On the mean Euler characteristic of Gorenstein toric contact manifolds

(joint with Leonardo Macarini, UFRJ, Brazil)

I) "Convex Hamiltonian energy surfaces and their periodic trajectories", Ekeland - Hofer, 1987

- $\mathbb{R}^{2(n+1)}$, $(x_0, \dots, x_n, y_0, \dots, y_n) \equiv (x, y)$

$$H : \mathbb{R}^{2(n+1)} \longrightarrow \mathbb{R} \quad \rightsquigarrow \quad \begin{cases} \dot{x} = \partial H / \partial y \\ \dot{y} = -\partial H / \partial x \end{cases}$$

<u>Problem</u>: existence and multiplicity of periodic trajectories in a given fixed energy smooth hypersurface $M^{2n+1} := H^{-1}(c)$.

<u>Note</u>: $X_H = \left(\frac{\partial H}{\partial y}, -\frac{\partial H}{\partial x}\right) \in \operatorname{Ker}\left(dx \wedge dy|_M\right) \equiv$ characteristic foliation

i.e.

simple periodic trajectories of X_H on M $\xleftrightarrow{1:1}$ simple closed characteristics of M

- Non-degenerate periodic orbit γ $\xrightarrow[\text{along } \gamma]{\text{lin. flow}}$ index $\mu(\gamma) \in \mathbb{Z}$ (Conley - Zehnder)

$\longrightarrow$ mean index

$$\boxed{\Delta(\gamma) := \lim_{N \to \infty} \frac{\mu(\gamma^N)}{N} \in \mathbb{R}}$$

<u>Note</u>: $\Delta(\gamma) > 0$ on convex $M^{2n+1} \subset \mathbb{R}^{2(n+1)}$.

- **Theorem** (Ekeland-Hofer, 1987)

  If $M^{2n+1} \subset \mathbb{R}^{2(n+1)}$ is convex and has finitely many simple closed characteristics $\gamma_1, \dots, \gamma_m$, then

  $$\boxed{\sum_{\ell=1}^{m} \frac{\sigma(\gamma_\ell)}{\Delta(\gamma_\ell)} = \frac{1}{2}} \quad \text{(resonance relation)}$$

  where $\sigma(\gamma_\ell) = \pm 1$ or $\pm 1/2$ according to the parity of $\mu(\gamma_\ell)$ $\begin{pmatrix} \text{even} \leftrightarrow + \\ \text{odd} \leftrightarrow - \end{pmatrix}$ and $\mu(\gamma_\ell^2) - \mu(\gamma_\ell)$ $\begin{pmatrix} \text{even} \leftrightarrow 1 \\ \text{odd} \leftrightarrow 1/2 \end{pmatrix}$.

"Cor.": A generic convex $M^{2n+1} \subset \mathbb{R}^{2(n+1)}$ has ∞-many simple closed characteristics.

II) Contact manifolds and Reeb flows

- $\boxed{(M^{2n+1}, \xi)}$, $\xi = \ker(\alpha)$ with $\alpha \in \Omega^1(M)$ s.t. $\alpha \wedge (d\alpha)^n =$ volume form
- Reeb vector field $R_\alpha \in \mathfrak{X}(M, \xi)$: $R_\alpha \in \ker(d\alpha)$ and $\alpha(R_\alpha) = 1$.
  $\longrightarrow$ $\boxed{\text{Reeb flow}}$

Note: any $f : M^{2n+1} \longrightarrow \mathbb{R}^+$ gives rise to another Reeb vector field $R_{f\alpha} \in \mathfrak{X}(M, \xi)$ and Reeb flow.

Examples:

- Star-shaped Hamiltonian flows in $\mathbb{R}^{2(n+1)}$ are Reeb flows on (S^{2n+1}, ξ_{std}).
- geodesic flows are Reeb flows on (ST^*N, ξ_{std}).

- Topological condition $\boxed{c_1(M,\xi)=0}$ $\Rightarrow$ can define index $\mu(\gamma)$ and $\boxed{\text{mean index } \Delta(\gamma)}$ of any closed Reeb orbit.

- <u>Theorem</u> (Ginzburg – Kerman, 2010)

  If the Reeb flow of R_α is non-degenerate with finitely many simple closed orbits $\gamma_1, \dots, \gamma_m$, with positive mean index, then

$$\boxed{\sum_{\ell=1}^{m} \frac{\sigma(\gamma_\ell)}{\Delta(\gamma_\ell)} = \chi(M,\xi)} \quad \text{(resonance relation)}$$

  where $\chi(M,\xi)$ is the mean Euler characteristic, an invariant of the contact manifold (M,ξ).

<u>Notes:</u>

- $\chi(M,\xi)$ is the mean Euler characteristic of cylindrical contact homology $\equiv$ Morse homology for the action functional $\mathcal{A}_\alpha(\gamma) = \int_\gamma \alpha$ on the loop space of M.
- Resonance relation of Ekeland-Hofer $\Leftrightarrow \chi(S^{2n+1}, \xi_{std}) = 1/2$.
- Kwon and van Koert, 2016: given any $x \in \mathbb{Q}$ there is a (Stein fillable) contact structure ξ_x on S^5 such that $\chi(S^5, \xi_x) = x$.

III) Gorenstein toric contact manifolds

- Good toric contact manifolds (Lerman, 2003)

$$\boxed{(M^{2n+1}, \xi) \circlearrowleft \mathbb{T}^{n+1}}$$

are contact reductions of

$$\mathbb{C}^d \setminus \{0\} \supset (S^{2d-1}, \xi_{std}) \circlearrowleft \mathbb{T}^d$$

with respect to a subtorus

$$K := \operatorname{Ker}(\underline{\beta} : \mathbb{T}^d \twoheadrightarrow \mathbb{T}^{n+1})$$

<u>Examples</u>: (anti-) canonical circle bundles over monotone toric symplectic manifolds (and orbifolds).

- Surjective homomorphism $\underline{\beta}$ is determined by d primitive vectors $\nu_1, \dots, \nu_d \in \mathbb{Z}^{n+1} \subset \mathbb{R}^{n+1} = \operatorname{Lie}(\mathbb{T}^{n+1})$
- Gorenstein toric contact manifold $\overset{\text{def}}{\Longleftrightarrow}$ $\boxed{c_1(M, \xi) = 0}$

$\Longleftrightarrow$ $\nu_1, \dots, \nu_d$ can be chosen of the form

$$\nu_j := (v_j, 1) \text{ with } v_j \in \mathbb{Z}^n, \; j = 1, \dots, d.$$

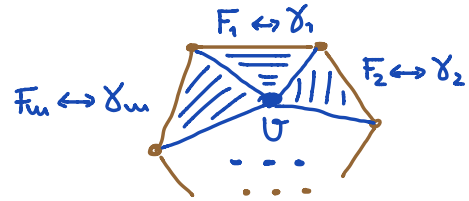
- $\boxed{D := \operatorname{conv}(v_1, \dots, v_d)} \subset \mathbb{R}^n$

$\equiv$ toric diagram of <u>the</u> Gorenstein toric contact manifold (M_D^{2n+1}, ξ_D)

- <u>Theorem</u> (A.- Macarini, 2016)

$$\boxed{\chi(M_D, \xi_D) = \frac{n! \, \operatorname{vol}(D)}{2}}$$

"Proof:" Any "irrational" interior point v of D determines a non-degenerate toric Reeb vector whose simple closed orbits are in 1-1 correspondence with the facets of D:



Moreover,

$$\boxed{\frac{1}{\Delta(\gamma_\ell)} = \frac{n!}{2}\,\mathrm{vol}\left(\mathrm{conv}(v, F_\ell)\right)}, \quad \ell = 1, \dots, m.$$

Result follows from Ginzburg-Kerman's resonance relation, since $\sigma(\gamma_\ell) = +1$, $\ell = 1, \dots, m$.

Q.E.D.

- Several interesting applications...

A particularly nice one, using a result of Batyrev-Dais from 1996, is the following:

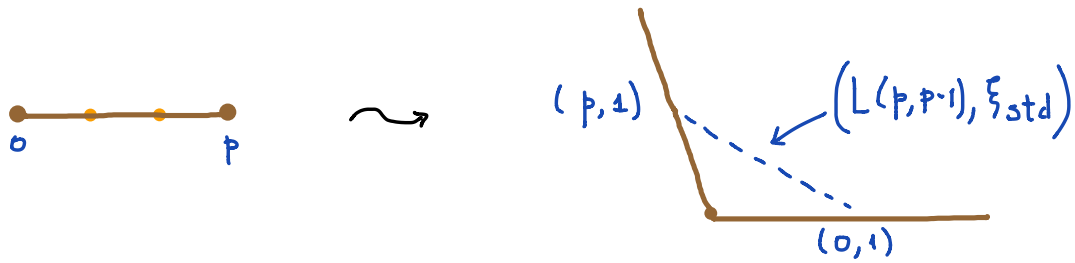
$$\boxed{2\chi(M_D, \xi_D) = \text{Euler characteristic of any nepant (i.e. } c_1 = 0\text{) toric symplectic filling.}}$$

In particular, the 1987 resonance relation of Ekeland-Hofer reflects that $\boxed{\chi(\text{ball}) = 1}$!

In fact, the underlying contact manifold of any convex hypersurface in $\mathbb{R}^{2(n+1)}$ is (S^{2n+1}, ξ_{std}), which admits the standard symplectic ball $B^{2(n+1)} \subset \mathbb{R}^{2(n+1)}$ as a nepant toric symplectic filling.

Examples

a)



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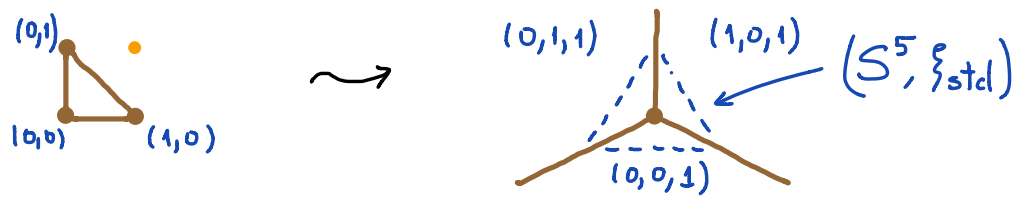
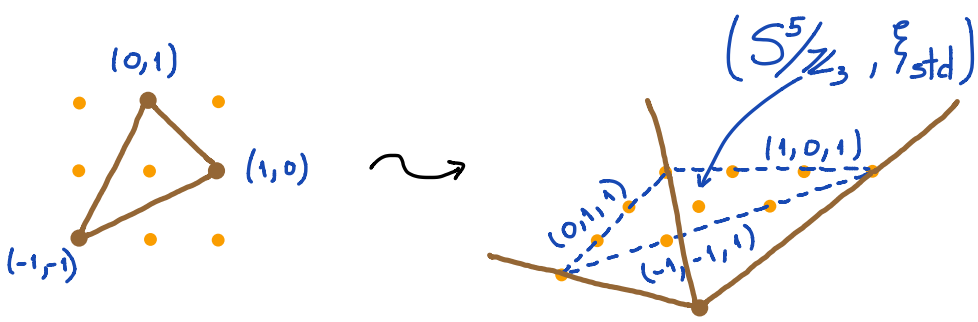


c)



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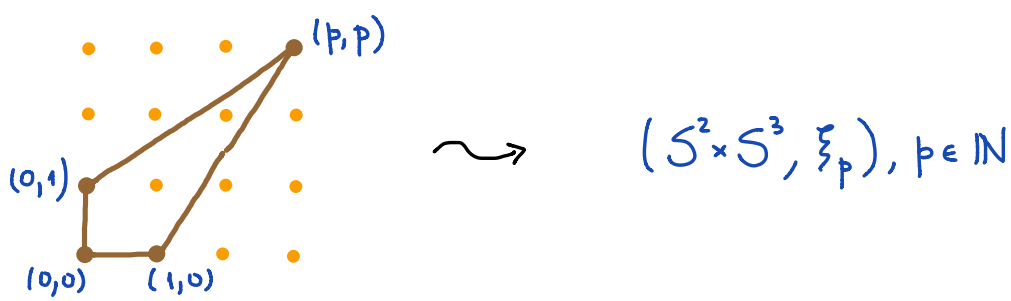
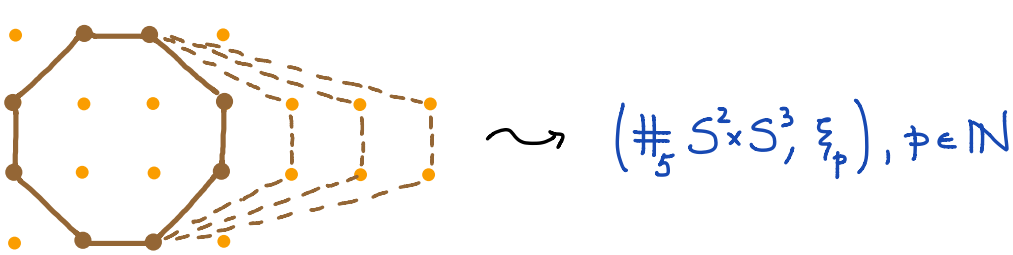
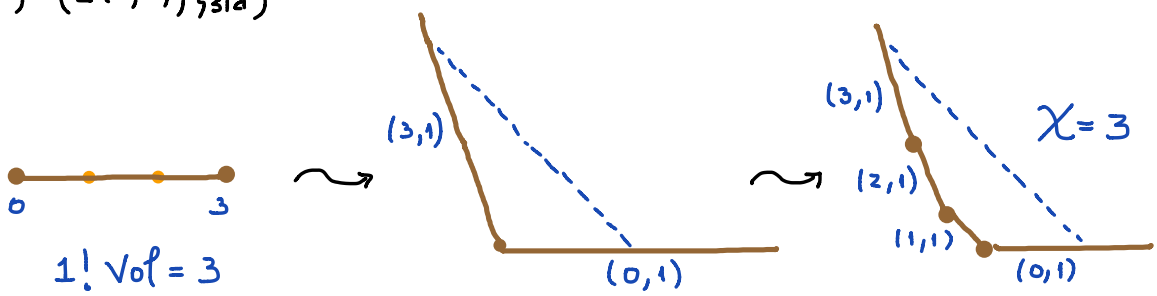


e)



Examples:

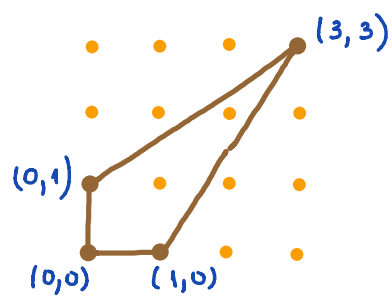
a) $(L(3,2), \xi_{std})$



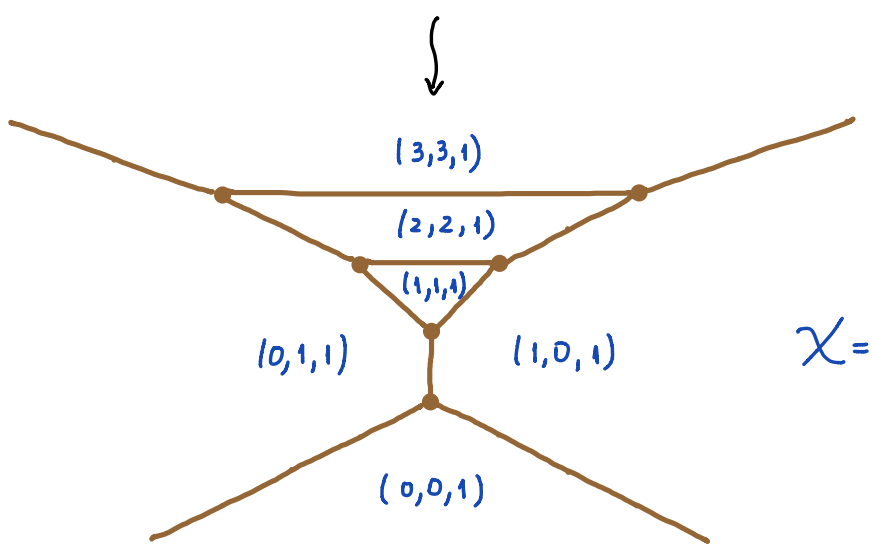
$1!\,\mathrm{Vol} = 3$

$\chi = 3$

b) $(S^2 \times S^3, \xi_3)$



$2!\,\mathrm{Vol} = 6$



$\chi = 6$