

# ON THE INVARIANT SPECTRUM OF $S^1$ -INVARIANT METRICS ON $S^2$

MIGUEL ABREU AND PEDRO FREITAS

ABSTRACT. A theorem of J. Hersch (1970) states that for any smooth metric on  $S^2$ , with total area equal to  $4\pi$ , the first nonzero eigenvalue of the Laplace operator acting on functions is less than or equal to 2 (this being the value for the standard round metric). For metrics invariant under the standard  $S^1$ -action on  $S^2$ , one can restrict the Laplace operator to the subspace of  $S^1$ -invariant functions and consider its spectrum there. The corresponding eigenvalues will be called invariant eigenvalues, and the purpose of this paper is to analyze their possible values.

We first show that there is no general analogue of Hersch's theorem, by exhibiting explicit families of  $S^1$ -invariant metrics with total area  $4\pi$  where the first invariant eigenvalue ranges through any value between 0 and  $\infty$ . We then restrict ourselves to  $S^1$ -invariant metrics that can be embedded in  $\mathbf{R}^3$  as surfaces of revolution. For this subclass we are able to provide optimal upper bounds for all invariant eigenvalues. As a consequence, we obtain an analogue of Hersch's theorem with an optimal upper bound (greater than 2 and geometrically interesting). This subclass of metrics on  $S^2$  includes all  $S^1$ -invariant metrics with non-negative Gauss curvature.

One of the key ideas in the proofs of these results comes from symplectic geometry, and amounts to the use of the moment map of the  $S^1$ -action as a coordinate function on  $S^2$ .

## 1. INTRODUCTION

Let  $S^2$  be the 2-sphere and  $g$  any smooth Riemannian metric on it. Denote by  $\lambda(g)$  the first non-zero eigenvalue of the Laplace operator defined by  $g$ , acting on functions on  $S^2$ . In 1970, J.Hersch [H] showed that

$$\lambda(g) \leq \frac{8\pi}{\text{Area}_g(S^2)} .$$

If we scale the metric so that its total volume is the standard  $4\pi$ , then Hersch's theorem becomes

$$(1.1) \quad \lambda(g) \leq 2 .$$

This was generalized to surfaces of higher genus by P.Yang and S.T.Yau [YY] and to Kähler metrics on projective complex manifolds by J.-P.Bourguignon, P.Li and S.T.Yau [BLY]

The purpose of this paper is to analyze when and how results of this type (i.e. upper bounds for eigenvalues) can be obtained in an invariant setting. We consider

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only smooth metrics  $g$  on  $S^2$  with total area  $4\pi$  and invariant under the standard  $S^1$ -action (fixing the North and South poles). Denote by  $\lambda_j(g)$ , with

$$0 = \lambda_0(g) < \lambda_1(g) < \lambda_2(g) < \cdots ,$$

the invariant eigenvalues of the Laplace operator defined by  $g$  (i.e. with  $S^1$ -invariant eigenfunctions). What can be said about the possible values of  $\lambda_j(g)$ ?

Our first result says that there are no general restrictions on  $\lambda_1(g)$  (and so, no general analogue of Hersch's theorem).

**Theorem 1.** *Within the class of smooth  $S^1$ -invariant metrics  $g$  on  $S^2$  with total area  $4\pi$ , the first invariant eigenvalue  $\lambda_1(g)$  can be any number strictly between zero and  $\infty$ .*

*The same is true within the subclass of those metrics that have fixed Gauss curvature at the poles:  $K_g(N) = K_g(S) = 1$ .*

Our second result gives optimal upper bounds on the possible values of the invariant eigenvalues for the subclass of  $S^1$ -invariant metrics on  $S^2$  that are isometric to a surface of revolution in  $\mathbb{R}^3$ . In particular, it includes an analogue of Hersch's theorem within this geometrically interesting subclass of metrics.

**Theorem 2.** *Within the class of smooth  $S^1$ -invariant metrics  $g$  on  $S^2$  with total area  $4\pi$  and corresponding to a surface of revolution in  $\mathbb{R}^3$ , we have that*

$$\lambda_j(g) < \frac{\xi_j^2}{2}, \quad j = 1, \dots ,$$

where  $\xi_j$  is the  $((j+1)/2)^{\text{th}}$  positive zero of the Bessel function  $J_0$  if  $j$  is odd, and the  $(j/2)^{\text{th}}$  positive zero of  $J'_0$  if  $j$  is even. These bounds are optimal.

In particular,

$$\lambda_1(g) < \frac{\xi_1^2}{2} \approx 2.89.$$

As a consequence of this last theorem, we have the following

**Corollary 3.** *Let  $g$  be any  $S^1$ -invariant metric on  $S^2$  with total area  $4\pi$  and non-negative Gauss curvature. Then the eigenvalues  $\lambda_j$  satisfy the same upper bounds as in the previous theorem. These bounds are optimal.*

*Proof.* Either apply Theorem 2, using the fact that any  $S^1$ -invariant metric on  $S^2$  with non-negative scalar curvature is isometric to a compact surface of revolution in  $\mathbb{R}^3$  (see e.g. [KW] and the references therein), or see at the end of §4 why the proof of Theorem 2 also proves this corollary (without using the above fact).  $\square$

Since the problem we are dealing with is essentially a 1-dimensional eigenvalue problem, one might think that the size of the invariant eigenvalues would be related to the length of a geodesic meridian joining the North and South poles (which is the same as the *diameter* of  $S^2$  with the given  $S^1$ -invariant metric): *small diameter*  $\leftrightarrow$  *large eigenvalues* and *large diameter*  $\leftrightarrow$  *small eigenvalues*. However we will see in §5 that this is not the case. Although the problem is 1-dimensional, it has curvature coming from the underlying 2-sphere and that plays an important role.

The simplest family of  $S^1$ -invariant metrics for which  $\lambda_1(g) \rightarrow \infty$ , considered in the proof of Theorem 1, does have the property that the corresponding diameter  $D(g)$  tends to zero (while the total volume is always  $4\pi$ !). However, we will see that it is possible to change this particular family of metrics in two different ways:

- (i) in the first we keep the property that  $D(g) \rightarrow 0$  but change the first invariant eigenvalue so that now  $\lambda_1(g)$  also tends to zero;
- (ii) in the second we keep the property that  $\lambda_1(g) \rightarrow \infty$  but change the diameter so that now  $D(g)$  also tends to  $\infty$ .

We will also see in the paper (§4) why the subclass of  $S^1$ -invariant metrics corresponding to surfaces of revolution in  $\mathbb{R}^3$  is much more rigid than the full class of all  $S^1$ -invariant metrics on  $S^2$ . We will quantify that rigidity very precisely, and conclude that among surfaces of revolution in  $\mathbb{R}^3$  with total area  $4\pi$ , the one maximizing all invariant eigenvalues also minimizes the diameter and consists of the union of two flat discs of area  $2\pi$  each (a singular surface).

The above minimization property of the diameter of the double flat disc is just a generalization in this particular  $S^1$ -invariant setting of the following conjecture of Alexandrov: for every  $(S^2, g)$  with non-negative curvature the ratio  $\text{Area}_g(S^2)/D^2(g)$  is bounded from above by  $\pi/2$ , and this value is only attained in the singular case of the double flat disc. According to [B] this conjecture is still open. The  $S^1$ -invariant case suggests that another possible interesting statement for it would be obtained by replacing “with non-negative curvature” with “isometric to a closed surface in  $\mathbb{R}^3$ ” (recall that, by [N], any  $(S^2, g)$  with positive curvature is isometric to a closed surface in  $\mathbb{R}^3$ ).

It will also become clear in the paper (see in particular §4) that as one deforms the standard sphere towards the union of two flat discs, through a family of positive curvature ellipsoids of revolution intuitively obtained by “pressing” the North and South poles against each other, the first invariant eigenvalue increases from 2 to the limiting value  $\frac{\xi_1^2}{2} \approx 2.89$ . Due to Hersch’s Theorem and to the fact that the first noninvariant eigenvalue has multiplicity two, it follows that for small deformations of this type the first invariant eigenvalue (which is larger than two) is the third eigenvalue in the full spectrum. Since for the standard sphere the first eigenvalue is equal to two and has multiplicity three, we obtain that any of these slightly deformed metrics have their third eigenvalue (in the full spectrum) larger than the third eigenvalue of the standard sphere. In this way, we have obtained examples of metrics with positive curvature which provide a negative answer to the following question raised by S.T. Yau (Problem #71 in [Y]).

*Let  $0 = \mu_0(g) < \mu_1(g) \leq \dots \leq \mu_m(g) \leq \dots$  be the (full) spectrum of any  $(S^2, g)$ . Is  $\mu_m(g) \leq \mu_m(\text{standard})$ ,  $\forall m$ ?*

We note that another example providing a negative answer to this question had already been given by Engman in [E1], but in that case the metric had some negative curvature.

The proofs of Theorems 1 and 2 are an illustration of the usefulness of symplectic coordinates for some problems in Riemannian geometry. In dimension 2, any Riemannian metric determines a symplectic form (just the corresponding area form), and symplectic coordinates are any coordinates  $(x, y)$  on which this symplectic form is the standard  $dx \wedge dy$ . The existence of a circle action of isometries, determines a particularly nice choice of symplectic coordinates (i.e. action/angle coordinates) and these turn out to be very convenient for the problem at hand.

This remark describes the simplest particular case of a more general picture valid for any symplectic toric manifold (i.e. a symplectic manifold of dimension  $2n$

with an effective Hamiltonian action of the  $n$ -torus  $T^n$ ). The interested reader can look in [G], where this general picture was used for the first time (to describe some Kähler metrics on toric varieties), and [A].

In the  $S^2$ -setting, M. Engman has also used this type of coordinates to derive spectral properties of  $S^1$ -invariant metrics. In particular, he proves in [E1] that the first invariant eigenvalue can be arbitrarily large (using a trace formula, while we use a Hardy type inequality to prove Theorem 1), and in [E2] he obtains the value 3 as an upper bound for the first invariant eigenvalue of surfaces of revolution in  $\mathbb{R}^3$ . We thank Rafe Mazzeo for pointing out M. Engman's work to us.

The paper is organized as follows. In §2 we describe in detail the particular coordinates on  $S^2$  that will be used, and how all the relevant Riemannian quantities can be expressed in this way. Theorem 1 is proved in §3, while §4 is devoted to the proof of Theorem 2. Examples that show the general independence between the first invariant eigenvalue and the diameter are given in §5. In the Appendix we present the explicit solutions of a variational second order ODE that comes up naturally in §3.

## 2. PRELIMINARIES AND BACKGROUND

**2.1. Description of invariant metrics.** Let  $S^2 \subset \mathbb{R}^3$  be the standard sphere of radius 1, with an  $S^1$ -action given by rotation around the vertical axis. An equivariant version of the Uniformization Theorem says that there is only one  $S^1$ -invariant conformal structure on  $S^2$ . More explicitly, given any  $S^1$ -invariant metric  $g$  on  $S^2$  there is an  $S^1$ -equivariant diffeomorphism  $\varphi : S^2 \rightarrow S^2$  such that  $\varphi^*(g)$  is pointwise conformally equivalent to the standard metric  $g_0$  on  $S^2$ . This means that  $\varphi^*(g)$  is compatible with the standard ( $S^1$ -invariant) complex structure  $j_0$  on  $S^2$ . Since we are in real dimension 2, this is equivalent to saying that  $\varphi^*(g)$  is a Kähler metric on the complex manifold  $(S^2, j_0)$ .

We conclude that in order to study spectral properties of  $S^1$ -invariant metrics on  $S^2$ , it is enough to consider only  $S^1$ -invariant Kähler metrics on the complex manifold  $(S^2, j_0)$ . Moreover, by multiplying a given metric by an appropriate constant, we may consider only metrics with a fixed total volume (say  $4\pi$ ). We will now give an explicit description of this class of metrics.

Let  $g$  be an  $S^1$ -invariant Kähler metric on  $(S^2, j_0)$ , with associated Kähler (or area) form denoted by  $\omega$ , and such that  $\text{Vol}_g(S^2) = \int_{S^2} \omega = 4\pi$ . Because  $S^2$  is simply connected, the  $S^1$ -action is Hamiltonian with respect to  $\omega$  and we denote a corresponding Hamiltonian function (or moment map) by  $H : S^2 \rightarrow \mathbb{R}$ . A consequence of the Duistermaat-Heckman theorem in this very simple setting is that the push-forward by  $H$  of the measure determined on  $S^2$  by  $\omega$  is the measure  $\mu$  on  $\mathbb{R}$  given by

$$\mu(A) = 2\pi \cdot m(A \cap H(S^2)), \quad A \subset \mathbb{R},$$

where  $m$  is standard Lebesgue measure. Hence we have that the image interval  $H(S^2)$  has length 2. Since  $H$  is only determined up to a constant, we will normalize it so that  $H(S^2) = [-1, 1] \subset \mathbb{R}$  (this is equivalent to requiring that  $\int_{S^2} H \cdot \omega = 0$ ). This moment polytope  $P = [-1, 1]$  is determined by the affine functions

$$l_1(x) = 1 + x \quad \text{and} \quad l_2(x) = 1 - x$$

in the sense that  $x \in P$  if and only if  $l_i(x) \geq 0$ ,  $i = 1, 2$ , and  $x \in P^\circ \equiv$  interior of  $P = (-1, 1)$  if and only if  $l_i(x) > 0$ ,  $i = 1, 2$ . These two affine functions will be relevant below.

The inverse image  $H^{-1}(P^\circ)$  (i.e. the sphere minus the two poles fixed by the  $S^1$ -action) is the complex torus  $M = \mathbb{C}/2\pi\sqrt{-1}\mathbb{Z}$ , and  $S^1 = \mathbb{R}/2\pi\mathbb{Z}$  acts on  $M$  by the action:

$$S^1 \times M \rightarrow M, (t, z) \mapsto z + \sqrt{-1}t$$

where  $z = u + \sqrt{-1}v \in M$  and  $t \in S^1$ . If  $\omega$  is an  $S^1$ -invariant Kähler form on  $M$ , for which the  $S^1$  action is Hamiltonian, then there exists a function  $F = F(u)$ ,  $u = \text{Re } z$ , such that  $\omega = 2\sqrt{-1}\partial\bar{\partial}F$  and the moment map  $H : M \rightarrow \mathbb{R}$  is given by  $H(z) = dF/du$ .

It follows that  $\omega$  can be written in the form

$$(2.1) \quad \frac{\sqrt{-1}}{2} \frac{d^2 F}{du^2} dz \wedge d\bar{z}$$

and the restriction to  $\mathbb{R}(= \text{Re } \mathbb{C})$  of the Kähler metric is the Riemannian metric

$$(2.2) \quad \frac{d^2 F}{du^2} du^2 .$$

Under the Legendre transform given by the moment map

$$(2.3) \quad x = \frac{dF}{du} = H ,$$

this is the pull-back of the metric

$$\frac{d^2 G}{dx^2} dx^2$$

on  $P^\circ$ , where the potential  $G$  is the Legendre function dual to  $F$  (up to a linear factor in  $x$ ). More explicitly, the inverse of the Legendre transform (2.3) is

$$(2.4) \quad u = \frac{dG}{dx} + a ,$$

with  $a \in \mathbb{R}$  a constant. It follows from (2.3) and (2.4) that

$$(2.5) \quad \frac{d^2 G}{dx^2} = \left( \frac{d^2 F}{du^2} \right)^{-1} \text{ at } x = \frac{dF}{du} .$$

For the standard metric  $g_0$  on  $S^2$ , the function  $G_0$  obtained in this way is (see [G])

$$G_0 = \frac{1}{2} \sum_{k=1}^2 l_k(x) \log l_k(x) = \frac{1}{2} [(1+x) \log(1+x) + (1-x) \log(1-x)]$$

and the metric induced on  $P^\circ$  is

$$g_0(x) dx^2 \quad \text{with} \quad g_0(x) = \frac{d^2 G_0}{dx^2} = \frac{1}{1-x^2} .$$

For any other  $S^1$ -invariant Kähler metric  $g$  on  $S^2$ , with total volume  $4\pi$ , we have that the corresponding volume form  $\omega$  satisfies

$$\omega - \omega_0 = \frac{\sqrt{-1}}{2} \partial\bar{\partial}\phi ,$$

where  $\omega_0$  denotes the volume form of the standard metric  $g_0$  and  $\phi$  is a smooth real-valued  $S^1$ -invariant function on  $S^2$ . It follows from the above construction that the associated potential  $G$  is given by

$$G = G_0 + \Phi ,$$

with  $\Phi \in C^\infty(P)$ , and hence the metric induced by  $g$  on  $P^\circ$  is of the form

$$g(x) dx^2 \quad \text{with} \quad g(x) = \frac{d^2 G}{dx^2} = \frac{1}{1-x^2} + h(x) ,$$

where  $h \in C^\infty(P)$  is such that  $g(x) > 0$  for  $x \in P^\circ$ .

We conclude from this construction that the space of all  $S^1$ -invariant Kähler metrics  $g$  on  $S^2$  with total volume  $4\pi$  may be identified with the space of all functions

$$(2.6) \quad h \in C^\infty(P) \text{ such that } h(x) > -\frac{1}{1-x^2}, \quad x \in P^\circ .$$

**2.2. The Laplacian.** If  $f = f(u)$  is an  $S^1$ -invariant function on  $M = H^{-1}(P^\circ)$ , its Laplacian with respect to the Kähler metric defined by (2.1) is given by

$$(2.7) \quad \Delta f = - \left( \frac{d^2 F}{du^2} \right)^{-1} \frac{d^2 f}{du^2}$$

(note that this is not the same as the Laplacian of the restriction of  $f$  to  $\mathbb{R}(= \text{Re } \mathbb{C})$  with respect to the Riemannian metric defined by (2.2)). To write this operator in terms of the moment map coordinate  $x$ , we note that it follows from (2.3) and (2.5) that

$$(2.8) \quad \frac{d}{du} = \frac{dx}{du} \frac{d}{dx} = \frac{d^2 F}{du^2} \frac{d}{dx} = \left( \frac{d^2 G}{dx^2} \right)^{-1} \frac{d}{dx} = \frac{1}{g(x)} \frac{d}{dx}$$

at  $x = \frac{dF}{du}$ . Hence the Laplacian is given in the  $x$  coordinate by

$$(2.9) \quad \Delta f = - \frac{d}{dx} \left( \frac{1}{g} \frac{df}{dx} \right) = - \left( \frac{1}{g} f' \right)' .$$

**2.3. The invariant eigenvalues.** Given any  $S^1$ -invariant Kähler metric  $g$  on  $(S^2, j_0)$ , with volume form  $\omega$  and total volume  $4\pi$ , and a function  $f \in C^\infty(S^2)$ , let

$$(2.10) \quad \|f\|_g^2 = \int_{S^2} f^2(x) \omega + \int_{S^2} |\nabla f|_g^2 \omega .$$

The completion of  $C^\infty(S^2)$  with respect to the above norm is the Sobolev space  $L_1^2(S^2)$ . The Laplacian  $\Delta_g$  is a self-adjoint elliptic operator on  $L_1^2(S^2)$  with a discrete non-negative spectrum.

Because the push-forward (by the moment map  $H$ ) of the measure determined on  $S^2$  by  $\omega$  is simply  $2\pi$  times the Lebesgue measure on the polytope  $P$ , we have that for  $S^1$ -invariant functions  $f \in C^\infty(S^2)$  the above norm can be written in terms of the  $x$  coordinate as

$$(2.11) \quad \|f\|_g^2 = 2\pi \left( \int_{-1}^1 f^2(x) dx + \int_{-1}^1 \frac{(f'(x))^2}{g(x)} dx \right) ,$$

where  $g(x) dx^2$  is the metric induced by  $g$  on  $P^\circ$ . As in any Riemannian manifold, different metrics induce equivalent norms and we denote by  $X$  the completion of  $C^\infty(P)$  with respect to any of them. Then the Laplacian defined by (2.9) is a self-adjoint operator on  $(X, \|\cdot\|_g)$ , having discrete non-negative spectrum consisting

exactly of the eigenvalues  $\lambda_j(g)$  of  $\Delta_g$  on  $L_1^2(S^2)$  with  $S^1$ -invariant eigenfunctions  $f_{g,j}$  (i.e. the  $S^1$ -invariant spectrum).

By the Min-Max principle we then have that the nontrivial invariant eigenvalues  $0 < \lambda_1 < \lambda_2 < \dots$ , are given by

$$(2.12) \quad \lambda_j = \lambda_j(g) = \inf_{f \in X_{g,j}} \frac{\int_{-1}^1 \frac{(f'(x))^2}{g(x)} dx}{\int_{-1}^1 f^2(x) dx}, \quad j = 1, \dots,$$

where  $X_{g,j} = \{f \in X : \int_{-1}^1 f(x)f_{g,k} dx = 0, k = 0, \dots, j-1, \text{ and } f \neq 0\}$  ( $f_{g,0} \equiv 1$ ). A nice feature of the use of the moment map coordinate  $x$  is that  $X_1 \equiv X_{g,1}$  and the denominator of the above quotient do not depend on the metric  $g$ , since the relevant measure on the polytope  $P$  is always  $2\pi$  times Lebesgue measure.

**2.4. The diameter.** The diameter  $D(g)$  of  $(S^2, g)$  is equal to the length of any geodesic meridian joining the North and South poles. This can be computed as the length of the polytope  $P$  with respect to the metric induced by  $g$ , and hence is given by

$$(2.13) \quad D(g) = \int_{-1}^1 \sqrt{g(x)} dx .$$

**2.5. The Gauss curvature.** The Gauss curvature  $K$  of the Kähler metric defined by (2.1) is given by

$$K = -\frac{1}{2} \left( \frac{d^2 F}{du^2} \right)^{-1} \frac{d^2 \log(d^2 f/du^2)}{du^2},$$

which, using (2.5) and (2.8), can be written in terms of the moment map coordinate  $x = dF/du$  as

$$\begin{aligned} K &= -\frac{1}{2} g \frac{1}{g} \frac{d}{dx} \left[ \frac{1}{g} \frac{d}{dx} \log(1/g) \right] \\ &= -\frac{1}{2} \left( \frac{1}{g} \right)'' . \end{aligned}$$

Hence, non-negative Gauss curvature amounts to  $(1/g)'' \leq 0$ .

**2.6. Example.** The standard metric  $g_0$  on  $S^2$  is given on  $P^\circ$  by

$$g_0 = \frac{dx^2}{1-x^2} .$$

The corresponding Laplacian on  $S^1$ -invariant functions is

$$\Delta f = -((1-x^2)f')' \quad \text{for } f \in C^\infty(P) ,$$

with invariant eigenfunctions the well-known Legendre polynomials and invariant spectrum  $\lambda_n = n(n+1)$ ,  $n \in \mathbb{N}$ . In particular this implies that

$$(2.14) \quad \frac{\int_{-1}^1 (1-x^2) [f'(x)]^2 dx}{\int_{-1}^1 f^2(x) dx} \geq \lambda_1(g_0) = 2 \quad \text{for any } f \in X_1 ,$$

an inequality that will be used later on.

The diameter and Gauss curvature are given by

$$D_0 = \int_{-1}^1 \frac{1}{\sqrt{1-x^2}} dx = \pi \quad \text{and} \quad K_0(x) = -\frac{1}{2}(1-x^2)'' = 1$$

as expected.

### 3. PROOF OF THEOREM 1

We saw in the previous section that any smooth  $S^1$ -invariant metric on  $S^2$  is determined by a positive function  $g \in C^\infty(P^\circ)$ ,  $P^\circ = (-1, 1)$ , of the form

$$(3.1) \quad g(x) = \frac{1}{1-x^2} + h(x)$$

with  $h \in C^\infty(P)$ ,  $P = [-1, 1]$ . We also saw that the relevant Riemannian information (Laplacian,  $\lambda_1$ , scalar curvature) can be written explicitly in terms of  $\bar{g} = 1/g$ . From (3.1) we get that  $\bar{g}$  is of the form

$$(3.2) \quad \bar{g}(x) = \frac{1}{g(x)} = (1-x^2)[1 + (1-x^2)\bar{h}(x)]$$

where  $\bar{h} \in C^\infty(P)$  is such that  $\bar{g}(x) > 0$ ,  $x \in P^\circ$ . Note that  $\bar{g} \in C^\infty(P)$ . The functions  $h$  and  $\bar{h}$  are related to each other by

$$\bar{h}(x) = -\frac{h(x)}{1 + (1-x^2)h(x)} \quad \text{and} \quad h(x) = -\frac{\bar{h}(x)}{1 + (1-x^2)\bar{h}(x)}$$

and they both satisfy the inequality

$$h(x), \bar{h}(x) > -\frac{1}{1-x^2}, \quad x \in P^\circ.$$

Note that any function  $\bar{g} \in C^\infty(P)$  of the form given by (3.2) satisfies

$$(3.3) \quad \bar{g}(-1) = 0 = \bar{g}(1) \quad \text{and} \quad \bar{g}'(-1) = 2 = -\bar{g}'(1).$$

**3.1. Large first invariant eigenvalue.** It follows from (2.12) that in order to make  $\lambda_1$  large one should choose the function  $g$  small, and hence  $\bar{g}$  as large as possible. It is then clear from (3.2) that the simplest way to achieve that is to choose

$$\bar{h}(x) = \mu = \text{constant} > 0$$

and analyze what happens to  $\lambda_1$  as  $\mu$  tends to  $\infty$ .

Hence we will consider the family of metrics

$$g_\mu(x) = \frac{1}{1-x^2} - \frac{\mu}{1+(1-x^2)\mu} = \frac{1}{(1-x^2)(1+(1-x^2)\mu)} > 0,$$

with  $0 < \mu \in \mathbb{R}$ , for which we have that

$$\begin{aligned} \lambda_1(g_\mu) &= \inf_{f \in X_1} \frac{\int_{-1}^1 \frac{1}{g_\mu(x)} [f'(x)]^2 dx}{\int_{-1}^1 f^2(x) dx} \\ &= \inf_{f \in X_1} \left\{ \mu \frac{\int_{-1}^1 (1-x^2)^2 [f'(x)]^2 dx}{\int_{-1}^1 f^2(x) dx} + \frac{\int_{-1}^1 (1-x^2) [f'(x)]^2 dx}{\int_{-1}^1 f^2(x) dx} \right\} \\ &\geq \mu \inf_{f \in X_1} \frac{\int_{-1}^1 (1-x^2)^2 [f'(x)]^2 dx}{\int_{-1}^1 f^2(x) dx} + 2, \end{aligned}$$

this last inequality being valid because of (2.14). Thus, to prove that  $\lambda_1(g_\mu) \rightarrow \infty$  as  $\mu \rightarrow \infty$ , it is enough to show that this last quotient is bounded away from zero. In order to do this, we shall consider the inequality

$$(3.4) \quad \int_0^1 (1-x^2)^{2p} [f'(x)]^2 dx \geq C \int_0^1 f(x)^2 dx.$$

From results in [OK] it follows that it holds if and only if  $p$  is less than or equal to 1. Their proof is based on Minkowski's and Hölder's inequalities. Here we follow a different approach which allows us to obtain the optimal constant in the case where  $p$  is one. We also recover the optimal constant when  $p$  is  $1/2$ , which corresponds to Legendre's equation.

**Lemma 3.1.** *Let  $f \in X_1$  satisfy  $f(0) = 0$ . Then, for all  $p \in [0, 1]$  there exists  $C = C(p) \geq 1$  such that inequality (3.4) is satisfied.*

*Furthermore,  $C(1/2) = 2$  (Legendre's equation) and  $C(1) = 1$  are optimal constants in (3.4). For all other  $p \in [0, 1]$  we have that the optimal constant  $C_{opt}$  satisfies  $C_{opt}(p) \geq m$ , where  $m$  is the infimum of the function  $F : (0, 1) \rightarrow \mathbb{R}$  defined by*

$$F(x) = 2 + \frac{1-x^2}{x^2} [1 - (1-x^2)^{1-2p}].$$

*Proof.* From

$$0 \leq \int_0^1 \left[ -\frac{(1-x^2)^{1-p}}{x} f(x) + (1-x^2)^p f'(x) \right]^2 dx$$

and integrating by parts we obtain that

$$\int_0^1 (1-x^2)^{2p} [f'(x)]^2 dx \geq \int_0^1 F(x) f^2(x) dx \geq m \int_0^1 f^2(x) dx.$$

In the case where  $p = 1/2$ , we get  $F(x) \equiv 2$ . Since for  $f(x) = x$  we have equality, it follows that  $C_{opt}(1/2) = 2$ . When  $p = 1$ , we have that  $F(x) \equiv 1$ . The solution of the corresponding Euler-Lagrange equations which is given in the Appendix suggests that we now consider the sequence of functions defined by

$$f_\varepsilon(x) = \frac{x}{\sqrt{1 + \varepsilon - x^2}}$$

to get

$$\int_0^1 (1-x^2)^2 [f'_\varepsilon(x)]^2 dx = \frac{1+\varepsilon}{4} - \frac{8+\varepsilon}{8(1+\varepsilon)^2} + \frac{8+8\varepsilon+3\varepsilon^2}{8(1+\varepsilon)^{5/2}} \operatorname{atanh}\left(\frac{1}{\sqrt{1+\varepsilon}}\right)$$

and

$$\int_0^1 f_\varepsilon^2(x) dx = -1 + \sqrt{1+\varepsilon} \operatorname{atanh}\left(\frac{1}{\sqrt{1+\varepsilon}}\right).$$

From this it follows that

$$\lim_{\varepsilon \rightarrow 0^+} \frac{\int_0^1 (1-x^2)^2 [f'_\varepsilon(x)]^2 dx}{\int_0^1 f_\varepsilon^2(x) dx} = 1$$

and so  $C_{opt}(1) = 1$ . □

**Theorem 3.2.** *Let  $f \in X_1$ . Then*

$$(3.5) \quad \int_{-1}^1 (1-x^2)^{2p} [f'(x)]^2 dx \geq C \int_{-1}^1 f(x)^2 dx,$$

where  $C = C(p)$  is as in Lemma 3.1. In particular,  $C(1) = 1$  is optimal.

*Proof.* Assume that  $f$  has zero average. Then

$$\begin{aligned} \int_{-1}^1 (1-x^2)^{2p} [f'(x)]^2 dx &= \int_{-1}^1 (1-x^2)^{2p} \{[f(x) - f(0)]'\}^2 dx \\ &\geq C \int_{-1}^1 [f(x) - f(0)]^2 dx, \end{aligned}$$

where the inequality follows from Lemma 3.1. We thus have that

$$\begin{aligned} \int_{-1}^1 (1-x^2)^{2p} [f'(x)]^2 dx &\geq C \int_{-1}^1 f^2(x) dx - 2Cf(0) \int_{-1}^1 f(x) dx + 2Cf^2(0) \\ &\geq C \int_{-1}^1 f^2(x) dx. \end{aligned}$$

□

These results enable us to conclude that for the family of metrics considered above, the first invariant eigenvalue satisfies  $\lambda_1 = \lambda_1(g_\mu) \geq \mu + 2 \rightarrow \infty$  as  $\mu \rightarrow \infty$ .

The Gauss curvature  $K_\mu$  of this family of metrics is given by (see §2.5)

$$K_\mu(x) = -\frac{1}{2} [(1-x^2)(1+\mu(1-x^2))]'' = 1 + 2\mu(1-3x^2).$$

In particular, at the poles fixed by the  $S^1$ -action we have that the curvature blows up:

$$K_\mu(1) = K_\mu(-1) = 1 - 4\mu \rightarrow -\infty \text{ as } \mu \rightarrow \infty.$$

It is however possible to make the first invariant eigenvalue arbitrarily large while keeping the curvature at the poles fixed. Consider the family of metrics defined by

$$\bar{g}_\rho(x) = (1-x^2)[1 + \rho(1-x^2)^2],$$

and let us analyze what happens when  $\rho \rightarrow \infty$ . The Gauss curvature at the poles is fixed (equal to 1) while the first invariant eigenvalue is given by

$$\lambda_1(g_\rho) = \inf_{f \in X_1} \left\{ \frac{\int_{-1}^1 (1-x^2) [f'(x)]^2 dx}{\int_{-1}^1 f^2(x) dx} + \rho \frac{\int_{-1}^1 (1-x^2)^3 [f'(x)]^2 dx}{\int_{-1}^1 f^2(x) dx} \right\}.$$

Although we now have that

$$\inf_{f \in X_1} \frac{\int_{-1}^1 (1-x^2)^3 [f'(x)]^2 dx}{\int_{-1}^1 f^2(x) dx} = 0,$$

we can use Cauchy-Schwartz to obtain

$$\left( \int_{-1}^1 (1-x^2) [f'(x)]^2 dx \right) \cdot \left( \int_{-1}^1 (1-x^2)^3 [f'(x)]^2 dx \right) \geq \left( \int_{-1}^1 (1-x^2)^2 [f'(x)]^2 dx \right)^2$$

which, together with (2.14) and Lemma 3.1, easily implies that

$$\lambda_1(g_\rho) \geq \sqrt{4+2\rho} \rightarrow \infty \text{ as } \rho \rightarrow \infty.$$

**3.2. Small first invariant eigenvalue.** We shall now see that it is possible to choose the metric in such a way that the first invariant eigenvalue becomes arbitrarily small. This can be achieved with a family of metrics “dual” to the previous one. We choose

$$h(x) \equiv \nu > 0$$

and consider the family of metrics defined by

$$g_\nu(x) = \frac{1}{1-x^2} + \nu,$$

for constant  $\nu$ . The first invariant eigenvalue of the Laplacian on  $S^2$  corresponding to this family is given by

$$\begin{aligned} \lambda_1(g_\nu) &= \inf_{f \in X_1} \frac{\int_{-1}^1 \frac{1-x^2}{1+\nu(1-x^2)} [f'(x)]^2 dx}{\int_{-1}^1 f^2(x) dx} \\ &< \frac{1}{\nu} \inf_{f \in X_1} \frac{\int_{-1}^1 [f'(x)]^2 dx}{\int_{-1}^1 f^2(x) dx} = \frac{\pi^2}{4\nu}, \end{aligned}$$

and so we get that  $\lambda_1(g_\nu) \rightarrow 0$  as  $\nu \rightarrow \infty$ .

A calculation shows that the Gauss curvature  $K_\nu$  of this family of metrics is positive everywhere, tends to zero (as  $\nu \rightarrow \infty$ ) for every  $x \in (-1, 1)$ , while at the poles we have that

$$K_\nu(1) = K_\nu(-1) = 1 + 4\nu \rightarrow \infty \text{ as } \nu \rightarrow \infty.$$

The fact that one can get  $\lambda_1$  arbitrarily small with a family of metrics with fixed curvature at the poles is left as an easy exercise to the reader.

## 4. PROOF OF THEOREM 2

We now want to consider  $S^1$ -invariant metrics  $g$  on  $S^2$ , that correspond to closed surfaces of revolution in  $\mathbb{R}^3$ . Such a surface is obtained by revolving a profile curve

$$t \mapsto (0, p(t), q(t)), \text{ for } 0 \leq t \leq \ell,$$

about the third coordinate axis. We have necessarily that  $p(0) = 0 = p(\ell)$  and  $p(t) > 0$  for  $0 < t < \ell$ . If the curve is parametrized by arclength, which we will assume, then we have in addition  $\dot{p}(0) = 1 = -\dot{p}(\ell)$  and

$$(4.1) \quad (\dot{p}(t))^2 + (\dot{q}(t))^2 = 1, \text{ for all } 0 \leq t \leq \ell.$$

Assuming that the total volume of such a surface is  $4\pi$ , then the normalized moment map  $H$  of the  $S^1$ -action, with respect to the induced area form, “projects” the surface to the moment polytope  $P = [-1, 1] \subset \mathbb{R}$ . This can be seen as the relation between the arclength coordinate  $t$  and the moment map coordinate  $x$ :

$$(4.2) \quad x = x(t) = H(t), \quad t \in [0, \ell], \quad x \in P.$$

Under the above relation, the metric induced by  $\mathbb{R}^3$  on the surface of revolution gives rise to a metric  $g$  on  $P^\circ = (-1, 1)$ ,

$$g = g(x) dx^2$$

of the form considered in §2.1. This function  $g(x)$  determines the inverse of (4.2) by

$$(4.3) \quad t = t(x) = \int_{-1}^x \sqrt{g(s)} ds.$$

**Proposition 4.1.** *The relation between  $\bar{g} = 1/g$  and  $p$  is given by*

$$\bar{g}(x) = p(t(x))^2, \quad x \in P.$$

*Proof.* The curvature  $K$  of the surface of revolution is given in the arclength coordinate  $t$  by [O]

$$K(t) = -\frac{\ddot{p}(t)}{p(t)},$$

while in the moment map coordinate  $x$  we have that (see §2.5)

$$K(x) = -\frac{1}{2}\bar{g}''(x).$$

This means that

$$(4.4) \quad 2\ddot{p}(t(x)) = p(t(x)) \cdot \bar{g}''(x).$$

Defining a function  $\bar{p} : P \rightarrow \mathbb{R}$  by  $\bar{p}(x) = p(t(x))$ , we get from (4.3) and (4.4) the following differential equation relating  $\bar{p}$  and  $\bar{g}$ :

$$(4.5) \quad \bar{g}\bar{p}'' + \bar{g}'\bar{p}' = \frac{1}{2}(\bar{p}\bar{g}'' + \bar{g}'\bar{p}').$$

Given  $\bar{p}$ , one easily checks that the unique solution  $\bar{g}$  of (4.5) satisfying

$$\bar{g}(-1) = 0 = \bar{g}(1) \text{ and } \bar{g}'(-1) = 2 = -\bar{g}'(1)$$

is

$$\bar{g} = \bar{p}^2$$

as required.  $\square$

**Corollary 4.2.** *For a surface of revolution in  $\mathbb{R}^3$  with total area  $4\pi$ , the corresponding metric on the moment polytope  $P$ ,*

$$g = g(x) dx^2 = \frac{1}{\bar{g}(x)} dx^2 ,$$

is such that

$$|\bar{g}'(x)| \leq 2, \forall x \in P .$$

*Proof.* From  $\bar{g} = \bar{p}^2$  we have that

$$\bar{g}' = 2\bar{p}\bar{p}' = 2\bar{p}(p \circ t)' = 2\bar{p}(\dot{p} \circ t)t' = 2\bar{p}(\dot{p} \circ t) \frac{1}{\sqrt{\bar{g}}} = 2(\dot{p} \circ t) .$$

Hence, using (4.1), we get

$$|\bar{g}'(x)| = 2|\dot{p}(t(x))| \leq 2, \forall x \in P .$$

□

We now know that, for a smooth closed surface of revolution in  $\mathbb{R}^3$  with total area  $4\pi$ , the corresponding function  $\bar{g} \in C^\infty(P)$  satisfies

$$(4.6) \quad \bar{g}(-1) = 0 = \bar{g}(1), \bar{g}'(-1) = 2 = -\bar{g}'(1) \text{ and } |\bar{g}'(x)| \leq 2, \forall x \in P .$$

Any such  $\bar{g}$  is clearly less than or equal to the “tent” function

$$\bar{g}_{\max}(x) = 2(1 - |x|).$$

We shall now consider the invariant spectrum of this limit problem. The values of  $\lambda_j(g_{\max})$  can be explicitly determined in two different ways: one more geometric, the other more analytic.

In the more geometric way, one interprets  $g_{\max}$  as being the singular metric on  $S^2$  consisting of two flat discs, each of area  $2\pi$ , glued along a singular equator. In fact, the scalar curvature of  $g_{\max}$  is zero where defined (i.e.  $P \setminus \{0\}$ ) because  $\bar{g}_{\max}$  is a polynomial of degree 1 in  $[-1, 0)$  and  $(0, 1]$ . Moreover, one can see  $g_{\max}$  as the limit of the family of surfaces of revolution in  $\mathbb{R}^3$  consisting of ellipsoids of total area  $4\pi$  squeezed between two horizontal planes moving towards each other (the associated family of functions  $\bar{g}$  interpolates increasingly from  $\bar{g}_0$  to  $\bar{g}_{\max}$ ). Because this is a family of “even” metrics (i.e. invariant under the antipodal map or, equivalently, the corresponding functions  $g$  and  $\bar{g}$  are even on  $P$ ), the associated invariant eigenfunctions alternate between being odd and even, and hence either themselves or their first derivative vanish at  $x = 0$  (the equator of the surface of revolution). In the limit, and restricting attention to just one of the hemispheres, the Laplace operator of this family of metrics tends to the Euclidean Laplacian on a disc of area  $2\pi$ , while the invariant eigenfunctions tend to its radially symmetric eigenfunctions with either Dirichlet or Neumann boundary conditions. Hence we have that

$$\lambda_j(g_{\max}) = \frac{\xi_j^2}{2}, \quad j = 1, \dots ,$$

where  $\xi_j$  is the  $((j+1)/2)^{\text{th}}$  positive zero of the Bessel function  $J_0$  if  $j$  is odd, and the  $(j/2)^{\text{th}}$  positive zero of  $J'_0$  if  $j$  is even. In particular,

$$\lambda_1(g_{\max}) = \frac{\xi_1^2}{2} \approx 2.89.$$

To obtain the same result from a more analytic perspective, we consider the differential equation directly, that is,

$$(4.7) \quad \begin{cases} [2(1-x)f']' + \lambda f = 0, & x \in [0, 1] \\ f(0) = 0 \text{ or } f'(0) = 0, \end{cases}$$

where the extra boundary condition in  $f$  stems from the fact mentioned above that we are considering an even metric and thus the eigenfunctions alternate between being odd and even.

We now consider the change of variables defined by  $t = \sqrt{2(1-x)}$ , which transforms this to the equation corresponding to the standard Dirichlet/Neumann problem on the disk of radius  $\sqrt{2}$ . We are thus led to the equation

$$t^2 \ddot{f} + t \dot{f} + \lambda t^2 f = 0,$$

with the conditions  $f(\sqrt{2}) = 0$  or  $f'(\sqrt{2}) = 0$ . This equation has  $f(t) = J_0(\sqrt{\lambda}t)$  as a solution and we thus obtain the above mentioned values for  $\lambda_j(g_{\max})$ .

The proof of Theorem 2 is now a direct consequence of the monotonicity principle (derived from Poincaré's principle). Given any closed surface of revolution in  $\mathbb{R}^3$  with total area  $4\pi$ , the corresponding function  $\bar{g} \in C^\infty(P)$  satisfies (4.6), and hence  $\bar{g} \leq \bar{g}_{\max}$  on  $P$ . This means that

$$\frac{\int_{-1}^1 \bar{g}(x)(f'(x))^2 dx}{\int_{-1}^1 f^2(x) dx} \leq \frac{\int_{-1}^1 2(1-|x|)[f'(x)]^2 dx}{\int_{-1}^1 f^2(x) dx} \text{ for any } f \in X,$$

which by the monotonicity principle implies the eigenvalue inequalities

$$\lambda_j(g) \leq \lambda_j(g_{\max}).$$

Moreover, because for a smooth surface of revolution we have the strict inequality  $\bar{g} < \bar{g}_{\max}$  somewhere on  $P^\circ$ , it is not too difficult to show that in this case the eigenvalue inequalities are strict.

Finally, these results allow us to prove Corollary 3 for metrics with non-negative Gauss curvature. For such metrics, the corresponding function  $\bar{g}$  satisfies (see §2.5)

$$\bar{g}''(x) \leq 0, \quad \forall x \in P,$$

and, as an easy consequence, conditions (4.6). Hence, their invariant eigenvalues are also less than the corresponding ones for  $g_{\max}$ , and this is again an optimal bound since the family of ellipsoids considered above (degenerating to the union of two flat discs) has positive curvature.

The minimizing property of the diameter of the double flat disc among surfaces of revolution in  $\mathbb{R}^3$  follows easily from (2.13) and (4.6):

$$D(g) = \int_{-1}^1 \sqrt{g} dx = \int_{-1}^1 \frac{1}{\sqrt{\bar{g}}} dx > \int_{-1}^1 \frac{1}{\sqrt{\bar{g}_{\max}}} dx = D(g_{\max}) = 2\sqrt{2}.$$

## 5. INDEPENDENCE OF $\lambda_1$ AND $D$

In the case of the metric given in § 3.1, which was used to obtain a large first invariant eigenvalue, it is an exercise in calculus to show that

$$D(g_\mu) = \int_{-1}^1 \sqrt{g_\mu(x)} dx = \int_{-1}^1 \frac{dx}{\sqrt{(1-x^2)(1+(1-x^2)\mu)}} \rightarrow 0$$

as  $\mu$  goes to infinity. Thus, in this case we have that the diameter goes to zero while  $\lambda_1$  goes to infinity. On the other hand, it is straightforward to check that in the case where  $\lambda_1$  was made to be arbitrarily small, the corresponding diameter was unbounded.

As pointed out in the Introduction, these examples together with the fact that the invariant spectrum is given by a one-dimensional eigenvalue problem, suggest a natural relation between the diameter of the surface and the value of the first invariant eigenvalue, in the sense that making the diameter large would give small values of  $\lambda_1$  and vice-versa. However, this is not necessarily the case, as we shall now see. In order to do this, we need a result from [OK] already mentioned in § 3.1. A version adequate for our purposes is the following.

**Theorem 5.1.** *Let  $g$  be a continuous even function on  $(-1, 1)$ . The inequality*

$$\int_{-1}^1 \frac{1}{g(x)} [f'(x)]^2 dx \geq C \int_{-1}^1 f^2(x) dx$$

*holds for every odd function  $f \in X_1$  if and only if*

$$A = \sup_{x \in (0,1)} \left[ (1-x) \int_0^x g(t) dt \right]$$

*is finite. Furthermore, the optimal constant  $C$  satisfies  $A \leq 1/C \leq 2A$ .*

For a more general version of this result, as well as for a proof, see [OK].

In the case where  $g$  is even and corresponds to a metric on the sphere, we know that  $A$  is finite, and the optimal constant corresponds to the first nontrivial eigenvalue, since for even  $g$ 's the corresponding first invariant eigenfunction is odd.

By appropriate changes of the family of metrics used in § 3.1, we will now give examples showing that it is possible to have both the diameter and  $\lambda_1$  either very small or very large.

*Example 1.* [ $D$  and  $\lambda_1$  both small] Consider the function

$$g_\mu(x) = \frac{1}{(1-x^2)[1+(1-x^2)\mu]} + \frac{\mu^{1-\alpha}}{(1+\mu x^2)^2},$$

where  $\alpha$  is a constant in  $(0, 1/2)$ . In this case we have

$$\begin{aligned} D(g_\mu) &= 2 \int_0^1 \left[ \frac{1}{(1-x^2)(1+(1-x^2)\mu)} + \frac{\mu^{1-\alpha}}{(1+\mu x^2)^2} \right]^{1/2} dx \\ &\leq 2 \int_0^1 \frac{1}{\sqrt{(1-x^2)(1+(1-x^2)\mu)}} + \frac{\mu^{\frac{1-\alpha}{2}}}{1+\mu x^2} dx \rightarrow 0 \end{aligned}$$

as  $\mu \rightarrow \infty$  (using the fact that  $\alpha > 0$ ). Thus we have that the diameter converges to zero.

On the other hand, the value of  $A$  is now given by

$$\begin{aligned}
A &= \sup_{x \in (0,1)} \left\{ (1-x) \int_0^x \left[ \frac{1}{(1-t^2)(1+(1-t^2)\mu)} + \frac{\mu^{1-\alpha}}{(1+\mu t^2)^2} \right] dt \right\} \\
&\geq \sup_{x \in (0,1)} \left\{ \mu^{1-\alpha} (1-x) \int_0^x \frac{1}{(1+\mu t^2)^2} dt \right\} \\
&= \sup_{x \in (0,1)} \left\{ \frac{1-x}{2} \left[ \frac{\mu^{1-\alpha} x}{1+\mu x^2} + \mu^{\frac{1}{2}-\alpha} \arctan(\sqrt{\mu}x) \right] \right\} \\
&\geq \frac{1}{4} \mu^{\frac{1}{2}-\alpha} \arctan(\sqrt{\mu}/2)
\end{aligned}$$

which goes to infinity as  $\mu$  goes to infinity, provided  $\alpha < 1/2$ . Since  $\lambda_1 \leq 1/A$ , we have that  $\lambda_1$  goes to zero.

*Example 2.* [ $D$  and  $\lambda_1$  both large] Consider the function

$$g_\mu(x) = \frac{1}{(1-x^2)(1+(1-x^2)\mu)} + \frac{1}{\log \mu} \left( \frac{1}{(1+1/\mu)^2 - x^2} \right)^2.$$

In this case,

$$\begin{aligned}
D(g_\mu) &= 2 \int_0^1 \left[ \frac{1}{(1-x^2)[1+(1-x^2)\mu]} + \frac{1}{\log \mu} \left( \frac{1}{(1+1/\mu)^2 - x^2} \right)^2 \right]^{1/2} dx \\
&\geq \frac{2}{\log^{1/2} \mu} \int_0^1 \frac{1}{[(1+1/\mu)^2 - x^2]} dx \\
&= \frac{\mu}{1+\mu} \left[ \frac{\log(2+1/\mu)}{\log^{1/2} \mu} + \log^{1/2} \mu \right]
\end{aligned}$$

which goes to infinity as  $\mu$  goes to infinity.

For  $A$  we now have

$$\begin{aligned}
A &= \sup_{x \in (0,1)} \left\{ (1-x) \int_0^x \left[ \frac{1}{(1-t^2)(1+(1-t^2)\mu)} + \frac{1}{\log \mu} \left( \frac{1}{(1+1/\mu)^2 - t^2} \right)^2 \right] dt \right\} \\
&\leq \sup_{x \in (0,1)} \left\{ (1-x) \int_0^x \frac{1}{(1-t^2)(1+(1-t^2)\mu)} dt \right\} + \\
&\quad + \sup_{x \in (0,1)} \left\{ \frac{(1-x)}{\log \mu} \int_0^x \frac{1}{((1+1/\mu)-t)^2} dt \right\} \rightarrow 0
\end{aligned}$$

as  $\mu \rightarrow \infty$ , by a simple calculation (note that the first term corresponds to the family used in § 3.1). Since  $\lambda_1 \geq 1/2A$  we conclude that  $\lambda_1$  still goes to infinity for this family of metrics.

#### APPENDIX A. SOLUTIONS OF THE EULER-LAGRANGE EQUATIONS

In the case where  $p$  is equal to 1 (in Lemma 3.1), the second order differential equation associated with the minimization problem under study is given by

$$(A.1) \quad -\frac{d}{dx} \left[ (1-x^2)^2 \frac{df}{dx} \right] = \lambda f.$$

It turns out that this equation can actually be solved explicitly for all real values of the parameter  $\lambda$ . To see this, we begin by writing it as

$$(1-x^2)^2 \frac{d^2 f}{dx^2} - 4x(1-x^2) \frac{df}{dx} + \lambda f = 0, \quad x \in (-1, 1).$$

Following [CL], we now look for solutions of the form

$$f(x) = (1+x)^r [\alpha_0 + \alpha_1(1+x) + \dots]$$

from which we obtain the indicial equation

$$r^2 + r + \lambda/4 = 0$$

with two solutions

$$r_{\pm} = \frac{-1 \pm i\omega}{2}, \quad \omega = \sqrt{\lambda-1}.$$

We then see that

$$f(x) = (x+\omega)(1-x)^{r-}(1+x)^{r+}$$

satisfies (A.1). Due to the symmetry of the problem,  $f(-x)$  is also a solution. The form of the (real) solutions will now depend on the sign of  $1-\lambda$ .

In the case where  $\lambda > 1$ , we have that  $r_{\pm} = (-1 \pm i\omega)/2$  and it is possible to obtain from the general form of the solution that:

$$f_1(x) = \frac{1}{\sqrt{1-x^2}} \left[ x \cos \left[ \frac{\omega}{2} \log \left( \frac{1-x}{1+x} \right) \right] - \omega \sin \left[ \frac{\omega}{2} \log \left( \frac{1-x}{1+x} \right) \right] \right]$$

and

$$f_2(x) = \frac{1}{\sqrt{1-x^2}} \left[ \omega \cos \left[ \frac{\omega}{2} \log \left( \frac{1-x}{1+x} \right) \right] + x \sin \left[ \frac{\omega}{2} \log \left( \frac{1-x}{1+x} \right) \right] \right].$$

are two linearly independent real solutions (odd and even, respectively).

When  $\lambda = 1$  we get that

$$f_1 = \frac{x}{\sqrt{1-x^2}} \quad \text{and} \quad f_2 = \frac{1}{\sqrt{1-x^2}} \left[ \frac{x}{2} \log \left( \frac{1-x}{1+x} \right) + 1 \right]$$

are odd and even solutions, respectively.

When  $\lambda < 1$ , and letting now  $\gamma = -i\omega = \sqrt{1-\lambda}$ , we obtain the two solutions

$$f_1(x) = (x+\gamma)(1+x)^{r-}(1-x)^{r+} + (x-\gamma)(1+x)^{r+}(1-x)^{r-}$$

and

$$f_2(x) = (x+\gamma)(1+x)^{r-}(1-x)^{r+} - (x-\gamma)(1+x)^{r+}(1-x)^{r-}.$$

From this we see that there are no nontrivial solutions of the Euler–Lagrange equations lying in the space  $X$ .

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DEPARTAMENTO DE MATEMÁTICA, INSTITUTO SUPERIOR TÉCNICO, AV.ROVISCO PAIS, 1049-001 LISBOA, PORTUGAL

*E-mail address:* mabreu@math.ist.utl.pt, pfreitas@math.ist.utl.pt