

Contact Invariants
of
Gorenstein Toric Contact Manifolds,
the Ehrhart Polynomial
and Chen-Ruan Cohomology
(joint with L. Macarini and M. Moreira)

① Gorenstein Toric Contact Manifolds

- $(M^{2n+1}, \xi) \hookrightarrow \mathbb{T}^{n+1}$, i.e. $S(M)^{z(n+1)} \hookrightarrow \mathbb{T}^{n+1}$
is a toric symplectic cone
- Constructed as contact reductions of
 $(S^{2d-1} \subset \mathbb{C}^d \setminus \{0\}, \xi_{\text{std}}) \hookrightarrow \mathbb{T}^{d-1}$ by
 $K := \ker(\beta: \mathbb{T}^{d-1} \longrightarrow \mathbb{T}^{n+1})$

- Determined by $v_j \in \mathbb{Z}^{n+1}$, $j=1, \dots, d$, which are also the defining normals of

moment cone $C \equiv$ image of moment map $\mu: S(M)^{\mathbb{Z}^{n+2}} \longrightarrow \prod_{i \in I} \mathbb{T}^{n+1} \cong \mathbb{R}^{n+1}$

- Gorenstein, i.e. $C_1(\xi) = 0$, implies w.l.o.g. that $v_j := (u_j, 1) \in \mathbb{Z}^{n+1}$ w/ $u_j \in \mathbb{Z}^n$.

- Smoothness requires that each facet of $D := \text{conv}(u_1, \dots, u_d) \subset \mathbb{R}^n$ is

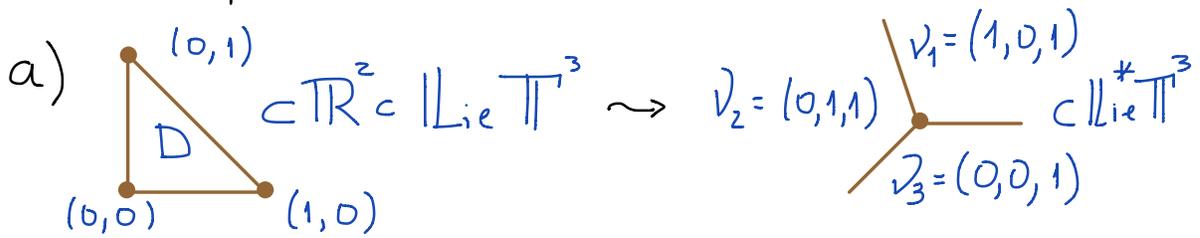
$\text{Aff}(n, \mathbb{Z})$ -equivalent to $\text{conv}(e_1, \dots, e_n)$

with $\{e_1, \dots, e_n\} =$ standard \mathbb{Z} -basis of \mathbb{Z}^n ,

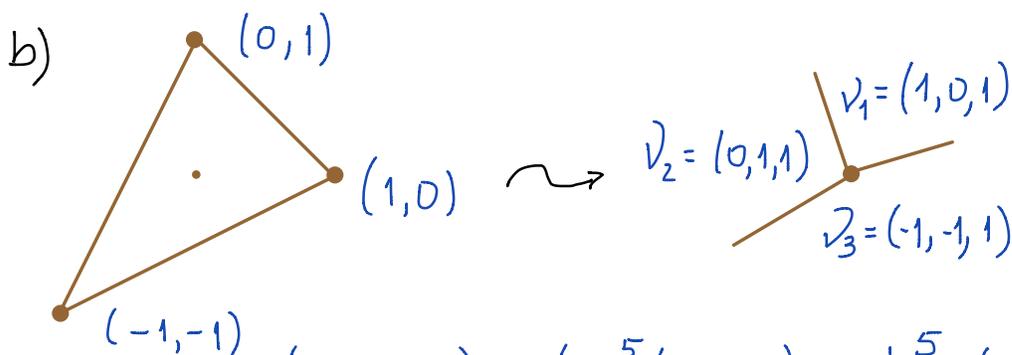
i.e. $D \subset \mathbb{R}^n$ is a **toric diagram**

- Notation: $D \rightsquigarrow (M_D, \xi_D)$

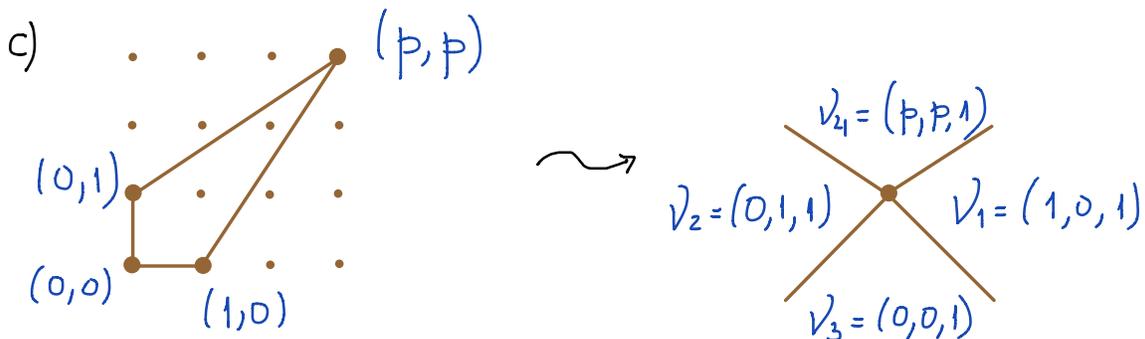
• Examples



$$(M_D, \xi_D) = (S^5, \xi_{\text{std}})$$



$$(M_D, \xi_D) = (S^5 / \mathbb{Z}_3, \xi_{\text{std}}) = L^5(1,1,1)$$



$$(M_D, \xi_D) = (S^2 \times S^3, \xi_p), \quad p \in \mathbb{N}$$

Note: $p=1$ gives unit co-sphere bundle of S^3

② HC_{*}(M_D, ξ_D)

- [Martelli - Sparks - Yau, 2006]

$\text{int}(D) \subset \mathbb{R}^n$ parametrizes (normalized)

toric Reeb vector fields: $U = (r_1, \dots, r_n) \in \text{int}(D)$

$$\rightsquigarrow V = (r_1, \dots, r_n, 1) \in \text{Lie}(\mathbb{T}^{n+1}) \rightsquigarrow R_V$$

- [A. - Macarini]

R_V non-deg. $\Leftrightarrow r_j$'s irrational & \mathbb{Q} -indep.

In that case:

(i) simple closed R_V -orbits $\xleftrightarrow{1:1}$ facets of D

$$\gamma_1, \dots, \gamma_m$$

(ii) $\deg(\gamma_j^l) := \mu_{\text{CZ}}(\gamma_j^l) + n - 2 \in \underline{2\mathbb{Z}_0^+}$

$$j = 1, \dots, m, \quad l \in \mathbb{N}$$

(iii) $cb_k(D, \nu) := \# \{ \text{closed } R_\nu\text{-orbits with degree} = k \}$

should be a contact invariant of (M_D, ξ_D)

(= rank $HC_k(M_D, \xi_D) \equiv$ contact Betti #)

[ESH $_+^*$, Kwon-vanKoert, McLean-Ritter]

OK at least when (M_D, ξ_D) has a

crepant ($C_1=0$) toric symplectic

filling, e.g. when $n=2$.

Notation: $cb_k(D) = \text{contact Betti #}$

III $cb_*(D)$ and the Ehrhart polynomial

• $L_D(t) := \#(D \cap \frac{1}{t} \mathbb{Z}^n)$, $t \in \mathbb{Z}^+$
 $= \sum_{k=0}^n c_k(D) t^k$, w/ $c_0(D)=1$ and $c_n(D)=\text{vol}(D)$

[Stanley] $= \sum_{k=0}^n \delta_k(D) \binom{t+n-k}{n}$, w/ $\delta_k(D) \in \mathbb{Z}_0^+$

- Thm. [A. - Macarini - Moreira]

$$cb_{2k}(D) - cb_{2(k-1)}(D) = \delta_{n-k}(D)$$

Cor.

(1) $cb_{2k}(D) = n! \text{vol}(D), \forall k \geq n$

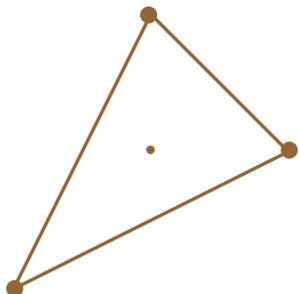
[because $c_n = \frac{1}{n!} \sum_{k=0}^n \delta_k$], in particular

mean Euler characteristic = $\frac{n!}{2} \text{vol}(D)$.

(2) $cb_{2(n-1)}(D) = n! \text{vol}(D) - 1$ [$\delta_0 = 1$]

(3) $cb_0(D) = \#(\text{int } D \cap \mathbb{Z}^n)$ [= δ_n]

- Example: Cor. $\Rightarrow cb_*(D)$ when $D \subset \mathbb{R}^2$.

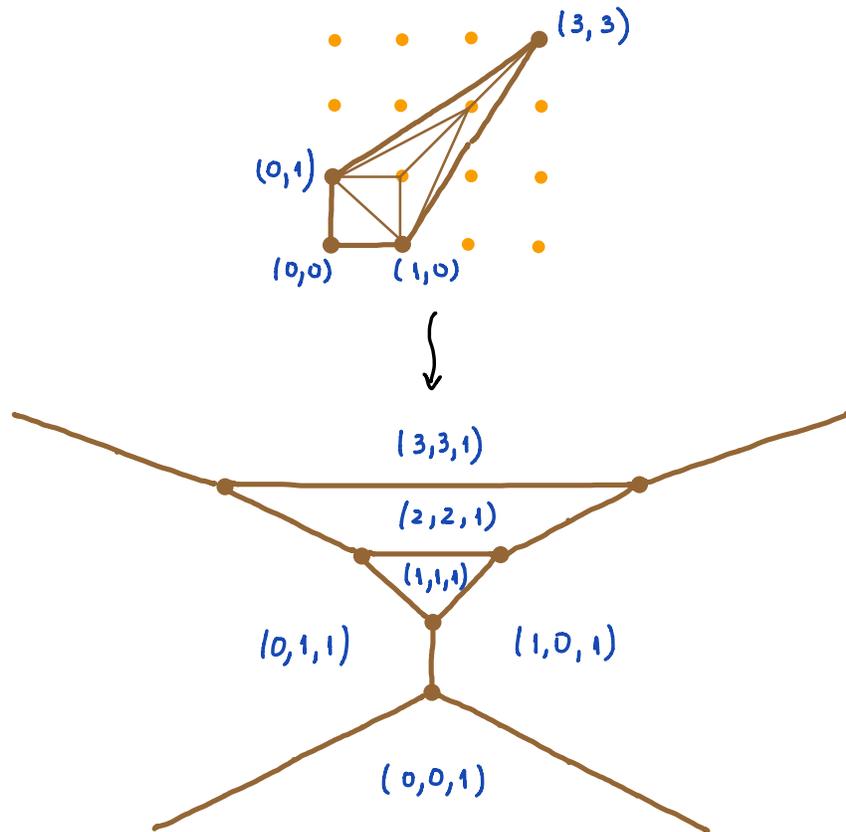


$$\Rightarrow cb_* \left(L_3^5(1,1,1) \right) = \begin{cases} 1, & * = 0 \\ 2, & * = 2 \\ 3, & * = 2k \geq 4 \\ 0, & \text{otherwise} \end{cases}$$

④ $cb_*(D)$ and orbifold resolutions/fillings

- Triangulation \mathcal{T} of $D \rightsquigarrow \text{fan } \Sigma$ over \mathcal{T}
 \rightsquigarrow (partial) resolution X_Σ of the toric isolated singularity at vertex of $S(M_D)$
- Prop. Σ admits a strictly convex support function Ψ , hence (X_Σ, ω_Ψ) is a toric symplectic orbifold with $c_1 = 0$.
In other words, every Gorenstein toric contact manifold admits a crepant toric orbifold symplectic filling.

- Smooth example



- Thm. [Batyrev - Dais 1996, Stapledon 2008]

$$\dim H_{orb}^*(X_{\Sigma}; \mathbb{Q}) = \begin{cases} \delta_j(D), & * = 2j, j \in \mathbb{N}_0 \\ 0, & \text{otherwise} \end{cases}$$

$* \in \mathbb{Q}$

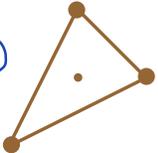
$$H_{orb}^* \equiv \text{Chen-Ruan cohomology}$$

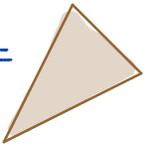
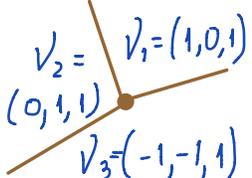
Cor.

$$HC_*(M_{D, \xi_D}) \cong \bigoplus_{k \geq 0} H_{orb}^{2n-*+2k}(X_{\Sigma}; \mathbb{Q})$$

- Remark: McLean-Ritter have similar result for isolated finite quotient singularities, which overlaps with this corollary when M_D is a Lens space, i.e. D is a simplex.

- Example

$$D \sim M_D = L_3^5(1,1,1)$$


$$a) \tau = \text{triangle} \Rightarrow X_\Sigma = \mathbb{C}^3 / \mathbb{Z}_3$$



$$\Rightarrow H_{\text{orb}}^*(\mathbb{C}^3 / \mathbb{Z}_3) = \bigoplus_{g \in \mathbb{Z}_3} H^{*-2 \text{Lg}}((\mathbb{C}^3)^g / \mathbb{Z}_3)$$

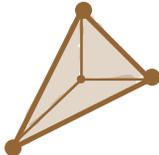
$$= \underbrace{H^*(\text{pt})}_{F_1} \oplus \underbrace{H^{*-2}(\text{pt})}_{F_{1/3}} \oplus \underbrace{H^{*-4}(\text{pt})}_{F_{2/3}}$$

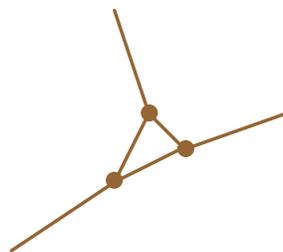
$$= \begin{cases} \mathbb{Q}, & * = 0, 2, 4 \\ 0, & \text{otherwise} \end{cases}$$

⇒

| | | | | | | | |
|-----------|---|---|---|---|---|----|-----|
| $*$ | 0 | 2 | 4 | 6 | 8 | 10 | ... |
| F_1 | 0 | 0 | 1 | 1 | 1 | 1 | ... |
| $F_{1/3}$ | 0 | 1 | 1 | 1 | 1 | 1 | ... |
| $F_{2/3}$ | 1 | 1 | 1 | 1 | 1 | 1 | ... |
| $cb_*(D)$ | 1 | 2 | 3 | 3 | 3 | 3 | ... |

b)

$\tau =$  $\Rightarrow X_\Sigma = (\mathcal{O}(\pm 3) \rightarrow \mathbb{C}P^2)$



$\Rightarrow H_{nb}^*(X_\Sigma) = H^*(\mathbb{C}P^2)$, i. e.

$H_{nb}^*(\mathbb{C}^3/\mathbb{Z}_3) \cong H^*(\text{Res}(\mathbb{C}^3/\mathbb{Z}_3))$

and one gets the same $cb_*(D)$,

as expected.

⑤ $cb_*(D)$ and prequantizations

- $(M_D, \xi_D, R_D) \longrightarrow M_D/S^1 =: B$

rational, gen. S^1 -action

B = monotone compact symplectic toric orbifold

$\xleftrightarrow{1:1}$ r -Gorenstein polytopes, i.e.

$P \subset \mathbb{R}^n$ s.t. rP is reflexive

Note: when B is smooth we have that $H_{\text{odd}}(B) = 0$, $c_1(B, \omega) = r[\omega]$ and the minimal Chern number of $B = r \cdot |\pi_1(M)|$.

- Thm. [Bourgeois 2002]

When B is smooth,

$$HC_*(M, \xi) = \bigoplus_{k \geq 0} H_{*-2rk-2(r-1)}(B; \mathbb{Q})$$

- Thm. [A. - Macarini - Moreira]

$$HC_*(M_D, \xi_D) = \bigoplus_{k \geq 0} \bigoplus_{0 < T \leq 1} F_T^{*-2rT + 2 - 2rk} (B)$$

where $H_{orb}^*(B; \mathbb{Q}) = \bigoplus_{0 < T \leq 1} F_T^*(B)$

with $F_T^*(B) = H^{*-2L_T}(B_T; \mathbb{Q})$,

$B_T := M^T/S^1 = \underline{\text{twisted sector}}$ of
 $T \in S^1 = \mathbb{R}/\mathbb{Z}$, $0 < T \leq 1$,

and $L_T \in \mathbb{Q}$ is its degree shifting number.

"Proof": Assume B_T connected and let

$$|B_T| := \mu_{RS}(M^T) - \frac{1}{2} \dim(B_T) + n - 2$$

where $\mu_{RS}(M^T)$ is the Robbin-Salamon index of any Reeb orbit in M^T .

Then

$$|B_T| = 2L_T + 2rT - 2 \quad (\in \mathbb{Z})$$

Q.E.D.

Cor. When $r=1$ one gets that

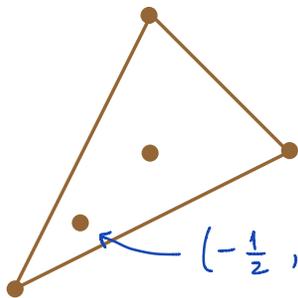
$$\delta_j(D) = \sum_{\substack{\uparrow \\ 2j-2 < q < 2j}} \dim H_{orb}^q(B; \mathbb{Q}) =: \dim H_{orb}^{2j}(B; \mathbb{Q})$$

[Stableton 2008]

and so $HC_*(M, \xi) = \bigoplus_{k \geq 0} H_{orb}^{2n-*+2k}(B; \mathbb{Q})$

• Example

$$L_3^5(1,1,1) = \left\{ \begin{array}{l} \text{prequant. of } (\mathbb{C}P^3, 3\omega_{FS}) \\ (r=1) \end{array} \right.$$



$$\left\{ \begin{array}{l} \text{prequant. of } \mathbb{C}P^2(4,1,1) \\ (r=2) \quad || \end{array} \right.$$

$$L_3^5(1,1,1) / \langle (-\frac{1}{2}, -\frac{1}{2}, 1) \rangle$$

$\mathbb{C}P^2(4,1,1)$

$$F_1^* = \begin{cases} \mathbb{Q}, & * = 0, 2, 4 \\ 0, & \text{otherwise} \end{cases}$$

$$F_{i/4}^* = \begin{cases} \mathbb{Q}, & * = i \\ 0, & \text{otherwise} \end{cases}$$

$i=1,2,3$

and

$$H_{orb}^* = \begin{cases} \mathbb{Q}, & * = 0, 1, 3, 4 \\ \mathbb{Q} \oplus \mathbb{Q}, & * = 2 \\ 0, & \text{otherwise} \end{cases}$$

$HC_*(L^5_3(1,1,1))$

| $* =$ | 0 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | ... |
|-----------|---|---|---|---|---|----|----|----|-----|
| F_1 | 0 | 1 | 1 | 2 | 1 | 2 | 1 | 2 | ... |
| $F_{1/4}$ | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | ... |
| $F_{2/4}$ | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | ... |
| $F_{3/4}$ | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | ... |
| cb_* | 1 | 2 | 3 | 3 | 3 | 3 | 3 | 3 | ... |

- Further work [Moreira, MS_c thesis, 2019]
- G finite, $(X=X/G, \omega)$ global quotient compact symplectic Calabi-Yau orbifold
- $H \in C_G(X)$, $J \in \mathcal{J}_G(X, \omega)$
- (zH, J) regular for $z > 0$ small enough
- $\Lambda =$ rational universal Novikov ring

$$\Rightarrow HF_*(X, zH, J; \Lambda) \cong H_{orb}^{n-*}(X; \Lambda)$$