrica 3
" geométrica 2

$$J = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$(A+aI)v_2 = v_1 \iff \begin{bmatrix} a & 1 \\ -a^2 & -a \end{bmatrix} v_2 = \begin{bmatrix} 1 \\ -a \end{bmatrix} \Leftarrow v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$S = [v_1\ v_2] = \begin{bmatrix} 1 & 0 \\ -a & 1 \end{bmatrix} \qquad J = \begin{bmatrix} -a & 1 \\ 0 & -a \end{bmatrix} \qquad S^{-1} = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}$$

$$A = SJS^{-1} \Rightarrow e^{At} = Se^{Jt}S^{-1}$$

$$e^{Jt} = ? \qquad J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \qquad J = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$$

$$J = \lambda I + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \lambda I + N \qquad N = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$N^2 = NN = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$N^3 = N^2N = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0$$

N diz-se nilpotente

$$e^{Jt} = I + Jt + \frac{J^2t^2}{2!} + \frac{J^3t^3}{3!} + \dots + \frac{J^nt^n}{n!} + \dots$$

$$= I + (\lambda I + N)t + (\lambda^2 I + 2\lambda IN + N^2)\frac{t^2}{2!}$$

$$+ (\lambda^3 I + 3\lambda^2 N + 3\lambda N^2)\frac{t^3}{3!} +$$

$$\dots + \left(\lambda^n I + n\lambda^{n-1}N + \frac{n(n-1)}{2}\lambda^{n-2}N^2\right)\frac{t^n}{n!} + \dots$$

$$= e^{\lambda t}I + Nt\left(1 + \frac{2t}{2!}\lambda + \frac{3\lambda^2t^2}{3!} + \dots + \lambda^{n-1}\frac{n\,t^{n-1}}{n!} + \dots\right)$$

$$+ \frac{N^2t^2}{2}\left(1 + \frac{3\cdot 2t}{3!}\lambda + \dots + \frac{n(n-1)\lambda^{n-2}t^{n-2}}{n!} + \dots\right)$$

$$= e^{\lambda t}I + Nte^{\lambda t} + \frac{N^2t^2}{2}e^{\lambda t}$$

$$= e^{\lambda t}\left(I + Nt + \frac{N^2t^2}{2}\right)$$

Outra forma de chegar ao mesmo resultado

$$e^{Jt} = e^{(\lambda I + N)t} = e^{\lambda It + Nt} = e^{\lambda It}e^{Nt}$$

$$= e^{\lambda t}I\,e^{Nt} = e^{\lambda t}e^{Nt}$$

$$= e^{\lambda t}\left(I + Nt + \frac{N^2 t^2}{2}\right)$$

$$= e^{\lambda t}\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & t & 0 \\ 0 & 0 & t \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & t^2/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right)$$

$$= e^{\lambda t}\begin{bmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$$

$$e^{\lambda I t} = \begin{bmatrix} e^{\lambda t} & 0 & 0 \\ 0 & e^{\lambda t} & 0 \\ 0 & 0 & e^{\lambda t} \end{bmatrix} = e^{\lambda t} I$$