

COMPATIBLE COMPLEX STRUCTURES ON SYMPLECTIC RATIONAL RULED SURFACES

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In fond memory of Raoul Bott

Abstract

In this article, we study the topology of the space \mathcal{F}_ω of complex structures compatible with a fixed symplectic form ω , using the framework of Donaldson. By comparing our analysis of the space \mathcal{F}_ω with results of McDuff on the space \mathcal{F}_ω of compatible almost complex structures on rational ruled surfaces, we find that \mathcal{F}_ω is contractible in this case.

We then apply this result to study the topology of the symplectomorphism group of a rational ruled surface, extending results of Abreu and McDuff.

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1. Introduction

The work of Gromov on J -holomorphic curves in [Gr] has provided tools for understanding the topology of the group of symplectomorphisms of certain 4-dimensional

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symplectic manifolds. Gromov began the study of the group of symplectomorphisms on $S^2 \times S^2$ with the standard symplectic form $\sigma \oplus \sigma$. He showed that this group is homotopy equivalent to a semidirect product of $\mathbb{Z}/2$ with $\mathrm{SO}(3) \times \mathrm{SO}(3)$. Later Abreu [A] continued this study by analyzing the structure of the group of symplectomorphism on $S^2 \times S^2$ with symplectic form $\omega = \lambda\sigma \oplus \sigma$, $1 < \lambda \leq 2$. This work was extended by Abreu and McDuff [AM] to arbitrary λ , leading to a complete calculation of the rational cohomology ring of the classifying space of the symplectomorphism group (modulo a mistake that is corrected below in Theorem 1.3). There are corresponding results for the structure of the symplectomorphism group of the nontrivial S^2 -bundle over S^2 (for an arbitrary symplectic form). The basic idea in the work mentioned above is to analyze the action of the symplectomorphism group on the contractible space of compatible almost complex structures \mathcal{J}_ω .

In [D] Donaldson showed that the action of the symplectomorphism group on \mathcal{J}_ω is Hamiltonian, and the moment map for this action is the Hermitian scalar curvature of the corresponding almost Kähler metric. He also showed that, restricted to the space \mathcal{J}_ω of compatible integrable structures, this action fits into the general framework of infinite-dimensional geometric invariant theory, going back to Atiyah and Bott [AB2] (see [AGK] for more discussion of this point of view). Therefore, in principle, the norm square of the moment map should induce a stratification of \mathcal{J}_ω with critical points being the extremal Kähler metrics compatible with the symplectic form. By work of Calabi [CI], for rational ruled surfaces these metrics correspond to a finite collection of Hirzebruch surfaces. In particular, this suggests that each stratum in \mathcal{J}_ω should be homotopy equivalent to a single orbit of the symplectomorphism group. A stratification of \mathcal{J}_ω with similar properties had been established by Abreu and studied in detail by McDuff [Mc1]. This indicated that the hypothetical stratification on \mathcal{J}_ω determined by the moment map could be the one induced via the inclusion $\mathcal{J}_\omega \subset \mathcal{J}_\omega$.

We begin our study of the space \mathcal{J}_ω by analyzing the stratification induced by the above inclusion. We show that each stratum V contains an orbit of the symplectomorphism group, corresponding to a Hirzebruch surface F with a standard Kähler metric, which is weakly equivalent to it. The stratification of \mathcal{J}_ω has the advantage that its gluing data can be easily understood via Kodaira-Spencer deformation theory. By comparing our stratification of \mathcal{J}_ω with that of Abreu and McDuff, we prove the following.

THEOREM 1.1

The inclusion of the space of compatible integrable complex structures into the space of all compatible almost complex structures, $\mathcal{J}_\omega(M) \subset \mathcal{J}_\omega(M)$, is a weak homotopy equivalence for a rational ruled surface M . In particular, the space $\mathcal{J}_\omega(M)$ is weakly contractible.

As far as we are aware, and besides the obvious case of real 2-dimensional surfaces, this is the first example where the topology of the space of compatible integrable complex structures on a symplectic manifold has been understood.

Recall that any rational ruled surface is diffeomorphic to either $S^2 \times S^2$, the trivial S^2 -bundle over S^2 , or $S^2 \tilde{\times} S^2$, the nontrivial S^2 -bundle over S^2 . Work of Taubes, Liu and Li, and Lalonde and McDuff (see [LM] for detailed references) implies that any symplectic form on one of these smooth manifolds is “standard,” that is, diffeomorphic to a scalar multiple of $\omega_\lambda = \lambda\sigma \oplus \sigma$, $1 \leq \lambda \in \mathbb{R}$, on $S^2 \times S^2$, or to any chosen symplectic form ω_λ , $0 < \lambda \in \mathbb{R}$, on $S^2 \tilde{\times} S^2$ such that $[\omega_\lambda](E) = \lambda$ and $[\omega_\lambda](F) = 1$, where E denotes the homology class of the exceptional divisor under the natural identification $S^2 \tilde{\times} S^2 \cong \mathbb{P}^2 \# \overline{\mathbb{P}^2}$ and F denotes the homology class of the fiber.

Complex deformation theory gives us a good understanding of the way the strata of \mathcal{J}_ω glue and allows us to express the symplectomorphism group as an iterated homotopy pushout of certain compact subgroups (see Theorem 5.5). This, in turn, immediately gives the integral cohomology groups of the classifying space of the symplectomorphism groups of rational ruled surfaces. Let G_λ denote the group of symplectomorphisms of $S^2 \times S^2$ with symplectic form $\omega_\lambda = \lambda\sigma \oplus \sigma$, where $1 < \lambda \in \mathbb{R}$ lies between the integers $0 < \ell < \lambda \leq \ell + 1$. Let BG_λ denote the classifying space of G_λ . We have the following.

THEOREM 1.2

The integral cohomology groups of BG_λ in the untwisted case are given by

$$H^*(BG_\lambda; \mathbb{Z}) = H^*(BSO(3) \times BSO(3); \mathbb{Z}) \oplus \bigoplus_{i=1}^{\ell} \Sigma^{4i-2} H^*(BS^1 \times BSO(3); \mathbb{Z}),$$

where Σ denotes the suspension of graded abelian groups.

The relevance of the compact Lie groups appearing in the previous statement becomes clear in Sections 3–5. If we work away from the prime 2, we can compute the ring structure.

THEOREM 1.3

The cohomology of BG_λ in the untwisted case with coefficients in the ring $R = \mathbb{Z}[1/2]$ is given by the following free module over the ring $R[x, y]$ on generators b_i, a_j , $0 \leq i < \ell, 0 \leq j \leq \ell$:

$$H^*(BG_\lambda; R) = R[x, y]\langle a_0, b_0, a_1, b_1, a_2, \dots, a_\ell \rangle,$$

where $a_0 = 1$, the degree of the elements x, y is 4, the degree of b_k is $4k + 2$, and that of a_k is $4k$. Moreover, as a ring, we may identify $H^*(BG_\lambda, R)$ as the subring of

$$H^*(BG_\lambda; \mathbb{Q}) = \frac{\mathbb{Q}[x, y, z]}{\langle z \prod_{i=1}^\ell (z^2 + i^4x - i^2y) \rangle},$$

where the degree of z is 2, and the elements a_k and b_k are identified, respectively, with the elements

$$\frac{z^2}{(2k)!} \prod_{i=1}^{k-1} (z^2 + i^4x - i^2y) \quad \text{and} \quad \frac{z}{(2k+1)!} \prod_{i=1}^k (z^2 + i^4x - i^2y).$$

If G_λ denotes the group of symplectomorphisms of $S^2 \times S^2$, with symplectic form ω_λ as above, where $0 < \lambda \in \mathbb{R}$ lies between the integers $0 \leq \ell < \lambda \leq \ell + 1$, we have the following.

THEOREM 1.4

The cohomology groups of the space BG_λ in the twisted case are given by

$$H^*(BG_\lambda; \mathbb{Z}) = \bigoplus_{i=0}^\ell \Sigma^{4i} H^*(BU(2); \mathbb{Z}).$$

In particular, we see that the cohomology of BG_λ is torsion free in the twisted case.

THEOREM 1.5

The rational cohomology ring of BG_λ in the twisted case is given by

$$H^*(BG_\lambda; \mathbb{Q}) = \frac{\mathbb{Q}[x, y, z]}{\langle \prod_{i=0}^\ell (-z^2 + (2i+1)^4x - (2i+1)^2y) \rangle},$$

where the degree of the class z is 2, and that of the classes x and y is 4.

We bring to the attention of the reader the relations in the rational cohomology of BG_λ . In [AM], both in the twisted and untwisted cases, the decomposable term z^2 was missing in each of the factors that make up the relation. In addition, in the twisted case, the nondecomposable terms in the factors have powers of odd integers as coefficients instead of all integers. The source of these inaccuracies in [AM] is discussed in Remark 5.19.

Regarding previous articles on the topology of symplectomorphism groups of rational ruled surfaces, the results of this article depend logically only on the analysis of the stratification of \mathcal{J}_ω in [Mc1], the computation of the homotopy type of the

strata in [A] (or [AM]), and (just for the computation of the ring structures) the fact, proved in [Mc2], that the homotopy colimit of certain inclusions $BG_\lambda \rightarrow BG_{\lambda+\epsilon}$ is the classifying space of the group of fiberwise diffeomorphisms.

Organization of the article

The body of the article is divided into four sections and three appendices.

Section 2 is further divided into four subsections. In the first part, we begin by developing our framework to study complex deformation theory of a complex 4-manifold M . In the next part, we specialize to a symplectic 4-manifold (M, ω) . Let \mathcal{J}_ω be the space of almost complex structures on M compatible with ω , and let \mathcal{I}_ω be the compatible integrable structures. Given a Kähler structure $J_0 \in \mathcal{I}_\omega$, we interpret the normal bundle in \mathcal{I}_ω to the space of equivalent Kähler structures, in terms of deformation theory. In the next subsection, we study the action of the diffeomorphism group on the space of compatible complex structures for an arbitrary symplectic manifold (M, ω) . Using this we show that, under certain conditions, the space of diffeomorphic compatible structures is homotopy equivalent to an orbit of the symplectomorphism group. Finally, in the last part of this section, we analyze the inclusion $\iota : \mathcal{I}_\omega \subset \mathcal{J}_\omega$. The space \mathcal{I}_ω admits a map from the moduli space \mathcal{M} of J -holomorphic curves. We derive sufficient cohomological conditions for \mathcal{M} to meet ι transversally. In the case of rational ruled surfaces, this implies that the map ι is transverse to the strata introduced by Abreu. Some technical lemmas in Section 2 have been banished to Appendices A and B. We feel that many of the results in this section should be well known to experts, but we have not been able to find convenient references in the literature.

In Section 3, we study rational ruled surfaces in detail. This section is divided into two parts. In the first part, we describe a stratification on the space \mathcal{I}_ω of compatible almost complex structures, for a fixed symplectic form ω , previously studied in [A], [AM], [Mc1]. In the second part, we use results from the previous section to show that this induces an analogous stratification on the corresponding space \mathcal{I}_ω of the compatible complex structures. We then conclude with the proof of Theorem 1.1.

Section 4 is dedicated to studying the deformation theory of Hirzebruch surfaces. We review the construction of Hirzebruch surfaces by Kähler reduction and use it to identify their Kähler automorphism groups. We also relate standard bases for the maximal tori of the automorphism groups to the bases yielding standard Delzant polygons as images of the moment map. We then use the fixed point formula for elliptic complexes of Atiyah and Bott to determine the isotropy representations of the symplectomorphism group on the normal bundle to the various strata in \mathcal{I}_ω . These results are applied in Section 5 to study the topology of the symplectomorphism group.

In Section 5, we use the results of Sections 3 and 4 to express the classifying space of the symplectomorphism group as a finite iterated homotopy pushout of classifying spaces of certain Kähler isometry subgroups (leaving some technicalities for Appendix C). Theorems 1.2 and 1.4 follow immediately. We then use the loop maps from G_λ to the group of fiberwise diffeomorphisms defined in [Mc2] together with the classification of Hamiltonian S^1 -actions on four-manifolds (see [K1]) to compute the rational cohomology rings described in Theorems 1.3 and 1.5 (see Theorem 5.17 and Remark 5.18). A key step is understanding the relation between the cohomology of the various Kähler isometry groups (Proposition 5.15). Finally, we use the computation in [HHH] of the T^2 -equivariant cohomology groups of $\Omega SU(2)$ to find the cohomology ring of the classifying space of the fiberwise diffeomorphism group away from 2, and we use this to complete the proof of Theorem 1.3.

Conventions

Throughout this article, we work with infinite-dimensional Fréchet manifolds that are locally modeled on an inverse limit of Banach spaces. Standard theorems such as the inverse function theorem do not hold automatically in the Fréchet setting. Therefore, we need to say a few words about the context in which the transversality arguments in this article need to be interpreted. All the Fréchet manifolds we work with can naturally be interpreted as inverse limits of Banach manifolds. For example, the space of smooth sections of a bundle over a smooth manifold is the intersection over k of the corresponding Banach manifolds of C^k -sections. For each individual k , all the transversality arguments we use hold in the infinite-dimensional context. The validity of the results stated in the smooth setting should therefore be interpreted as the validity of the corresponding result for each Banach manifold indexed by k . Statements about the homotopy type of the corresponding Fréchet manifold can be derived from the fact that the successive inclusions between the Banach manifolds are weak equivalences (see [P]). We illustrate this with an example in Remark 2.2.

2. General facts on compatible complex structures

The goal of this section is to set up a geometric framework and establish some facts regarding the space of compatible integrable complex structures \mathcal{I}_ω and its inclusion in the space of compatible almost complex structures \mathcal{J}_ω on a symplectic 4-manifold (M, ω) .

We begin by describing cohomological conditions under which the space \mathcal{I}_ω is a submanifold of \mathcal{J}_ω (see Theorem 2.3). We then show that for a given $J \in \mathcal{I}_\omega$, the intersection $(\text{Diff}(M) \cdot J) \cap \mathcal{I}_\omega$ is weakly equivalent to the orbit of J under the symplectomorphism group, as long as the Kähler isometry group of (J, ω) is a deformation retract of the complex automorphisms of J preserving the cohomology class of ω (see Corollary 2.6). Finally, we find conditions under which the projection

map from the space of J -holomorphic curves to \mathcal{J}_ω is transverse to \mathcal{J}_ω . In the case of rational ruled surfaces, this implies that Abreu and McDuff’s stratification of \mathcal{J}_ω induces a stratification of \mathcal{J}_ω (see Theorem 2.9).

Complex structures

In this subsection, we review the classical deformation theory of Kodaira and Spencer (see [Ko]) from the point of view that we adopt in the next subsection to study deformations of compatible complex structures.

Given an almost complex manifold (M, J) , the Nijenhuis tensor $N_J \in \Omega_J^{0,2}(TM) = \Omega_J^{0,2}(M) \otimes \Omega^0(TM)$ is a $(0, 2)$ -form on (M, J) with values in (TM, J) that measures the nonintegrability of J (see Definition A.1 in Appendix A). If one considers the space \mathcal{J} of all almost complex structures on the manifold M , then the Nijenhuis tensor N can be seen as a section of the natural vector bundle $\Omega^{0,2}(TM)$ over \mathcal{J} , whose fiber over a point $J \in \mathcal{J}$ is $\Omega_J^{0,2}(TM)$:

$$N : \mathcal{J} \rightarrow \Omega^{0,2}(TM), \quad J \mapsto (J, N_J).$$

The space \mathcal{J} of integrable complex structures on M is the zero-set of this Nijenhuis section N . As usual, it is a submanifold of \mathcal{J} if the Nijenhuis section N is transversal to the zero section.

The vector bundle $\Omega^{0,2}(TM)$ is a canonical summand of the trivial bundle over \mathcal{J} with fiber $\Omega^2(TM)$:

$$\Omega^{0,2}(TM) \oplus (\Omega^{2,0}(TM) \oplus \Omega^{1,1}(TM)) \cong \Omega^2(TM) \times \mathcal{J} \rightarrow \mathcal{J}.$$

This means, in particular, that $\Omega^{0,2}(TM)$ has a natural connection ∇ , given by projection of the trivial connection on $\Omega^2(TM) \times \mathcal{J}$:

$$\nabla \cdot = (d \cdot)^{0,2}.$$

Since $T\mathcal{J} \cong \Omega^{0,1}(TM)$, we have the fact that ∇N can be regarded as a bundle map

$$\nabla N : \Omega^{0,1}(TM) \rightarrow \Omega^{0,2}(TM).$$

In particular, if for a given $J \in \mathcal{J}$, that is, a $J \in \mathcal{J}$ for which $N_J \equiv 0$, the map

$$\nabla N_J : \Omega_J^{0,1}(TM) \rightarrow \Omega_J^{0,2}(TM)$$

is surjective, then the Nijenhuis section is transversal to the zero section at this $J \in \mathcal{J}$.

As proved in Corollary A.9, the map ∇N is essentially the $\bar{\partial}$ -operator (see Definition A.5). More precisely,

$$\nabla N_J = (-2J)\bar{\partial}_J, \quad \forall J \in \mathcal{J}.$$

The following proposition is then immediate.

PROPOSITION 2.1

If M is a 4-dimensional manifold, $J \in \mathcal{J}$ is an integrable complex structure on M , and the cohomology group $H_J^{0,2}(TM) = 0$, then \mathcal{J} is a submanifold of \mathcal{J} in the neighborhood of J with tangent space

$$T_J \mathcal{J} = \ker \{ \bar{\partial} : \Omega_J^{0,1}(TM) \rightarrow \Omega_J^{0,2}(TM) \} \subset \Omega_J^{0,1}(TM) = T_J \mathcal{J}.$$

The group $\text{Diff}(M)$ of diffeomorphisms of M acts naturally on \mathcal{J} via

$$\varphi^*(J) := (d\varphi)^{-1} J (d\varphi), \quad \forall \varphi \in \text{Diff}(M), J \in \mathcal{J}.$$

The induced infinitesimal action is given by the Lie derivative

$$\mathcal{L} : \Omega^0(TM) \rightarrow \Omega^{0,1}(TM) = T \mathcal{J},$$

which, as proved in Proposition A.4, can be written for a given $X \in \Omega^0(TM)$ and at a given $J \in \mathcal{J}$ as

$$\mathcal{L}_X J = (2J)(\bar{\partial}X) + \frac{1}{2}J(X \lrcorner N_J) \in \Omega_J^{0,1}(TM) = T_J \mathcal{J}.$$

The action of $\text{Diff}(M)$ on \mathcal{J} preserves \mathcal{J} . Since $\bar{\partial}$ commutes with J (see Proposition A.3), the previous formula implies that the tangent space to an orbit $\text{Diff}(M) \cdot J$ at a point $J \in \mathcal{J}$ is given by

$$T_J (\text{Diff}(M) \cdot J) = \text{im} \{ \bar{\partial} : \Omega_J^0(TM) \rightarrow \Omega_J^{0,1}(TM) \}.$$

It then follows from Proposition 2.1 that the moduli space of infinitesimal deformations of $J \in \mathcal{J}$ is given by $H_J^{0,1}(TM)$ and

$$T_J \mathcal{J} \cong T_J (\text{Diff}(M) \cdot J) \oplus H_J^{0,1}(TM).$$

Remark 2.2

As promised, we elaborate on the transversality argument here.

We may see N as a section of the bundle $\Omega_k^{0,2}(TM)$ with base being the space \mathcal{J}_{k+1} (the space of C^{k+1} -almost complex structures), and fiber $\Omega_k^{0,2}(TM, J)$ (the space of C^k -forms of type (0,2) with values in TM) over $J \in \mathcal{J}_{k+1}$. This bundle supports a natural connection, and the same proof shows that ∇N can be identified with the $\bar{\partial}$ -operator

$$\nabla N = \bar{\partial} : \Omega_{k+1}^{0,1}(TM, J) \longrightarrow \Omega_k^{0,2}(TM, J).$$

Let $H_k^{0,n}(TM, J)$ be the cohomology groups of the Dolbeault complex

$$\Omega_{k+2}^0(TM, J) \xrightarrow{\bar{\partial}} \Omega_{k+1}^{0,1}(TM, J) \xrightarrow{\bar{\partial}} \Omega_k^{0,2}(TM, J) \longrightarrow 0.$$

Notice that $H_k^{0,n}(TM, J)$ is isomorphic to $H^{0,n}(TM, J)$, for $J \in \mathcal{J}$, and for all $n, k \geq 0$ since the complex above is a resolution of the sheaf of holomorphic vector fields on M by fine sheaves. Hence we may apply the implicit function theorem for each k to show that the zero locus of N is a Banach manifold $\mathcal{J}_{k+1} \subset \mathcal{J}_{k+1}$ modeled on the Banach space of closed C^{k+1} -forms of type $(0, 1)$.

Now, let $\text{Diff}_{k+2}(M)$ denote the group of C^{k+2} -diffeomorphisms of M . The action map of $\text{Diff}_{k+2}(M)$ on \mathcal{J}_{k+1} is C^1 , so we may differentiate the action. Repeating the proof of the previous claim, the tangent space to the orbit of the group $\text{Diff}_{k+2}(M)$ at $J \in \mathcal{J} \cap \mathcal{J}_{k+1}$ can be identified with the space of exact C^{k+1} -forms of type $(0, 1)$. Therefore, the space of infinitesimal deformations of J inside \mathcal{J}_{k+1} can be naturally identified with the cohomology group $H^{0,1}(TM, J)$ (which is independent of k).

Compatible complex structures

Now, let (M, ω) be a symplectic 4-manifold, and let \mathcal{J}_ω be the contractible submanifold of \mathcal{J} consisting of almost complex structures on M compatible with ω . Given $J \in \mathcal{J}_\omega$, denote by

$$h_J(\cdot, \cdot) \equiv \omega(\cdot, J\cdot) - i\omega(\cdot, \cdot)$$

the hermitian metric on TM induced by the pair (ω, J) . We may use h_J to identify $T_J\mathcal{J} = \Omega_J^{0,1}(TM)$ with the space $T_J^{0,2}(M) \equiv \Omega_J^{0,1}(M) \otimes \Omega_J^{0,1}(M)$ of complex $(0, 2)$ -tensors, via

$$\Omega_J^{0,1}(TM) \ni A(\cdot) \leftrightarrow \theta_A(\cdot, \cdot) \equiv h_J(A\cdot, \cdot) \in T_J^{0,2}.$$

Under this identification, the subspace $T_J\mathcal{J}_\omega = S\Omega_J^{0,1}(TM) \subset \Omega_J^{0,1}(TM) = T_J\mathcal{J}$ can be identified with the subspace of complex symmetric $(0, 2)$ -tensors $S_J^{0,2}(M) \subset T_J^{0,2}(M)$:

$$A \in T_J\mathcal{J}_\omega \Leftrightarrow AJ + JA = 0 \quad \text{and} \quad \omega(A\cdot, \cdot) = -\omega(\cdot, A\cdot) \Leftrightarrow \theta_A \in S_J^{0,2}(M).$$

The quotient may therefore be identified with the space of $(0, 2)$ -forms on M :

$$T_J\mathcal{J}/T_J\mathcal{J}_\omega = \Omega_J^{0,1}(TM)/S\Omega_J^{0,1}(TM) \cong T_J^{0,2}(M)/S_J^{0,2}(M) = \Omega_J^{0,2}(M).$$

Given a compatible integrable complex structure $J \in \mathcal{J}_\omega \subset \mathcal{J}_\omega$, consider the following sequence of chain complexes with exact columns, where the above identifications

are taken into account and commutativity of the lower left corner is proved in Appendix B:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & S\Omega_J^{0,1}(TM) & \xrightarrow{\bar{\partial}} & \Omega_J^{0,2}(TM) & \longrightarrow & 0 \\
 \downarrow & & \downarrow & & \downarrow & & \\
 \Omega^0(TM) & \xrightarrow{\bar{\partial}} & \Omega_J^{0,1}(TM) & \xrightarrow{\bar{\partial}} & \Omega_J^{0,2}(TM) & \longrightarrow & 0 \\
 \downarrow & & \downarrow & & \downarrow & & \\
 \Omega_J^{0,1}(M) & \xrightarrow{\bar{\partial}} & \Omega_J^{0,2}(M) & \xrightarrow{\bar{\partial}} & 0 & \longrightarrow & 0
 \end{array}$$

The snake lemma yields a long exact sequence of holomorphic cohomology groups:

$$\begin{aligned}
 0 \longrightarrow H_J^0(TM) \longrightarrow cl\Omega_J^{0,1}(M) \xrightarrow{\delta} clS\Omega_J^{0,1}(TM) \longrightarrow H_J^{0,1}(TM) \\
 \longrightarrow H_J^{0,2}(M) \longrightarrow SH_J^{0,2}(TM) \longrightarrow H_J^{0,2}(TM) \longrightarrow 0,
 \end{aligned}$$

where $cl\Omega_J^{0,1}(M)$ and $clS\Omega_J^{0,1}(TM)$ denote the closed $(0, 1)$ -forms in the respective spaces and the symmetric cohomology $SH_J^{0,2}(TM)$ is given by the cokernel of $\bar{\partial}$ restricted to the symmetric $(0, 2)$ -tensors $S\Omega_J^{0,1}(TM)$. A geometric interpretation of the maps in the above long exact sequence can be given as follows.

By analyzing the Nijenhuis tensor and its covariant derivative as before, one sees that \mathcal{F}_ω is a submanifold of \mathcal{F}_ω in the neighborhood of $J \in \mathcal{F}_\omega$ if the symmetric cohomology group $SH_J^{0,2}(TM)$ vanishes. Under this assumption, $clS\Omega_J^{0,1}(TM)$ can be identified with the tangent space $T_J\mathcal{F}_\omega$, and the image of $\delta : cl\Omega_J^{0,1}(M) \rightarrow clS\Omega_J^{0,1}(TM)$ identifies the tangent space to the intersection of the orbit $(\text{Diff}(M) \cdot J)$ with \mathcal{F}_ω . Moreover, the image of the cokernel of δ in $H_J^{0,1}(TM)$ identifies the moduli of infinitesimal deformations of $J \in \mathcal{F}_\omega \subset \mathcal{F}$ that can be realized in an ω -compatible way. We then have the following theorem.

THEOREM 2.3

If (M, ω) is a symplectic 4-dimensional manifold, $J \in \mathcal{F}_\omega$ is a compatible integrable complex structure on (M, ω) , and the cohomology groups $H_J^{0,2}(TM)$ and $H_J^{0,2}(M)$ are zero, then \mathcal{F}_ω is a submanifold of \mathcal{F}_ω in the neighborhood of J , with tangent space

$$T_J\mathcal{F}_\omega = \ker \{ \bar{\partial} : S\Omega_J^{0,1}(TM) \rightarrow \Omega_J^{0,2}(TM) \} \subset S\Omega_J^{0,1}(TM) = T_J\mathcal{F}_\omega.$$

Moreover, the moduli space of infinitesimal compatible deformations of J in \mathcal{F}_ω coincides with the moduli space of infinitesimal deformations of J in \mathcal{F} , that is, it is

given by $H_j^{0,1}(TM)$ and

$$T_J \mathcal{F}_\omega \cong T_J((\text{Diff}(M) \cdot J) \cap \mathcal{F}_\omega) \oplus H_j^{0,1}(TM).$$

Remark 2.4

If $H^1(M, \mathbb{R}) = 0$, then $c\ell\Omega_j^{0,1}(M) = \bar{\delta}\Omega^0(M, \mathbb{C}) = \Omega^0(M, \mathbb{C})/\mathbb{C}$ is naturally identified with the complexified Lie algebra of the symplectomorphism group $\text{Symp}(M, \omega)$. Moreover, the kernel of δ can be identified with the vector space of holomorphic vector fields on (M, J) . This agrees nicely with Donaldson’s formal picture, where

$$(\text{Diff}(M) \cdot J) \cap \mathcal{F}_\omega = (\text{Symp}(M, \omega)^\mathbb{C} \cdot J).$$

Diffeomorphic compatible complex structures

In this subsection, (M, ω) is an arbitrary symplectic manifold, not necessarily of dimension 4, and \mathcal{F}_ω denotes again the space of complex structures on M compatible with ω . We write

$$\text{Diff}_{[\omega]}(M) = \{\varphi \in \text{Diff}(M) \mid \varphi^*([\omega]) = [\omega] \in H^2(M; \mathbb{R})\} \tag{1}$$

for the subgroup of diffeomorphisms of M preserving the cohomology class of the symplectic form.

Let $J_0 \in \mathcal{F}_\omega$ denote a fixed complex structure compatible with ω . Define

$$U = \{J \in \mathcal{F}_\omega \mid J_0 = \varphi^* J \text{ for some } \varphi \in \text{Diff}_{[\omega]}(M)\} \subset \mathcal{F}_\omega,$$

$$\Omega = \{\eta \in \Omega^2(M) \mid d\eta = 0, [\eta] = [\omega] \text{ and } \eta \text{ is compatible with } J_0\}.$$

Ω is a contractible convex subset in $\Omega^2(M)$, and we want to describe the topology of U .

Define a map from Ω to the identity component $\text{Diff}_0(M)$ of the diffeomorphism group of M ,

$$\Psi : \Omega \rightarrow \text{Diff}_0(M),$$

as follows. Given $\eta \in \Omega$, let $\psi_t \in \text{Diff}_0(M)$ be the isotopy satisfying $\psi_t^*((1-t)\omega + t\eta) = \omega$ which is canonically determined by Moser’s method and the Riemannian metric given by (ω, J_0) . Then

$$\Psi(\eta) := \psi_1.$$

Note that, if $K = \text{Iso}(\omega, J_0)$ denotes the Kähler isometry group and $g \in K$, then we have $\psi_t(g^*\eta) = g\psi_t(\eta)$ and, in particular, $\Psi(g^*\eta) = g\Psi(\eta)$, $\forall \eta \in \Omega$. Moreover, $\Psi(\eta)^*(\eta) = \omega$, $\forall \eta \in \Omega$.

Denote by $\text{Hol}_{[\omega]}(J_0)$ the group of complex automorphisms of (M, J_0) that preserve the cohomology class $[\omega] \in H^2(M; \mathbb{R})$.

PROPOSITION 2.5

The map $\mu : \text{Symp}(M, \omega) \times \Omega \rightarrow U$ defined by

$$\mu(\phi, \eta) = (\phi^{-1})^*\Psi(\eta)^*J_0$$

is a principal $\text{Hol}_{[\omega]}(J_0)$ -bundle.

Proof

Suppose that $J \in U$, and let $\varphi \in \text{Diff}_{[\omega]}(M)$ be such that $\varphi^*(J) = J_0$. Setting $\eta = \varphi^*\omega$, we derive the following:

$$\begin{aligned} J_0^*\eta &= J_0^*\varphi^*\omega = (d\varphi \circ J_0)^*\omega = (J \circ d\varphi)^*\omega = \varphi^*J^*\omega = \varphi^*\omega = \eta, \\ \eta(X, J_0X) &= \omega((d\varphi)X, (d\varphi)J_0X) = \omega((d\varphi)X, J(d\varphi)X) > 0. \end{aligned}$$

So we conclude that η is compatible with J_0 . Since $[\eta] = [\omega]$, we see that $\eta \in \Omega$. Moreover, $\varphi\Psi(\eta) \in \text{Symp}(M, \omega)$ since $(\varphi\Psi(\eta))^*\omega = \Psi(\eta)^*\eta = \omega$, and we have

$$\mu(\varphi\Psi(\eta), \eta) = (\Psi(\eta)^{-1}\varphi^{-1})^*\Psi(\eta)^*J_0 = (\varphi^{-1})^*J_0 = J.$$

Hence, the map μ is surjective.

Now, given $J \in U$ and an element $(\phi, \eta) \in \mu^{-1}(J)$, consider $\varphi = \phi\Psi(\eta)^{-1}$. Then $\varphi \in \text{Diff}_{[\omega]}(M)$ and

$$\mu(\phi, \eta) = (\phi^{-1})^*\Psi(\eta)^*J_0 = J \Rightarrow J_0 = \varphi^*J.$$

Conversely, given $\varphi \in \text{Diff}_{[\omega]}(M)$ such that $J_0 = \varphi^*J$, we have that $(\varphi\Psi(\eta), \eta) \in \mu^{-1}(J)$, where $\eta = \varphi^*\omega$. Hence,

$$\mu^{-1}(J) \cong \{\varphi \in \text{Diff}_{[\omega]}(M) \mid J_0 = \varphi^*J\};$$

that is, the fibers of the map μ are torsors on the group $\text{Hol}_{[\omega]}(J_0)$. In fact, defining a right action of $\text{Hol}_{[\omega]}(J_0)$ on $\text{Symp}(M, \omega) \times \Omega$ by

$$(\phi, \eta) \cdot \varphi = (\phi\Psi(\eta)^{-1}\varphi\Psi(\varphi^*(\eta)), \varphi^*(\eta)),$$

we see that the fibers of the map μ are free orbits of this action. \square

COROLLARY 2.6

If $J_0 \in \mathcal{I}_\omega$ is such that the inclusion $\text{Iso}(\omega, J_0) \hookrightarrow \text{Hol}_{[\omega]}(J_0)$ is a weak homotopy equivalence, then the inclusion of the $\text{Symp}(M, \omega)$ -orbit of J_0 in U , that is,

$$\text{Symp}(M, \omega) / \text{Iso}(\omega, J_0) \hookrightarrow U,$$

is also a weak homotopy equivalence.

Proof

Indeed, we have

$$\text{Symp}(M, \omega) \xrightarrow{\sim} \text{Symp}(M, \omega) \times \{\omega\} \subset \text{Symp}(M, \omega) \times \Omega \xrightarrow{\mu} U,$$

which induces

$$\text{Symp}(M, \omega) / \text{Iso}(\omega, J_0) \xrightarrow{\sim} (\text{Symp}(M, \omega) \times \Omega) / \text{Iso}(\omega, J_0) \xrightarrow{\bar{\mu}} U,$$

where the fiber of $\bar{\mu}$ is weakly contractible by assumption. □

Remark 2.7

According to Calabi [C1], a source of examples for Corollary 2.6 is J_0 's determining extremal Kähler metrics, at least on manifolds where $\text{Hol}(J_0)$ and $\text{Iso}(\omega, J_0)$ are both connected.

Transversality

In this subsection, we study transversality properties of certain strata of compatible almost complex structures on a compact symplectic manifold (M, ω) , with respect to the inclusion $\mathcal{I}_\omega \subset \mathcal{J}_\omega$. These strata are characterized by the existence of certain pseudoholomorphic curves.

Recall that if (M, J) is an almost complex manifold and (Σ, j) is a Riemann surface, then a map $u \in \text{Map}(\Sigma, M)$ is called a *J-holomorphic curve* if

$$J \circ du = du \circ j,$$

and it is called a *simple J-holomorphic curve* if, in addition, it is not multiply covered (see [MS]).

Given a compact symplectic manifold (M, ω) , a compact Riemann surface (Σ, j) , and a homology class $A \in H_2(M, \mathbb{Z})$, consider the space

$$\mathcal{M}(A, \Sigma) = \{(u, J) \in \text{Map}(\Sigma, M) \times \mathcal{J}_\omega : u \text{ is a simple } J\text{-holomorphic curve with } u_*([\Sigma]) = A\}.$$

Denote by $\Omega^{0,1}(\Sigma, TM)$ the vector bundle over $\text{Map}(\Sigma, M) \times \mathcal{J}_\omega$ whose fiber over (u, J) is

$$\Omega_j^{0,1}(\Sigma, u^*TM) \equiv J \text{ antilinear 1-forms on } \Sigma \text{ with values in } u^*TM.$$

This vector bundle has a natural section, denoted by $\bar{\partial}$, given at $(u, J) \in \text{Map}(\Sigma, M) \times \mathcal{J}_\omega$ by

$$\bar{\partial}_{(u,J)} \equiv \bar{\partial}_J(u) \equiv \frac{1}{2}(du + J \circ du \circ j) \in \Omega_j^{0,1}(\Sigma, u^*TM).$$

The space $\mathcal{M}(A, \Sigma)$ is the zero set of this section.

The tangent space to $\text{Map}(\Sigma, M) \times \mathcal{J}_\omega$ at (u, J) is given by

$$T_{(u,J)}(\text{Map}(\Sigma, M) \times \mathcal{J}_\omega) = T_u \text{Map}(\Sigma, M) \oplus T_J \mathcal{J}_\omega = \Omega^0(\Sigma, u^*TM) \oplus S\Omega_j^{0,1}(TM).$$

At $(u, J) \in \bar{\partial}^{-1}(0) = \mathcal{M}(A, \Sigma)$, the vertical component of the derivative of the section $\bar{\partial}$,

$$D(\bar{\partial})_{(u,J)} : \Omega^0(\Sigma, u^*TM) \oplus S\Omega_j^{0,1}(TM) \rightarrow \Omega_j^{0,1}(\Sigma, u^*TM),$$

is surjective and given by

$$D(\bar{\partial})_{(u,J)}(\xi, \alpha) = \bar{\partial}_J(\xi) + u^*(\alpha),$$

where $\bar{\partial}_J(\xi)$ is given as in Definition A.2 of Appendix A (see [MS, Remark 3.1.2, Proposition 3.2.1]). Hence, the following holds.

PROPOSITION 2.8

$\mathcal{M}(A, \Sigma)$ is an infinite-dimensional submanifold of $\text{Map}(\Sigma, M) \times \mathcal{J}_\omega$, with tangent space given by

$$T_{(u,J)}\mathcal{M}(A, \Sigma) = \{(\xi, \alpha) \in \Omega^0(\Sigma, u^*TM) \oplus S\Omega_j^{0,1}(TM) : \bar{\partial}_J(\xi) + u^*(\alpha) = 0\}.$$

The image of the projection

$$\pi : \mathcal{M}(A, \Sigma) \rightarrow \mathcal{J}_\omega, \quad (u, J) \mapsto J$$

defines a subset $U_A \subset \mathcal{J}_\omega$ characterized by

$$J \in U_A \Leftrightarrow A \in H_2(M, \mathbb{Z}) \text{ can be represented by a simple } J\text{-holomorphic curve with domain } (\Sigma, j).$$

We now want to find conditions ensuring that the image U_A of π is transversal to $\mathcal{F}_\omega \subset \mathcal{F}_\omega$ at points $(u, J) \in \mathcal{M}(A, \Sigma)$ with $J \in \mathcal{F}_\omega$. It follows from Proposition 2.8 that

$$\pi_*(T_{(u,J)}\mathcal{M}(A, \Sigma)) = \{\alpha \in S\Omega_J^{0,1}(TM) : [u^*\alpha] = 0 \in H_J^{0,1}(\Sigma, u^*TM)\}.$$

Assume that the restriction map

$$u^* : cLS\Omega_J^{0,1}(TM) \rightarrow H_J^{0,1}(\Sigma, u^*TM)$$

is surjective. Then, given any $\gamma \in T_J\mathcal{F}_\omega = S\Omega_J^{0,1}(TM)$, there exists $\beta \in T_J\mathcal{F}_\omega = cLS\Omega_J^{0,1}(TM)$ such that $(\gamma - \beta) \in \pi_*(T_{(u,J)}\mathcal{M}(A, \Sigma))$. Combining this with the reasoning leading to Theorem 2.3, one gets the following result.

THEOREM 2.9

Let $(M, \omega, J \in \mathcal{F}_\omega)$ be a Kähler 4-manifold such that the cohomology groups $H_J^{0,2}(M)$ and $H_J^{0,2}(TM)$ are zero. Suppose that $(u, J) \in \mathcal{M}(A, \Sigma)$ is such that $u^ : H_J^{0,1}(TM) \rightarrow H_J^{0,1}(u^*(TM))$ is an isomorphism. Then $\pi : \mathcal{M}(A, \Sigma) \rightarrow \mathcal{F}_\omega$ is transversal at (u, J) to $\mathcal{F}_\omega \subset \mathcal{F}_\omega$, and the infinitesimal complement to the image U_A of π in a neighborhood of J can be identified with the moduli space of infinitesimal deformations $H_J^{0,1}(TM)$.*

3. Compatible complex structures on rational ruled surfaces

In this section, we describe the stratification on the space \mathcal{F}_ω of compatible almost complex structures on a rational ruled surface, previously studied in [A], [AM], and [Mc1], and we use results from Section 2 to show that it induces an analogous stratification on the corresponding space \mathcal{F}_ω of compatible complex structures. We then conclude with the proof of Theorem 1.1.

In the rest of the article, we denote by (M, ω_λ) the symplectic manifolds $S^2 \times S^2$, with the split symplectic form

$$\omega_\lambda = \lambda\sigma \oplus \sigma, \tag{2}$$

with $1 \leq \lambda \in \mathbb{R}$, as well as the nontrivial bundle $S^2 \tilde{\times} S^2$ with a symplectic form

$$\omega_\lambda \text{ satisfying } [\omega_\lambda](E) = \lambda, [\omega_\lambda](F) = 1, \tag{3}$$

with $0 < \lambda \in \mathbb{R}$, where F is the homology class of a fiber and E is the homology class of the exceptional divisor under the natural identification of $S^2 \tilde{\times} S^2$ with $\mathbb{P}^2 \# \mathbb{P}^2$.

We refer to the case $M = S^2 \times S^2$ as the *untwisted case* and to $M = S^2 \tilde{\times} S^2$ as the *twisted case*. We identify $H_2(S^2 \times S^2; \mathbb{Z})$ with $\mathbb{Z} \times \mathbb{Z}$ in the standard way and

$H_2(S^2 \tilde{\times} S^2; \mathbb{Z})$ with $\mathbb{Z} \times \mathbb{Z}$ via

$$H_2(S^2 \tilde{\times} S^2; \mathbb{Z}) \ni mE + nF \mapsto (m, n) \in \mathbb{Z} \times \mathbb{Z}. \tag{4}$$

Finally, we write \mathcal{J}_λ for the contractible space of almost complex structures on M compatible with ω_λ and G_λ for the symplectomorphism group of (M, ω_λ) .

Compatible almost complex structures

The following theorem plays a fundamental role in the results obtained in [A] and [AM] regarding the topology of G_λ . It also plays a fundamental role here. The most technical point, listed as (v) in the statement, was proved in [Mc1, Theorem 1.2] using gluing techniques for pseudoholomorphic spheres.

THEOREM 3.1

Let (M, ω_λ) be one of the symplectic manifolds defined above (see (2) and (3)). There is a stratification of the contractible space \mathcal{J}_λ of compatible almost complex structures of the form

$$\mathcal{J}_\lambda = U_0 \sqcup U_1 \sqcup \cdots \sqcup U_\ell,$$

with $\ell \in \mathbb{N}_0$ such that $\ell < \lambda \leq \ell + 1$ satisfying the following.

(i) We have

$$U_k \equiv \left\{ J \in \mathcal{J}_\lambda : (1, -k) \in H_2(M; \mathbb{Z}) \text{ is represented by a } J\text{-holomorphic sphere} \right\}.$$

(ii) U_0 is open and dense in \mathcal{J}_λ . For $k \geq 1$, U_k has codimension $4k - 2$ in \mathcal{J}_λ in the untwisted case and codimension $4k$ in the twisted case.

(iii) We have $\overline{U_k} = U_k \sqcup U_{k+1} \sqcup \cdots \sqcup U_\ell$.

(iv) Given a compatible almost complex structure $J_k \in U_k$ with isometry group

$$\text{Iso}(\omega_\lambda, J_k) \cong \begin{cases} \mathbb{Z}/2 \times (SO(3) \times SO(3)) & \text{if } \lambda = 1 \text{ and } k = 0 \text{ in the} \\ & \text{untwisted case,} \\ SO(3) \times SO(3) & \text{if } \lambda > 1 \text{ and } k = 0 \text{ in the} \\ & \text{untwisted case,} \\ S^1 \times SO(3) & \text{if } k \geq 1 \text{ in the} \\ & \text{untwisted case,} \\ U(2) & \text{in the twisted case,} \end{cases}$$

then the inclusion

$$(G_\lambda/\text{Iso}(\omega_\lambda, J_k)) \longrightarrow U_k, \quad [\psi] \longmapsto \psi_*(J_k)$$

is a weak homotopy equivalence.

- (v) Each U_k has a tubular neighborhood $NU_k \subset \mathcal{J}_\lambda$ which fibers over U_k as a ball bundle.

Remark 3.2

In [AM, Section 2], such $(S^2 \times S^2, \omega_\lambda, J_k)$ and $(S^2 \tilde{\times} S^2, \omega_\lambda, J_k)$ were explicitly constructed as Kähler reductions of \mathbb{C}^4 . This is reviewed in Section 4.

Compatible complex structures

Our goal now is to show that an analogous theorem holds for the space $\mathcal{J}_\lambda \subset \mathcal{J}_\lambda$ of compatible integrable complex structures on (M, ω_λ) .

For each $k \in \{0, 1, \dots, \ell\}$, with $\ell \in \mathbb{N}_0$ such that $\ell < \lambda \leq \ell + 1$, define

$$V_k \equiv U_k \cap \mathcal{J}_\lambda = \left\{ J \in \mathcal{J}_\lambda : (1, -k) \in H_2(M; \mathbb{Z}) \text{ is represented by a } J\text{-holomorphic sphere} \right\}.$$

It follows from standard complex geometry (see [BHPV, Proposition V.4.3(i)] or [Ca]) that any $J \in V_k$ is complex isomorphic to the Hirzebruch surface

$$F_n = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-n)) \quad \text{with } n = 2k \text{ if } M = S^2 \times S^2 \text{ and } n = 2k + 1 \text{ if } M = S^2 \tilde{\times} S^2,$$

by a diffeomorphism of M that acts as the identity in homology. (Here, $\mathcal{O}(-1)$ denotes the tautological line bundle over \mathbb{P}^1 and $\mathbb{P}(E)$ denotes the projectivization of a vector bundle E .) Moreover, Calabi proved in [C1] that there is a complex structure $J_k \in V_k$, unique up to the action of G_λ , for which $g_{\lambda,k} \equiv \omega_\lambda(\cdot, J_k \cdot)$ is an extremal Kähler metric, with Kähler isometry group $K_k \equiv \text{Iso}(\omega_\lambda, J_k)$ as in Theorem 3.1. These two facts together imply that

$$V_k = \left\{ J \in \mathcal{J}_\lambda \mid J_k = \varphi^* J \text{ for some } \varphi \in \text{Diff}_{[\omega_\lambda]}(M) \right\} \subset \mathcal{J}_\lambda.$$

THEOREM 3.3

The inclusions

$$G_\lambda/K_k \hookrightarrow V_k \hookrightarrow U_k$$

are weak homotopy equivalences.

Proof

Corollary 2.6 says that the left map is a weak homotopy equivalence, while Theorem 3.1(iv) says that the composite is a weak homotopy equivalence. It follows that the map $V_k \hookrightarrow U_k$ is also a weak homotopy equivalence. \square

Any Hirzebruch surface F_n satisfies $H^{0,2}(F_n) = H^{0,2}(TF_n) = 0$ (see [Ko]). Hence, it follows from Theorem 2.3 that \mathcal{J}_λ is an infinite-dimensional submanifold of \mathcal{J}_λ . Since U_0 is open in \mathcal{J}_λ , we have that V_0 is also open in \mathcal{J}_λ .

For each $k \in \{1, \dots, \ell\}$, consider the space

$$\mathcal{M}_k = \{(u, J) \in \text{Map}(S^2, M) \times \mathcal{J}_\lambda : u \text{ is a simple } J\text{-holomorphic sphere with } u_*([S^2]) = (1, -k) \in H_2(M; \mathbb{Z})\},$$

which by Proposition 2.8 is an infinite-dimensional submanifold of $\text{Map}(S^2, M) \times \mathcal{J}_\lambda$. Positivity of intersections and the adjunction inequality for pseudoholomorphic curves in almost complex 4-manifolds (see [MS, Theorems 2.6.3, 2.6.4]) imply that

- if $(u, J) \in \mathcal{M}_k$, then the simple J -holomorphic map $u : S^2 \rightarrow M$ is an embedding;
- if $(u_1, J), (u_2, J) \in \mathcal{M}_k$, then the simple J -holomorphic maps $u_1, u_2 : S^2 \rightarrow M$ have exactly the same image; that is, they differ only by a holomorphic reparametrization of S^2 given by an element in $PSL(2, \mathbb{C})$.

This means that the projection

$$\pi : \mathcal{M}_k \rightarrow U_k \subset \mathcal{J}_\lambda, \quad (u, J) \mapsto J$$

is a principal $PSL(2, \mathbb{C})$ -bundle map over U_k .

PROPOSITION 3.4

For any $(u, J) \in \mathcal{M}_k$, the map $u^* : H^{0,1}(TF_n) \rightarrow H^{0,1}(u^*(TF_n))$ is an isomorphism, where $n = 2k$ if $M = S^2 \times S^2$ and $n = 2k + 1$ if $M = S^2 \tilde{\times} S^2$.

Proof

Recall that $F_n = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-n))$. The inclusion map $u : \mathbb{P}^1 \rightarrow F_n$ corresponds to the zero section $\mathbb{P}(0 \oplus \mathcal{O}(-n))$. Let $v : \mathbb{P}^1 \rightarrow F_n$ denote the section at infinity: $\mathbb{P}(\infty \oplus \mathcal{O}(-n))$. Let $i : Z(0) \subset F_n$ be the complement of u , and let $j : Z(\infty) \subset F_n$ be the complement of v . Notice that the space $Z(0)$ is equivalent to the total space of the bundle $\mathcal{O}(n)$ over v , and $Z(\infty)$ is equivalent to $\mathcal{O}(-n)$ over u . We have a short exact sequence of sheaves of \mathcal{O}_{F_n} -modules:

$$0 \rightarrow \mathcal{O}_{F_n} \rightarrow j_*j^*\mathcal{O}_{F_n} \rightarrow v_*v^!\mathcal{O}_{F_n} \rightarrow 0,$$

where $v^!$ is a functor from \mathcal{O}_{F_n} -modules to $\mathcal{O}_{\mathbb{P}^1}$ -modules such that the stalk of $v^!\mathcal{S}$ at $x \in \mathbb{P}^1$ (identified with the image of v) is given by the quotient

$$\mathcal{S}_x \rightarrow (j_*j^*\mathcal{S})_x \rightarrow (v^!\mathcal{S})_x \rightarrow 0.$$

Hence, $v^!\mathcal{S}$ may be seen as the higher residues along the normal. Identifying $Z(0)$ with $\mathcal{O}(n)$, it follows that $v^!\mathcal{O}_{F_n} = \text{Sym}_+\mathcal{O}(n)$, where $\text{Sym}_+\mathcal{S}$ stands for the augmentation ideal in the symmetric algebra on \mathcal{S} . We may now tensor the above short exact sequence with TF_n to get the following:

$$0 \rightarrow \mathcal{O}(TF_n) \rightarrow j_*j^*\mathcal{O}(TF_n) \rightarrow v_*v^!\mathcal{O}(TF_n) \rightarrow 0.$$

In cohomology, we get the following exact sequence:

$$\dots \rightarrow H^1(TF_n) \rightarrow H^1(Z(\infty), j^*TF_n) \rightarrow H^1(\mathbb{P}^1, v^!TF_n) \rightarrow \dots$$

Since $v^!TF_n$ is the bundle $(\mathcal{O}(2) \oplus \mathcal{O}(n)) \otimes \text{Sym}_+\mathcal{O}(n)$, the last term in the above sequence is trivial. Therefore, the restriction map

$$j^* : H^1(TF_n) \rightarrow H^1(Z(\infty), j^*TF_n)$$

is an epimorphism. Now, recall that $Z(\infty)$ is the total space of the bundle $\mathcal{O}(-n)$ over u . Let $\pi : Z(\infty) \rightarrow \mathbb{P}^1$ be the projection map for this bundle. We get an isomorphism

$$\pi_* : H^*(Z(\infty), j^*TF_n) \rightarrow H^*(\mathbb{P}^1, \pi_*j^*TF_n) = H^*(\mathbb{P}^1, u^*TF_n \otimes \text{Sym } \mathcal{O}(n)).$$

The long exact sequence in cohomology now gives us an epimorphism:

$$u^* : H^1(Z(\infty), j^*TF_n) \rightarrow H^1(\mathbb{P}^1, u^*TF_n \otimes \text{Sym } \mathcal{O}(n)) \rightarrow H^1(\mathbb{P}^1, u^*TF_n).$$

Composing the above sequence of epimorphisms, we see that the required map u^* in the statement of the proposition is an epimorphism. Finally, notice that $u^*TF_n = \mathcal{O}(2) \oplus \mathcal{O}(-n)$; hence, the dimensions of $H^1(\mathbb{P}^1, u^*TF_n)$ and $H^1(TF_n)$ both equal $n - 1$ (see, e.g., [Ko, Example 6.2(b)(4), p. 309]). The proof follows. \square

Remark 3.5

See also [ALP, Lemma A.8] for a generalization of Proposition 3.4 to blow-ups of a rational ruled surface.

It follows from Theorem 2.9 that, for $k \in \{1, \dots, \ell\}$, each strata $U_k \subset \mathcal{J}_\lambda$ is transversal to $\mathcal{S}_\lambda \subset \mathcal{J}_\lambda$. Hence, the stratification of \mathcal{J}_λ induces by intersection a stratification of \mathcal{S}_λ of the form

$$\mathcal{S}_\lambda = V_0 \sqcup V_1 \sqcup \dots \sqcup V_\ell,$$

which satisfies the direct analogues of items (i)–(iv) in Theorem 3.1.

Since each stratum V_k , $k \in \{1, \dots, \ell\}$, is the transversal intersection of U_k and \mathcal{J}_λ , the tubular neighborhood $NU_k \subset \mathcal{J}_\lambda$ of Theorem 3.1(v) gives rise to a tubular neighborhood $NV_k \equiv NU_k \cap \mathcal{J}_\lambda$ of V_k in \mathcal{J}_λ , which fibers over V_k as a ball bundle. By Theorem 2.9, each of these balls can be identified with a neighborhood of zero in

$$H^{0,1}(TF_n) \cong \mathbb{C}^{n-1}$$

(see [Ko, Example 6.2(b)(4), p. 309]).

Proof of Theorem 1.1

It follows from the results stated in the previous two subsections that the inclusion $\mathcal{J}_\lambda \hookrightarrow \mathcal{J}$ is transversal to the stratification

$$\mathcal{J} = U_0 \sqcup U_1 \sqcup \dots \sqcup U_\ell.$$

By Theorem 3.3, the induced stratification

$$\mathcal{J}_\lambda = V_0 \sqcup V_1 \sqcup \dots \sqcup V_\ell$$

is such that the inclusions

$$V_k \hookrightarrow U_k \quad \text{and} \quad NV_k \setminus V_k \hookrightarrow NU_k \setminus U_k$$

are weak homotopy equivalences. Writing $U_{0i} = U_0 \sqcup \dots \sqcup U_i$ and $V_{0i} = V_0 \sqcup \dots \sqcup V_i$, assume inductively that the inclusion

$$U_{0(i-1)} \rightarrow V_{0(i-1)}$$

is a weak equivalence. Then we have a map of excisive triads

$$(V_{0i}, NV_i, V_{0(i-1)}) \rightarrow (U_{0i}, NU_i, U_{0(i-1)})$$

which is a weak equivalence when restricted to NV_i , $V_{0(i-1)}$, and their intersection $NV_i \setminus V_i$. It follows (see, e.g., [M, p. 80]) that

$$V_{0i} \rightarrow U_{0i}$$

is a weak equivalence, and the result now follows by induction.

Since \mathcal{J}_λ is contractible, saying that $\mathcal{J}_\lambda \subset \mathcal{J}$ is a weak equivalence is, of course, equivalent to the statement that \mathcal{J}_λ is weakly contractible. \square

Remark 3.6

Note that the arguments above also apply to the space of complex structures tamed by a symplectic form. Thus Theorem 1.1 still holds, with the word “compatible” replaced by “tame”.

4. Deformation of Hirzebruch surfaces

We use the notation fixed in the introduction of Section 3. In that section, we described a stratification of the space \mathcal{S}_λ of compatible integrable complex structures on (M, ω_λ) which is equivariant with respect to the action of G_λ .

Our aim in this section is to compute the representations of the isotropy groups $\text{Iso}(\omega_\lambda, J_k)$ of Theorem 3.1 on the links of the strata V_k containing J_k . According to Theorem 2.9, the links can be identified with $H^{0,1}(TF_n)$, where $n = 2k$ in the untwisted case and $n = 2k + 1$ in the twisted case (cf. discussion in Section 3). Atiyah and Bott’s fixed point theorem [AB2, Theorem II.4.12] reduces this calculation to a calculation of the isotropy representations of the maximal torus of $\text{Iso}(\omega_\lambda, J_k)$ on the tangent spaces to its (four) fixed points in M . In order to obtain these, we first describe a construction of (ω_λ, J_k) by Kähler reduction (which appears, e.g., in [AM, Section 2.3]). This construction also gives us a hold on the Kähler isometry groups in Theorem 3.1 and allows us to relate the standard basis for their maximal tori to the one arising naturally from the Kähler reduction procedure. All of this is necessary in the next section when we compute the cohomology of the classifying space of the symplectomorphism groups.

Kähler reduction

Kähler reduction of \mathbb{C}^4 by the action of the 2-torus T_n^2 acting via

$$(s, t) \cdot (z_1, \dots, z_4) = (s^n t z_1, t z_2, s z_3, s z_4)$$

at the values

$$\begin{cases} (\lambda + \frac{n}{2}, 1) & \text{if } n \text{ is even,} \\ (\lambda + \frac{n+1}{2}, 1) & \text{if } n \text{ is odd} \end{cases}$$

produces Kähler manifolds that are symplectomorphic to (M, ω_λ) , where $M = S^2 \times S^2$ if n is even and $M = S^2 \tilde{\times} S^2$ if n is odd (see [AM, Section 2.3] for a detailed explanation of this). As complex manifolds, these reductions are isomorphic to Hirzebruch surfaces F_n with projection onto the base \mathbb{P}^1 given by $[(z_1, z_2, z_3, z_4)] \mapsto [z_3 : z_4]$.

The inclusion of the torus T_n^2 in the standard torus $T^4 \subset U(4)$ is given by the matrix

$$\begin{bmatrix} n & 1 \\ 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}.$$

The connected component of the Kähler isometry group of F_n is

$$K(n) = Z_{U(4)}(T_n^2)/T_n^2 = \begin{cases} (U(2) \times U(2))/T_0^2 & \text{if } n = 0, \\ (T^2 \times U(2))/T_n^2 & \text{if } n \geq 1, \end{cases}$$

and so

$$K(n) = \begin{cases} SO(3) \times SO(3) & \text{if } n = 0 \text{ in the untwisted case,} \\ S^1 \times SO(3) & \text{if } n > 0 \text{ is even,} \\ U(2) & \text{otherwise.} \end{cases} \quad (5)$$

The maximal torus \mathbb{T} of $K(n)$ is T^4/T_n^2 . We identify it with T^2 via the composite

$$T^2 \xrightarrow{\phi} T^4 \rightarrow T^4/T_n^2 \quad (6)$$

with

$$\phi = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

We refer to this basis as the *moment map basis* for the maximal torus \mathbb{T} . We need to relate this basis for \mathbb{T} with the standard basis arising from the identification (5). First, note that

$$SO(3) = PU(2) = U(2)/\Delta(S^1), \quad (7)$$

and so there is a natural quotient map $U(2) \rightarrow SO(3)$. We take the standard torus inside $SO(3)$ to be the image of $U(1) \times 1 \subset U(2)$ under this quotient, and we make the obvious choice for basis of $T^2 \subset U(2)$.

For $n > 0$, we have $K(n) = U(2)_{z_3, z_4}/(\mathbb{Z}/n)$ which can be naturally identified with the groups described in (5) using the double cover $S^1 \times SU(2) \rightarrow U(2)$ and the canonical isomorphism $S^1/(\mathbb{Z}/k) \simeq S^1$. It is not hard to check that with respect to the standard bases described above and these identifications, the n -fold covering map $U(2)_{z_3, z_4} \rightarrow K(n)$ is given on maximal tori by the matrix

$$\begin{bmatrix} \frac{n+1}{2} & \frac{n-1}{2} \\ \frac{n-1}{2} & \frac{n+1}{2} \end{bmatrix} \text{ for } n \text{ odd,} \quad \text{and} \quad \begin{bmatrix} \frac{n}{2} & \frac{n}{2} \\ 1 & -1 \end{bmatrix} \text{ for } n > 0 \text{ even.} \quad (8)$$

LEMMA 4.1

The changes of basis from the moment map bases (6) to the standard bases for the maximal tori of $K(n)$ are given by the matrices

$$\begin{bmatrix} 1 & \frac{n+1}{2} \\ 1 & \frac{n-1}{2} \end{bmatrix} \text{ for } n \text{ odd,} \quad \begin{bmatrix} 1 & \frac{n}{2} \\ 0 & 1 \end{bmatrix} \text{ for } n > 0 \text{ even,} \quad \text{and} \quad \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \text{ for } n = 0.$$

Proof

Suppose that $n > 0$ is odd. The image of the circle $(0, n, 0, 0)$ in $K(n)$ equals the image of $(0, 0, 1, 1)$. This is the circle $(1, 1)$ in $U(2)_{z_3, z_4}$ which according to (8) is taken to the circle (n, n) in $K(n)$. Thus the first element of the moment map basis (6) is mapped to the standard diagonal in $K(n)$, and we get the first column of the first matrix in the statement. The other computations are similar. \square

Isotropy representations

We now want to compute the representation of $K(n)$ on $H^{0,1}(TF_n)$ which, according to Section 3, is the isotropy representation of $\text{Iso}(\omega_\lambda, J_k)$ on the link of the stratum $V_k \subset \mathcal{I}_\lambda$.

For this, we use the $K(n)$ -equivariant elliptic complex

$$\Omega^0(TF_n) \xrightarrow{\bar{\partial}} \Omega^{0,1}(TF_n) \xrightarrow{\bar{\partial}} \Omega^{0,2}(TF_n). \tag{9}$$

Since $H^{0,2}(TF_n) = 0$ (see [Ko, Example 6.2(b)(4), p. 309]), the index of this complex is the virtual representation

$$H^0(TF_n) - H^{0,1}(TF_n)$$

of $K(n)$. The Atiyah-Bott fixed point theorem (see [AB1, Theorem 4.12]) gives a formula for the character of this representation in terms of the characters of the isotropy representation of the maximal torus \mathbb{T} of $K(n)$ on the fibers of TF_n over the \mathbb{T} -fixed points. We start by determining the latter using the construction of F_n by Kähler reduction given in the previous subsection.

Using the basis (6), the moment map for the action of \mathbb{T} on F_n is given by the expression

$$[(z_1, \dots, z_4)] \mapsto \frac{1}{2}(|z_2|^2, |z_3|^2) \in \mathbb{R}^2 = \text{Lie}(\mathbb{T})^*,$$

and so its image is the moment polygon with vertices $A = (0, 0)$, $B = (1, 0)$, $C = (1, \mu)$, and $D = (0, \mu - n)$ in \mathbb{R}^2 , where

$$\mu = \begin{cases} \lambda + \frac{n}{2} & \text{if } n \text{ is even,} \\ \lambda + \frac{n+1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

The vertices of the moment polygon are the images of the four \mathbb{T} -fixed points which we also denote by A, B, C, D . The weights of the isotropy representation of the torus on the tangent space at a fixed point are determined by the primitive vectors in $\text{Hom}(\mathbb{T}, S^1) = \mathbb{Z}^2 \subset \mathbb{R}^2 = \text{Lie}(\mathbb{T})^*$ along the edges of the moment polygon which meet at the corresponding vertex. If we write x for the weight determined

by the vector $(1, 0)$ and y for the weight determined by the vector $(0, 1)$ and write the weights multiplicatively, then we see that the weights $w_i, i = 1, 2$, of the 2-dimensional isotropy representation at the fixed points are given by the following table:

	A	B	C	D	
w_1	x	$1/x$	$1/(xy^n)$	xy^n	(10)
w_2	y	y	$1/y$	$1/y$	

The character of the isotropy representation of \mathbb{T} on the tangent space at a fixed point is the sum of the two weights in the corresponding column. We can now prove the main result of this section.

THEOREM 4.2

Let $n > 1$. The representation of $K(n)$ on $H^{0,1}(TF_n)$ is given by

$$\begin{cases} \text{Det}^{-(n-3)/2} \otimes \text{Sym}^{n-2}(\mathbb{C}^2) & \text{if } n \text{ is odd,} \\ \text{Det} \otimes \text{Sym}^{n-2}(\mathbb{C}^2) & \text{if } n \text{ is even,} \end{cases}$$

where Det denotes the determinant representation of $U(2)$ if n is odd and the standard representation of the S^1 -factor if n is even; $\text{Sym}^k(\mathbb{C}^2)$ denotes the k th symmetric power of the defining representation of $U(2)$ if n is odd and the irreducible $(k+1)$ -dimensional representation of $SO(3)$ if n is even.

Proof

By [AB1, Theorem II.4.12], the topological index $I(n)$ of (9) is given by the following character* of \mathbb{T} :

$$I(n) = \sum \frac{w_1 + w_2}{(1 - 1/w_1)(1 - 1/w_2)} = \sum \frac{w_1 w_2 (w_1 + w_2)}{(1 - w_1)(1 - w_2)},$$

where the sum is taken over the four fixed points of \mathbb{T} and the weights are given by (10). Writing this in terms of the weights x, y yields

$$\begin{aligned} I(n) &= \frac{xy(x+y)}{(1-x)(1-y)} + \frac{y/x(1/x+y)}{(1-1/x)(1-y)} \\ &\quad + \frac{1/(xy^{n+1})(1/(xy^n)+1/y)}{(1-1/(xy^n))(1-1/y)} + \frac{xy^{n-1}(xy^n+1/y)}{(1-xy^n)(1-1/y)}, \end{aligned}$$

*Note that TX denotes the cotangent bundle of X in [AB1].

and writing this in linearly independent monomials, we obtain

$$\begin{aligned}
 I(0) &= 2 + y + \frac{1}{y} + x + \frac{1}{x}, & I(1) &= 2 + y + \frac{1}{y} + \frac{1}{xy}(1 + y), \\
 I(n) &= 2 + y + \frac{1}{y} + \frac{1}{xy^n}(1 + y + \dots + y^n) \\
 &\quad - xy(1 + y + \dots + y^{n-2}) \quad \text{for } n > 1.
 \end{aligned}$$

The negative terms are the negative of the character of the representation $H^{0,1}(TF_n)$, which thus has complex dimension $n - 1$ for $n > 0$, in accordance with the fact that $\dim_{\mathbb{C}} H^{0,1}(TF_n) = n - 1$ (see [Ko, Example 6.2(b)(4), p. 309]). The positive terms are the character of a representation of dimension $n + 5$, namely, the restriction to $K(n)$ of the adjoint representation of the complex automorphism group of F_n on its Lie algebra $H^0(TF_n)$.

Using Lemma 4.1, we can write the character of the representation $H^{0,1}(TF_n)$ in terms of the standard weights a and b of $\mathbb{T} \subset K(n)$, dual to the standard basis of the maximal torus. We have

$$a = \begin{cases} xy^{(n+1)/2} & \text{if } n > 0 \text{ is odd,} \\ xy^{n/2} & \text{if } n > 0 \text{ is even,} \end{cases} \quad b = \begin{cases} y^{(n-1)/2}x & \text{if } n > 0 \text{ is odd,} \\ y & \text{if } n > 0 \text{ is even.} \end{cases}$$

Hence for $n > 0$, the character of the representation of $K(n)$ on $H^{0,1}(TF_n)$ is given by

$$\begin{cases} (ab)^{-(n-3)/2}(b^{n-2} + ab^{n-1} + \dots + a^{n-2}) & \text{if } n > 1 \text{ is odd,} \\ a(b^{1-n/2} + \dots + 1 + \dots + b^{n/2-1}) & \text{if } n > 0 \text{ is even,} \end{cases} \tag{11}$$

which completes the proof. □

Remark 4.3

An alternate proof of Theorem 4.2 can be obtained by computing explicitly the action of the Lie algebra of global holomorphic vector fields $H^0(F_n; \Theta)$ on $H^1(F_n; \Theta)$ using explicit bases which can be found in [Ca, p. 19]. This approach also permits a better understanding of the normal links of the stratifications of \mathcal{S} and \mathcal{J} , a problem considered in [Mc1]. The induced stratification on the link is precisely the cone on the usual stratification of the appropriate \mathbb{P}^n by the secant varieties of the rational normal curve.*

*The authors thank Barbara Fantechi for bringing this stratification of \mathbb{P}^n to our attention.

5. The homotopy type of the symplectomorphism group

In this section, we apply the results of the previous sections to study the topology of the symplectomorphism groups of rational ruled surfaces.

Let G_λ denote the symplectomorphism group of the standard symplectic form ω_λ on $S^2 \times S^2$ or $S^2 \tilde{\times} S^2$, where $\lambda > 1$ in the untwisted case and $\lambda > 0$ in the twisted case.

If G is a topological group and X is a G -space, we write

$$X_{hG} = EG \times_G X \tag{12}$$

for the homotopy orbits (or Borel construction) of the G -action on X .

The homotopy decomposition of G_λ

In this section, we write $\mathcal{J}_\lambda = \mathcal{J}_{\omega_\lambda}$ and $\mathcal{I}_\lambda = \mathcal{I}_{\omega_\lambda}$. Let \mathcal{A}_λ denote the space of almost complex structures tamed by ω_λ , let $\mathcal{A}_\lambda^i \subset \mathcal{A}_\lambda$ denote the subspace of complex structures, and let $\Omega_{[\lambda]}$ denote the space of symplectic forms cohomologous to ω_λ .

We need to use the spaces of tame complex structures because the maps between the classifying spaces BG_λ for different values of λ are best understood in terms of these spaces (see (13); cf. [Mc2, Corollary 2.3]). Although maps between the G_λ were also defined in [AM, Section 4], the construction there produces only homotopy multiplicative maps and it is not apparent that they induce maps on the level of the classifying spaces.

We write $\mathcal{J}_{[\lambda]}, \mathcal{I}_{[\lambda]}, \mathcal{A}_{[\lambda]}, \mathcal{A}_{[\lambda]}^i$ for the analogous spaces where the symplectic form ω_λ is replaced with all symplectic forms in the cohomology class of ω_λ .

Let

$$\mathcal{K}_{[\lambda]} = \{(J, \omega) \in \mathcal{J}_{[\lambda]} \times \Omega_{[\lambda]} : J \text{ is compatible with } \omega\}.$$

$\mathcal{K}_{[\lambda]}^t$ denotes the analogous space with J tamed by ω , and $\mathcal{K}_{[\lambda]}^i$ and $\mathcal{K}_{[\lambda]}^{i,t}$ denote the analogous subspaces where J is required to be integrable.

The above spaces have a natural action of the subgroup of diffeomorphisms preserving the cohomology class $[\omega_\lambda]$ (cf. (1)). In our situation, this is the subgroup of diffeomorphisms inducing the identity on homology which we denote by $\text{Diff}_{[0]}$.

Finally, if S is a subset of the isomorphism classes of complex structures, we decorate the above spaces with a superscript S to indicate that the corresponding complex structures belong to S . In the case of not necessarily integrable complex structures, the superscript S indicates representability of the appropriate homology class by an embedded J -holomorphic curve.

PROPOSITION 5.1

Let S be a set of isomorphism classes of complex structures. The canonical maps

$$(\mathcal{J}_\lambda^S)_{hG_\lambda} \rightarrow (\mathcal{J}_{[\lambda]}^S)_{h\text{Diff}_{[0]}}$$

are weak equivalences. The same statement holds for tame and/or integrable complex structures.

Proof

We have a $\text{Diff}_{[0]}$ equivariant bundle

$$\mathcal{J}_\lambda^S \rightarrow \mathcal{K}_{[\lambda]}^S \xrightarrow{\pi_2} \Omega_{[\lambda]}.$$

As $\Omega_{[\lambda]} = \text{Diff}_{[0]} / G_\lambda$ by [LM, Theorem 2.4], we see that $E\text{Diff}_{[0]} \times_{\text{Diff}_{[0]}} \mathcal{K}_{[\lambda]}^S = E\text{Diff}_{[0]} \times_{G_\lambda} \mathcal{J}_\lambda^S$. The latter space is also a model for $(\mathcal{J}_\lambda^S)_{hG_\lambda}$; hence the inclusion

$$(\mathcal{J}_\lambda^S)_{hG_\lambda} \rightarrow (\mathcal{K}_{[\lambda]}^S)_{h\text{Diff}_{[0]}}$$

is a weak equivalence. We have a commutative diagram

$$\begin{array}{ccc} & & \mathcal{K}_{[\lambda]}^S \\ & \nearrow & \downarrow \pi_2 \\ \mathcal{J}_\lambda^S & \hookrightarrow & \mathcal{J}_{[\lambda]}^S \end{array}$$

and π_2 is a $\text{Diff}_{[0]}$ -equivariant map with contractible fibers, so the result follows from the homotopy invariance of homotopy orbits. □

Remark 5.2

Using the arguments in the proof of Theorem 1.1 (which corresponds to the case when S is the set of all complex structures), one can show that, in fact, all the inclusions between the four different spaces of complex structures associated to a symplectic form restricted to a set S of isomorphism classes induce weak equivalences. Hence, for each S , the eight spaces mentioned in the statement of Proposition 5.1 are all weakly equivalent. We do not need this, however.

LEMMA 5.3

Let S denote a set of isomorphism classes of complex structures on M . Then

- (i) $\mathcal{A}_{[\lambda]}^{i,S} = \mathcal{J}_{[\lambda]}^S$;
- (ii) $\mathcal{J}_{[\lambda]}^S$ is a finite union of $\text{Diff}_{[0]}$ -orbits, each containing one of the Kähler structures J_k of Theorem 3.1 constructed in Section 4.

Proof

Any complex structure $J \in \mathcal{A}_{[\lambda]}^i$ admits a J -holomorphic sphere in the homology class $(1, -k) \in H_2(M; \mathbb{Z})$ for some $k \in \{0, 1, \dots, \ell\}$ with $\ell < \lambda \leq \ell + 1$ as well as a J -holomorphic sphere representing the homology class $(0, 1)$. By standard results in complex geometry (see, e.g., [BHPV, Proposition V.4.3(i)]), it follows that (M, J) is complex isomorphic to the Hirzebruch surface F_{2k} or F_{2k+1} according to whether we are in the untwisted or twisted cases. Since these F_n admit a Kähler structure with symplectic form in the cohomology class $[\omega_\lambda]$ (constructed in Section 4), it follows that $J \in \mathcal{J}_{[\lambda]}$. This proves (i) and (ii). \square

Recall from [Mc2, Lemma 2.2] that $\mathcal{A}_{[\lambda]} \subset \mathcal{A}_{[\lambda+\epsilon]}$ for $\epsilon > 0$. The loop maps* $G_\lambda \rightarrow G_{\lambda+\epsilon}$ are defined by taking the homotopy orbits of $\text{Diff}_{[0]}$ along this inclusion:

$$BG_\lambda \simeq (\mathcal{A}_{[\lambda]})_{h\text{Diff}_{[0]}} \subset (\mathcal{A}_{[\lambda+\epsilon]})_{h\text{Diff}_{[0]}} \simeq BG_{\lambda+\epsilon}, \tag{13}$$

where the weak equivalences are a consequence of Proposition 5.1 in the case of tame complex structures.

Given Theorem 1.1 or, more precisely, Remark 3.6, and Proposition 5.1 (in the case of tame integrable complex structures), we may replace $\mathcal{A}_{[\mu]}$ in (13) by the subspaces $\mathcal{A}_{[\mu]}^i$ of integrable complex structures. By Lemma 5.3, this space does not change when μ varies in an interval of the form $[k, k + 1]$, and so we have the following result of McDuff.

PROPOSITION 5.4 ([Mc2, Theorem 1.4])

For $k < \lambda \leq \mu \leq k + 1$, the map $BG_\lambda \rightarrow BG_\mu$ is a weak equivalence.

We can now prove our main result concerning the homotopy type of BG_μ . First, we recall that the *homotopy pushout* of a diagram of spaces

$$X \xleftarrow{f} A \xrightarrow{g} Y$$

(or double mapping cylinder of f and g) is the quotient space

$$P = \left(X \coprod A \times [0, 1] \coprod Y \right) / \sim, \tag{14}$$

where \sim denotes the equivalence relation generated by $(a, 0) \sim f(a)$ and $(a, 1) \sim g(a)$. To compute the homology of the homotopy pushout, we have the Mayer-Vietoris

*There is a more geometric construction of homotopy multiplicative maps $G_\lambda \rightarrow G_{\lambda+\epsilon}$ in [AM, Section 4], but it is important for us that these are actually loop maps, that is, that they induce maps on the level of classifying spaces. This is not apparent from [AM, Section 4], which is why we use the construction in [Mc2].

sequence

$$\dots \rightarrow H_*(A) \xrightarrow{(f_*, g_*)} H_*(X) \oplus H_*(Y) \rightarrow H_*(P) \rightarrow \dots$$

obtained from the obvious decomposition of P and similarly for cohomology. We say that a square

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ g \downarrow & & \downarrow i \\ Y & \xrightarrow{j} & Q \end{array}$$

is a *homotopy pushout square* if if and jk are homotopic and the map $P \rightarrow Q$ canonically determined by such a homotopy is a weak equivalence.

As in Section 4, we write $K(n)$ for the connected component of the identity of the Kähler isometry group of the Hirzebruch surface F_n . Thus

$$K(n) = \begin{cases} SO(3) \times SO(3) & \text{if } n = 0 \text{ in the untwisted case,} \\ S^1 \times SO(3) & \text{if } n > 0 \text{ is even,} \\ U(2) & \text{otherwise.} \end{cases} \tag{15}$$

For the sake of simplifying the statement of the following theorem, in the untwisted case, for $0 < \lambda \leq 1$ we write G_λ for the connected component of the identity of the symplectomorphism group G_1 . By a theorem of Gromov [Gr, Theorem 0.3.C], the inclusion $K(0) \subset G_1$ is a weak equivalence and, in the twisted case, $K(1) \subset G_\lambda$ is a weak equivalence for $0 < \lambda \leq 1$.

THEOREM 5.5

Let $\lambda > 0$, and let ℓ be the integer such that $\ell < \lambda \leq \ell + 1$. There is a homotopy pushout square

$$\begin{array}{ccc} S_{hK(m)}^{2m-3} & \xrightarrow{\pi} & BK(m) \\ j \downarrow & & \downarrow \\ BG_\lambda & \xrightarrow{i} & BG_{\lambda+1} \end{array} \tag{16}$$

where

$$m = \begin{cases} 2\ell + 2 & \text{in the untwisted case,} \\ 2\ell + 3 & \text{in the twisted case,} \end{cases}$$

S^{2m-3} is the unit sphere of the representation of $K(m)$ described in Theorem 4.2, π is the canonical projection, and i is the map described in (13).

Proof

Consider the usual stratification of $\mathcal{F}_{\lambda+1}$:

$$\mathcal{F}_{\lambda+1} = V_0 \cup \dots \cup V_{\ell+1}.$$

Writing

$$V_{0k} = V_0 \cup \dots \cup V_k$$

and $NV_{\ell+1}$ for a tubular neighborhood of $V_{\ell+1}$, we have a homotopy pushout decomposition

$$\begin{array}{ccc} NV_{\ell+1} \setminus V_{\ell+1} & \longrightarrow & V_{\ell+1} \\ \downarrow & & \downarrow \\ V_{0\ell} & \longrightarrow & \mathcal{F}_{\lambda+1} \end{array} \tag{17}$$

Writing S for the set of Hirzebruch surfaces $\{F_i, \dots, F_{2\ell+i}\}$, where $i = 0$ in the untwisted case and $i = 1$ in the twisted case, Proposition 5.1 (in the case of compatible integrable complex structures) implies that the inclusion

$$(V_{0\ell})_{hG_{\lambda+1}} = (\mathcal{F}_{\lambda+1}^S)_{hG_{\lambda+1}} \subset (\mathcal{F}_{[\lambda+1]}^S)_{h\text{Diff}_{[0]}}$$

is a weak equivalence. By Lemma 5.3, we have

$$\mathcal{F}_{[\lambda+1]}^S = \mathcal{A}_{[\lambda+1]}^{i,S} = \mathcal{A}_{[\lambda]}^i,$$

and so another application of Proposition 5.1 to $\mathcal{F}_{\lambda+1}$ identifies the (homotopy class of the) $G_{\lambda+1}$ homotopy orbits of the bottom row in (17) with

$$(\mathcal{A}_{[\lambda]}^i)_{h\text{Diff}_{[0]}} \subset (\mathcal{A}_{[\lambda+1]}^i)_{h\text{Diff}_{[0]}}$$

which is exactly the map $i: BG_{\lambda} \rightarrow BG_{\lambda+1}$ in (13).

By Theorems 3.3 and 4.2, the projection $(NV_{\ell+1} \setminus V_{\ell+1}) \rightarrow V_{\ell+1}$ has the (weak) homotopy type of the projection $G_{\lambda+1} \times_{K(m)} S^{2m-3} \rightarrow G_{\lambda+1}/K(m)$ (see Proposition C.6 for more details).

The only reason why the theorem does not follow immediately by taking homotopy orbits of $G_{\lambda+1}$ on (17) is that $NV_{\ell+1}$ is not invariant under the $G_{\lambda+1}$ -action. Nevertheless, the slice theorem implies that for each compact set $W \subset G$ there is a smaller tubular neighborhood $NV'_{\ell+1}$ sent by the action of W into $NV_{\ell+1}$. This allows us to define an A_∞ -action of $G_{\lambda+1}$ on $NV_{\ell+1}$ which, as such, is equivalent to the left $G_{\lambda+1}$ -action on $G_{\lambda+1} \times_{K(m)} S^{2m-3}$ (see Appendix C, (A.1) for details). The result follows. □

We note the following immediate consequence of Theorem 5.5.

COROLLARY 5.6 ([Mc2, Theorem 1.4])

With m as in the statement of Theorem 5.5, for all $\mu > \lambda$ the map $BG_\lambda \rightarrow BG_\mu$ is $(2m - 3)$ -connected.

The cohomology groups of BG_λ

In the rest of the article, we use Theorem 5.5 to obtain results concerning the cohomology of BG_λ . First, we need to compute the effect of the map π in (16) on cohomology.

Observe that $B(S^1 \times S^1) = \mathbb{P}^\infty \times \mathbb{P}^\infty$, so a choice of basis for a 2-torus \mathbb{T} gives canonical generators $T_1, T_2 \in H^*(B\mathbb{T}; \mathbb{Z}) = \mathbb{Z}[T_1, T_2]$. Thus the standard bases for the maximal tori of the groups $K(n)$ introduced in Section 4 (see (7)) determine generators for $H^*(BK(n); R) = H^*(B\mathbb{T}; R)^{\mathbb{Z}/2}$, where $\mathbb{Z}/2$ denotes the Weyl group and $R = \mathbb{Z}[1/2]$ or \mathbb{Z} according to whether n is even or odd. This way we have

$$\begin{cases} H^*(BK(n); \mathbb{Z}) = \mathbb{Z}[A_n, X_n] & \text{if } n \text{ is odd,} \\ H^*(BK(n); \mathbb{Z}[\frac{1}{2}]) = \mathbb{Z}[\frac{1}{2}][A_n, X_n] & \text{if } n > 0 \text{ is even,} \end{cases}$$

where $|A_n| = 2$ and $|X_n| = 4$, while it is easy to see that

$$H^*(BK(n); \mathbb{Z}/2) = \mathbb{Z}/2[T, w_2, w_3] \quad \text{for } n > 0 \text{ even,}$$

with $|T| = |w_2| = 2$ and $|w_3| = 3$.

LEMMA 5.7

For $n > 1$, the Euler class e_n of the isotropy representation of $K(n)$ is given by

$$e_n = \begin{cases} \prod_{i=1}^{(n-1)/2} ((2i - 1)^2 X_n - i(i - 1) A_n^2) \in H^*(BK(n); \mathbb{Z}) & \text{if } n \text{ is odd,} \\ \prod_{i=1}^{n/2-1} (A_n^2 - i^2 X_n) \in H^*(BK(n); \mathbb{Z}[\frac{1}{2}]) & \text{if } n \text{ is even.} \end{cases}$$

With any coefficients R , the Euler class is a nonzero divisor in $H^*(BK(n); R)$.

Proof

The inclusion of the maximal torus

$$BS^1 \times BS^1 \rightarrow BK(n)$$

identifies $H^*(BK(n); R)$ with the ring of polynomials in $H^*(BS^1 \times BS^1; R) = R[T_1, T_2]$ invariant under the Weyl group, where $R = \mathbb{Z}$ or $\mathbb{Z}[1/2]$ according to whether n is odd or even. The Weyl group action switches T_1 and T_2 for n even and fixes T_1 and acts by -1 on T_2 for n odd.

If $\chi: S^1 \times S^1 \rightarrow S^1$ is the weight determined by $(k, l) \in \mathbb{Z}^2$, the Euler class of the line bundle

$$E(S^1 \times S^1) \times_{S^1 \times S^1} \mathbb{C} \rightarrow B(S^1 \times S^1),$$

where $S^1 \times S^1$ acts on \mathbb{C} via χ , is $kT_1 + lT_2 \in H^2(B(S^1 \times S^1); \mathbb{Z})$. Since the Euler class is multiplicative, this determines the Euler class of any representation of $K(n)$ once we know the character of the representation on the maximal torus. The formulas for the Euler class now follow from Theorem 4.2 or, more precisely, from the formulas (11) for the character of the isotropy representations on the maximal tori in terms of the standard weights.

For n odd, $H^*(BK(n); R)$ is a polynomial ring for any R , and so it suffices to check that $e_n \neq 0$. This is the case because $((2i - 1)^2, i(i - 1)) = 1$. For n even and any coefficients R , $H^*(BK(n); R)$ contains a polynomial ring $R[A_n]$ with $|A_n| = 2$ as a retract. Since the coefficient of A_n^{n-1} in the formula for e_n is 1, it follows that e_n is not a zero divisor. □

Proofs of Theorems 1.2 and 1.4

Consider the diagram (16). Lemma 5.7 says that given any coefficient ring R , the map $H^*(\pi; R)$ is the quotient map

$$H^*(BK(m); R) \rightarrow H^*(BK(m); R) / \langle e_m \rangle.$$

Thus the Mayer-Vietoris sequence associated to (16) splits into a short exact sequence

$$0 \rightarrow \langle e_m \rangle = \Sigma^{2m-2} H^*(BK(m); R) \rightarrow H^*(BG_{\lambda+1}; R) \rightarrow H^*(BG_{\lambda}; R) \rightarrow 0.$$

When R is a field, this splits; since the groups in the extension are finitely generated, it follows that the same is true over \mathbb{Z} . The statements of Theorems 1.2 and 1.4 follow by induction. □

Computation of the rational cohomology ring of BG_λ

Let FDiff denote the group of fiber-preserving diffeomorphisms of an S^2 -bundle over S^2 inducing the identity on homology. It is clear from the context whether the bundle in question is trivial or not. To compute the cohomology ring of BG_λ , we make use of the loop maps

$$G_\lambda \rightarrow \text{FDiff}$$

defined in [Mc2].

Writing $\mathcal{A}_{[\infty]}$ for the space of almost complex structures compatible with some symplectic form, McDuff shows (see [Mc2, Proof of Prop 1.1]) that the space $\mathcal{A}_{[\infty]}$ is $\text{Diff}_{[0]}$ -equivariantly weakly equivalent to the space $\text{Diff}_{[0]} / \text{FDiff}$ and, hence, taking homotopy orbits we have canonical maps as in (13),

$$BG_\lambda \simeq (\mathcal{A}_{[\lambda]})_{h\text{Diff}_{[0]}} \subset (\mathcal{A}_{[\infty]})_{h\text{Diff}_{[0]}} \simeq B\text{FDiff}.$$

By Theorem 1.1, we can replace $\mathcal{A}_{[\infty]}$ with its subspace $\mathcal{J}_{[\infty]}$ of complex structures. We begin by noting the following consequence of Theorem 5.5.

PROPOSITION 5.8

For all λ , the map $BG_\lambda \rightarrow B\text{FDiff}$ induces a surjection on cohomology with any coefficients. Moreover,

$$H^*(B\text{FDiff}) = \lim_{\lambda} H^*(BG_\lambda).$$

Proof

By Lemma 5.7 (cf. the proofs of Theorems 1.2 and 1.4), the map $H^*(BG_\mu) \rightarrow H^*(BG_\lambda)$ is surjective for all $\lambda < \mu$. Since

$$B\text{FDiff} = \text{hocolim}_\lambda BG_\lambda,$$

Lemma 5.6 implies that the connectivity of the maps $BG_\mu \rightarrow B\text{FDiff}$ tends to ∞ with μ . This proves the first statement. The second statement is obvious. \square

The problem that must be overcome in order to compute the cohomology ring of G_λ is that of understanding the effect of the map j in (16) on cohomology since, by Lemma 5.7,

$$\begin{array}{ccc}
 H^* S_{hK(m)}^{2m-3} & \xleftarrow{\pi^*} & H^* BK(m) \\
 \uparrow j^* & & \uparrow \\
 H^* BG_\lambda & \xleftarrow{i^*} & H^* BG_{\lambda+1}
 \end{array} \tag{18}$$

is a pushout (or, equivalently, pullback) square of graded abelian groups and hence a pullback square of graded rings.

We do this, for cohomology with rational coefficients, by making use of the commutative diagram

$$\begin{array}{ccc}
 S_{hK(m)}^{2m-3} & \xrightarrow{\pi} & BK(m) \\
 j \downarrow & & \downarrow \\
 BG_\lambda & \xrightarrow{i} & BG_{\lambda+1} \\
 & \searrow & \searrow \\
 & & B\text{FDiff}
 \end{array}$$

We begin by analyzing $H^*(B\text{FDiff}; \mathbb{Q})$. By a theorem of Smale, the inclusion $SO(3) \subset \text{Diff}^+(S^2)$ is a weak equivalence, so we do not distinguish between the two. There is a short exact sequence

$$\mathcal{S} \rightarrow \text{FDiff} \xrightarrow{e} SO(3),$$

where the map e takes a diffeomorphism to the projection of its action on the base and \mathcal{S} denotes the gauge group of the appropriate S^2 -bundle over S^2 . This yields a fiber sequence

$$B\mathcal{S} \rightarrow B\text{FDiff} \xrightarrow{Be} BSO(3). \quad (19)$$

Now $B\mathcal{S}$ is the component of the null map in

$$\text{Map}(S^2, BSO(3))$$

in the untwisted case, while in the twisted case it is the component of the essential map corresponding to the generator of $\pi_2(BSO(3)) = \mathbb{Z}/2$. The latter statement is a consequence of the fact that, in the twisted case, \mathcal{S} is the pullback of the diagram of groups

$$\text{Map}(D^2, SO(3)) \rightarrow \text{Map}(S^1, SO(3)) \leftarrow \text{Map}(D^2, SO(3))$$

with one of the maps given by restriction to the boundary and the other restriction followed by conjugation by a generator of $\pi_1(SO(3))$ together with the fact that the standard simplicial construction of the classifying space functor commutes with fibered products. The two components of $\text{Map}(S^2, BSO(3))$ are equivalent away from

the prime 2 (and hence rationally) since the degree 2 map of S^2 induces an equivalence

$$\text{Map}_1(S^2, BSO(3)) \rightarrow \text{Map}_0(S^2, BSO(3)).$$

LEMMA 5.9

We have $H^*(B\text{FDiff}; \mathbb{Q}) = \mathbb{Q}[A, X, Y]$ with $|A| = 2$ and $|X| = |Y| = 4$, that is, the rationalization of the classifying space $B\text{FDiff}_{\mathbb{Q}}$ is weakly equivalent to $K(\mathbb{Q}, 2) \times K(\mathbb{Q}, 4) \times K(\mathbb{Q}, 4)$.

Proof

We have a fiber sequence

$$\Omega S^3 \simeq \Omega_0^2 BSO(3) \rightarrow \text{Map}_0(S^2, BSO(3)) \rightarrow BSO(3).$$

Since $H^*(\Omega S^3; \mathbb{Q})$ is polynomial generated by a class in degree 2 and $H^*(BSO(3); \mathbb{Q})$ is a polynomial ring generated by a class in degree 4, the Serre spectral sequence collapses, and so $H^*(B\mathcal{S}; \mathbb{Q}) = \mathbb{Q}[A, Y]$. Note that, by the discussion preceding the statement, this is true in both twisted and untwisted cases. Finally, the spectral sequence of (19) also collapses. □

There is a canonical choice for the degree 2 generator and one of the degree 4 generators, namely, the pullback of the generator of $H^4(BSO(3); \mathbb{Q})$ under Be , but this is *not* the case for the remaining degree 4 generator. In order to continue the computation, we must choose well-defined generators in $H^*(B\text{FDiff}; \mathbb{Q})$.

By Corollary 5.6, in both twisted and untwisted cases, the map

$$BG_2 \rightarrow B\text{FDiff}$$

is at least 5-connected and hence induces an isomorphism on cohomology in degrees at most 4. We use this fact and the pushout decomposition in Theorem 5.5 to pick the generators of $H^4(B\text{FDiff}; \mathbb{Q})$.

By Theorem 4.2, the groups $K(2)$ and $K(3)$ act transitively on the unit sphere of their isotropy representations with isotropy groups $SO(3) \subset K(2) = S^1 \times SO(3)$ and $U(1) \times 1 \subset K(3) = U(2)$. Since for H a closed subgroup of G we have $(G/H)_{hG} = EG/H = BH$, in Theorem 5.5 we have

$$S^1_{hK(2)} = BSO(3), \quad S^3_{hK(3)} = BU(1),$$

and hence BG_2 is obtained by the homotopy pushouts

$$\begin{array}{ccc} BSO(3) & \xrightarrow{\pi} & BK(2) \\ j \downarrow & & \downarrow \\ BK(0) & \longrightarrow & BG_2 \end{array} \qquad \begin{array}{ccc} BS^1 & \xrightarrow{\pi} & BK(3) \\ j \downarrow & & \downarrow \\ BK(1) & \longrightarrow & BG_2 \end{array}$$

in the untwisted and twisted cases, respectively, with π the maps induced by the inclusions of the isotropy groups. The map j in the left square is the inclusion of the diagonal by Iglesias’s classification of $SO(3)$ -equivariant symplectic four-manifolds (see [I]; cf., also, [AG, Theorem 1.1(ii)]), while the map j on the right is described in Proposition 5.13. Regardless of what these maps are, the Mayer-Vietoris sequences of the above diagrams together with Corollary 5.6 imply that the inclusions

$$(BSO(3) \times BSO(3)) \vee BS^1 \xrightarrow{\psi} B\text{FDiff} \quad \text{and} \quad BK(1) \vee BSU(2) \xrightarrow{\psi} B\text{FDiff}, \tag{20}$$

where $S^1 \subset K(2) = S^1 \times SO(3)$ is the inclusion of the first factor and $SU(2) \subset K(3) = U(2)$ is the standard inclusion, induce isomorphisms on cohomology (even with integral coefficients) in degrees at most 4.

For the rest of this section, we use the following notation for the standard generators in $H^*(BK(n); \mathbb{Q})$:

$$H^*(BK(n); \mathbb{Q}) = \begin{cases} \mathbb{Q}[Y_0, X_0] & \text{if } n = 0, \\ \mathbb{Q}[A_n, X_n] & \text{if } n > 0, \end{cases} \tag{21}$$

with $|A_n| = 2$ and $|X_n| = |Y_0| = 4$. Note that X_0 corresponds to the second copy of $SO(3)$, the one that rotates the base of the fibration according to the identification of the groups $K(n)$ in Section 4.

Definition 5.10

Consider the maps ψ in (20). Let $X, Y \in H^4(B\text{FDiff}; \mathbb{Q})$ denote the unique classes such that*

$$\begin{aligned} \psi^*(X) &= \begin{cases} X_0 & \text{in the untwisted case,} \\ X_1 & \text{in the twisted case,} \end{cases} \\ \psi^*(Y) &= \begin{cases} Y_0 + A_2^2 & \text{in the untwisted case,} \\ X_3 & \text{in the twisted case,} \end{cases} \end{aligned}$$

and let $T \in H^2(B\text{FDiff}; \mathbb{Q})$ denote the unique class such that $\psi^*(T) = A_k$, with $k = 2$ in the untwisted case and $k = 1$ in the twisted case.

Remark 5.11

For n even, $X \in H^4(B\text{FDiff}; \mathbb{Q})$ is the canonical generator obtained by pulling back the generator of $H^4(BSO(3); \mathbb{Q})$ along the map Be in (19), while Y was chosen to make the formula for $H^*(BG_\lambda; \mathbb{Q})$ below to agree as much as possible with the one in [AM, Theorem 1.2].

*We write X_3 for the image of the class $X_3 \in H^4(BK(3); \mathbb{Q})$ in $H^4(BSU(2); \mathbb{Q})$, and we write A_2 for the image of the class $A_2 \in H^2(BK(2); \mathbb{Q})$ in $H^2(BS^1; \mathbb{Q})$ under the obvious inclusions.

For n odd, since $BK(n) \rightarrow B\text{FDiff} \xrightarrow{Be} BSO(3)$ identifies with the projection $BU(2) \rightarrow BSO(3)$ and hence sends the generator $Z \in H^4(BSO(3); \mathbb{Q})$ to $A_n^2 - 4X_n$, the canonical generator is the class $T^2 - 4(X + Y) \in H^4(B\text{FDiff}; \mathbb{Q})$. In this twisted case, this class does not generate $H^4(B\text{FDiff}; \mathbb{Z})$, and so we decided not to use this as one of the generators for $H^4(B\text{FDiff}; \mathbb{Q})$.

See also (30) and (31) for the relation between the generators in the untwisted case and other geometrically defined classes.

We now need to compute the effect of the inclusions

$$BK(n) \xrightarrow{\psi_n} B\text{FDiff}$$

on rational cohomology. By definition of the generators of $H^*(B\text{FDiff}; \mathbb{Q})$, we know the answer for $n = 0, 1$ and partly for $n = 2, 3$. To complete the computation, we use the fact that for each n, m with the same parity there are S^1 's inside $K(n)$ and $K(m)$ which are conjugate inside FDiff . This should have an elementary proof, but we have only been able to obtain one for $(n, m) = (0, 2k)$ (which suffices to compute the cohomology ring in the untwisted case). In order to handle the twisted and untwisted cases uniformly, we take a different tack and use Karshon's classification of S^1 -actions to find the conjugate circles inside G_λ .

We use the standard bases for the maximal tori $S^1 \times S^1 \subset K(n)$ defined in Section 4 (see (7)). A circle in $K(n)$ is now described by an integer vector $(a, b) \in \mathbb{Z}^2$. The following result is essentially [K2, Lemma 3], but we include a proof for the reader's convenience.

PROPOSITION 5.12

Given λ such that there are complex structures compatible with ω_λ isomorphic to F_k and F_l , there are S^1 -equivariant symplectomorphisms between

- for k and l odd,
 - F_k with the S^1 -action given by $((l + 1)/2, (l - 1)/2)$,
 - F_l with the S^1 -action given by $((k + 1)/2, (k - 1)/2)$; and
- for k and l even:
 - F_k with the S^1 -action given by $(l/2, 1)$,
 - F_l with the S^1 -action given by $(k/2, 1)$.

Proof

The Kähler reduction construction from Section 4 provides a standard picture for the moment polygon of the Hirzebruch surfaces F_k with vertices $(0, 0)$, $(1, 0)$, $(1, \mu)$, and $(0, \mu - n) \in \text{Lie}(\mathbb{T})^*$ in terms of the moment map basis described in (6) for the maximal torus \mathbb{T} of $K(k)$.

Using the change of basis[†] in $\text{Lie}(\mathbb{T})^*$,

$$\begin{bmatrix} 1 & 0 \\ m & 1 \end{bmatrix}$$

with $m = \pm(k - l)/2$, we can make the slopes of the nonvertical edges of the moment polygons for k and l agree as long as k and l have the same parity. It follows from [K1, Theorem 4.1] that F_k and F_l are S^1 -equivariantly symplectomorphic with respect to the circle actions corresponding to projection onto the y -axis (i.e., $(0, 1)$) in these bases.

Hence (with k, l of the same parity) the S^1 's given in the moment map bases by

$$\begin{bmatrix} \frac{l-k}{2} \\ 1 \end{bmatrix} \text{ on } F_k, \quad \begin{bmatrix} \frac{k-l}{2} \\ 1 \end{bmatrix} \text{ on } F_l$$

produce equivariantly symplectomorphic manifolds (e.g., the two polygons on the left in [K1, Figure 4, p. 9] correspond to the case $(k, l) = (0, 2)$).

Applying the change of basis of Lemma 4.1 to these vectors now completes the proof (bearing in mind that on F_0 the circles written $(a, 1)$ and $(-a, 1)$ are conjugate). □

We can now prove the analog of [AG, Theorem 1.1(ii), Corollary 4.5] in the twisted case.

PROPOSITION 5.13

In the twisted case,

$$BG_2 = \text{hocolim} \left(BU(2) \xleftarrow{B(2,1)} BS^1 \xrightarrow{B(1,0)} BU(2) \right). \tag{22}$$

Moreover,

$$H^*(BG_2; \mathbb{Q}) = \mathbb{Q}[T, X, Y]/Y(9X - 2T^2),$$

and $H^*(BG_2; \mathbb{Z})$ is the subring generated[‡] over \mathbb{Z} by T, X, Y , and $TY/3$.

Proof

By Theorem 4.2, the map $BS^1 \xrightarrow{B(1,0)} BU(2)$ is the inclusion of the isotropy group of the representation of $K(3)$ on the normal slice. Proposition 5.12 thus identifies the map j in Theorem 5.5 with $BS^1 \xrightarrow{B(2,1)} BU(2)$, and the first statement follows.

[†]Note that the corresponding change of basis for the tori is the transpose matrix.

[‡]There are infinitely many relations on the generators of $H^*(BG_2; \mathbb{Z})$ corresponding to the fact that some elements in $\langle Y(9X - 2T^2) \rangle$ are divisible by powers of 3. These divided classes have to be included as relations so as to not introduce torsion.

Using the Mayer-Vietoris sequence of (22), one checks that

$$\begin{aligned} T &= (A_1, 3A_3), \\ X &= (X_1, 2A_3^2), \\ Y &= (0, X_3) \in H^*(BK(1); \mathbb{Z}) \times H^*(BK(3); \mathbb{Z}) \end{aligned}$$

generate $H^*(BG_2; \mathbb{Z}) \subset H^*(BK(1); \mathbb{Z}) \times H^*(BK(3); \mathbb{Z})$ over \mathbb{Q} and that these together with $(0, A_3X_3)$ generate over \mathbb{Z} . The result follows. \square

We need the following simple computations, which are left as an exercise.

LEMMA 5.14

Writing $H^*(BS^1) = \mathbb{Q}[T]$, $H^*(BS^1 \times BSO(3); \mathbb{Q}) = H^*(BU(2); \mathbb{Q}) = \mathbb{Q}[A, X]$, and $H^*(BSO(3) \times BSO(3); \mathbb{Q}) = \mathbb{Q}[X, Y]$, the map $S^1 \xrightarrow{(a,b)} S^1 \times SO(3)$ induces the map

$$A \mapsto aT, \quad X \mapsto b^2T^2,$$

$S^1 \xrightarrow{(a,b)} U(2)$ induces

$$A \mapsto (a + b)T, \quad X \mapsto (ab)T^2,$$

and $S^1 \xrightarrow{(a,b)} SO(3) \times SO(3)$ induces

$$X \mapsto a^2T^2, \quad Y \mapsto b^2T^2.$$

We are now in a position to understand the effect on cohomology of the inclusions of the groups $K(n)$ in FDiff which are the crucial input for the computation of the rational cohomology rings.

PROPOSITION 5.15

Consider the inclusions $BK(n) \xrightarrow{\psi_n} B \text{FDiff}$.

- If $n > 0$ is even, then

$$\psi_n^*(T) = \frac{n}{2}A_n,$$

$$\psi_n^*(X) = X_n,$$

$$\psi_n^*(Y) = A_n^2 + \frac{n^2}{4}X_n.$$

- If $n > 0$ is odd, then

$$\begin{aligned}\psi_n^*(T) &= nA_n, \\ \psi_n^*(X) &= \frac{n^2 - 1}{4}A_n^2 + \left(1 - \frac{n^2 - 1}{8}\right)X_n, \\ \psi_n^*(Y) &= \frac{n^2 - 1}{8}X_n.\end{aligned}$$

Proof

For $n = 2$, we need only compute the coefficients of X_2 in $\psi_2^*(X)$ and $\psi_2^*(Y)$ (since the coefficients of A_2^2 are determined by definition of the generators X and Y). By Proposition 5.12, the groups $K(0)$ and $K(2)$ contain a circle in common written in the standard bases as $(1, 1)$ and $(0, 1)$. Therefore we have a homotopy commutative diagram

$$\begin{array}{ccc} BS^1 & \xrightarrow{B(0,1)} & BK(2) \\ \downarrow B(1,1) & & \downarrow \psi_2 \\ BK(0) & \xrightarrow{\psi_0} & B\text{FDiff} \end{array}$$

and so Lemma 5.14 implies that these coefficients are both 1 as the statement indicates.

For $n = 3$, we need only find the coefficient of A_3 in $\psi_3^*(T)$ and the coefficients of A_3^2 in $\psi_3^*(X)$ and $\psi_3^*(Y)$. By Proposition 5.12, we have a commutative diagram

$$\begin{array}{ccc} BS^1 & \xrightarrow{B(1,0)} & BK(3) \\ \downarrow B(2,1) & & \downarrow \psi_3 \\ BK(1) & \xrightarrow{\psi_1} & B\text{FDiff} \end{array} \tag{23}$$

and so Lemma 5.14 implies that $\psi_3^*(T) = 3A_3$, $\psi_3^*(X) = 2A_3^2$, and $\psi_3^*(Y) = X_3$ as the statement indicates.

For all higher n , Proposition 5.12 and the previous arguments tell us the effect on cohomology of $BS^1 \xrightarrow{(a,b)} BK(n) \xrightarrow{\psi_n} B\text{FDiff}$ for two independent vectors (a, b) , and so simple algebra yields the remaining formulas. \square

COROLLARY 5.16

The kernel of the map $\psi_n^* : H^*(B\text{FDiff}; \mathbb{Q}) \rightarrow H^*(BK(n); \mathbb{Q})$ is the ideal

- (i) $\langle T \rangle$ if $n = 0$,

- (ii) $\langle (n^4/16)X - (n^2/4)Y + T^2 \rangle$ if n is even,
- (iii) $\langle ((n^2 - 1)n^2/8)(X + Y) - n^2Y - \frac{(n^2-1)^2}{32}T^2 \rangle$ if n is odd.

The following result collects the statements concerning the rational cohomology ring in Theorems 1.3 and 1.5 (see Remark 5.18 concerning the formulas in the twisted case).

THEOREM 5.17

Let $\ell < \lambda \leq \ell + 1$. With the choice of generators indicated in Definition 5.10, the map $H^*(B \text{FDiff}; \mathbb{Q}) \rightarrow H^*(BG_\lambda; \mathbb{Q})$ is the quotient map

$$\mathbb{Q}[T, X, Y] \longrightarrow \mathbb{Q}[T, X, Y]/\langle R_\ell(T, X, Y) \rangle,$$

where

$$R_k(T, X, Y) = \begin{cases} T(X - Y + T^2) \cdots (k^4 X - k^2 Y + T^2) & \text{in the untwisted case,} \\ Y \cdots ((2k + 1)^2 \binom{k(k+1)}{2} U - Y) - \frac{k^2(k+1)^2}{2} T^2 & \text{in the twisted case,} \end{cases}$$

and $U = X + Y$.

Proof

The proof is by induction. The result is clear for $\ell = 0$. Assume $\ell \geq 0$; the result holds for $\lambda \in]\ell, \ell + 1]$. Then by Theorem 5.5 and Lemma 5.7, we have a pullback diagram of rings

$$\begin{array}{ccc} \mathbb{Q}[A_m, X_m]/\langle e_m \rangle & \xleftarrow{\pi^*} & \mathbb{Q}[A_m, X_m] \\ \uparrow j^* & & \uparrow \\ \mathbb{Q}[T, X, Y]/\langle R_\ell(T, X, Y) \rangle & \xleftarrow{\quad} & H^*(BG_{\lambda+1}; \mathbb{Q}) = \mathbb{Q}[T, X, Y]/I_{\ell+1} \end{array} \tag{24}$$

where we have used Proposition 5.8 to express $H^*(BG_{\lambda+1}; \mathbb{Q})$ as a quotient of $H^*(B \text{FDiff}; \mathbb{Q})$ by an ideal $I_{\ell+1}$, e_m denotes the Euler class calculated in Lemma 5.7, and $m = 2\ell + 2$ or $2\ell + 3$ according to whether we are in the untwisted or twisted case.

We must have $I_{\ell+1} \subset \langle R_\ell \rangle$. On the other hand, by Corollary 5.16, we also have

$$I_{\ell+1} \subset \begin{cases} \langle \frac{m^4}{16} X - \frac{m^2}{4} Y + T^2 \rangle & \text{if } m \text{ is even,} \\ \langle \frac{(m^2-1)m^2}{8} (X + Y) - m^2 Y - \frac{(m^2-1)^2}{32} T^2 \rangle & \text{if } m \text{ is odd.} \end{cases} \tag{25}$$

Since R_ℓ and the polynomials $k_m(T, X, Y)$ appearing in (25) are coprime, it follows that

$$I_{\ell+1} \subset \langle R_\ell k_m \rangle = \langle R_{\ell+1} \rangle.$$

Denoting by d the degree of R_ℓ , the pullback square (24) gives the following generating function for the graded ring $H^*(BG_{\lambda+1}; \mathbb{Q})$:

$$\chi = \frac{1}{(1-t^2)(1-t^4)} + \frac{1-t^d}{(1-t^2)(1-t^4)^2} - \frac{1-t^d}{(1-t^2)(1-t^4)}.$$

This simplifies to

$$\chi = \frac{1-t^{d+4}}{(1-t^2)(1-t^4)^2},$$

which is the generating function for $\mathbb{Q}[T, X, Y]/\langle R_{\ell+1}(T, X, Y) \rangle$. Hence $I_{\ell+1} = \langle R_{\ell+1} \rangle$. * \square

Remark 5.18

Making the change of variables

$$z = T, \quad x = 4U - T^2, \quad y = 4U + 32Y - 2T^2,$$

we have

$$(2k+1)^2 \left(\frac{k(k+1)}{2} U - Y \right) - \frac{k^2(k+1)^2}{2} T^2 = \frac{1}{32} (-z^2 + (2k+1)^4 x - (2k+1)^2 y),$$

and so using the generators x, y, z , we obtain the following formula for $H^*(BG_\lambda; \mathbb{Q})$ in the statement of Theorem 1.5 which is similar to the formula in the untwisted case:

$$H^*(BG_\lambda, \mathbb{Q}) = \frac{\mathbb{Q}[x, y, z]}{\langle \prod_{i=0}^{\ell} (-z^2 + (2i+1)^4 x - (2i+1)^2 y) \rangle} \quad \text{for } 0 \leq \ell < \lambda \leq \ell + 1.$$

The generators x and z have natural geometric descriptions (see Remark 5.11), but we have no geometric interpretation for y .

Remark 5.19

The ring structure obtained in Theorem 5.17 and Remark 5.18 differs from the one previously calculated in [AM, Theorem 1.2, Theorem 1.5]. There are two different reasons for the difference.

Checking that ψ_m^ does indeed send R_ℓ to the ideal generated by e_m (so that j^* is well defined) is a recommended confidence-building activity.

Regarding the multiples of T^2 in the factors that make up the relation, the problem can be traced to a misapplication of [AA, Theorem 5.4] in [AM, p. 1007]. Using the notation of [AA], $\overline{K}(d\mu)$ depends only on the value of $d\mu$ in the associated graded vector space of the filtration of the Sullivan model by word length. Thus the higher Whitehead products provide information only on the image of the relation in this associated graded vector space and have no bearing on the coefficients of the decomposable T^2 -terms (which have higher filtration).

In this way, one sees that the higher Whitehead products in $\pi_*(G_\lambda) \otimes \mathbb{Q}$ cannot be used exclusively to compute the ring structure in $H^*(BG_\lambda; \mathbb{Q})$.

The explanation for the remaining difference in the twisted case (regarding the coefficients of X and Y in the factors that make up the relation) lies in a mistake in [AM, Lemma 2.11]. Indeed, it follows easily from Proposition 5.15 that (with the notation of [AM]) we have

$$\alpha_k = (2k + 1)\alpha_0 \in \pi_1(G_\lambda^1),$$

$$\xi_k = \xi_0 + \frac{k(k + 1)}{2}\eta \in H_3(G_\lambda^1; \mathbb{Z}).$$

Computation of the cohomology ring away from 2 in the untwisted case

In this section, we calculate $H^*(BG_\lambda; \mathbb{Z}[1/2])$ in the untwisted case. This is done by combining Proposition 5.8 with the calculation of $H^*(B\text{FDiff}; \mathbb{Z}[1/2])$. Henceforth, all spaces are localized away from the prime 2. We write $R = \mathbb{Z}[1/2]$.

Since we are working away from the prime 2, it is easy to see that, in the untwisted case, FDiff is equivalent to the semidirect product

$$\mathcal{G} = SO(3) \ltimes \text{Map}(S^2, SU(2)),$$

where $SO(3)$ acts on $\text{Map}(S^2, SU(2))$ by precomposition of its standard action on $S^2 = \mathbb{C}P^1$.

THEOREM 5.20

Let $R = \mathbb{Z}[1/2]$. $H^*(B\mathcal{G}; R)$ is a free module over $R[x, y]$ on generators a_k, b_k with $k \geq 0$:

$$H^*(B\mathcal{G}, R) = R[x, y]\langle a_0, b_0, a_1, b_1, a_2, \dots \rangle,$$

where $a_0 = 1, |x| = |y| = 4, |b_k| = 4k + 2,$ and $|a_k| = 4k$. Moreover, $H^*(B\mathcal{G}, R)$ is isomorphic to the subring of $\mathbb{Q}[x, y, z],$ with $|z| = 2,$ when b_k and a_k are identified, respectively, with

$$\frac{z}{(2k + 1)!} \prod_{i=1}^k (z^2 + i^4 x - i^2 y), \quad \frac{z^2}{(2k)!} \prod_{i=1}^{k-1} (z^2 + i^4 x - i^2 y).$$

Remark 5.21

One can see that the groups of fiber-preserving diffeomorphisms for the twisted and untwisted bundles are equivalent away from the prime 2, and so Theorem 5.20 describes $H^*(B \text{FDiff}; \mathbb{Z}[1/2])$ also in the twisted case. We do not use this, however.

Proof of Theorem 5.20

Recall that $\mathcal{G} = SO(3) \times \text{Map}(S^2, SU(2))$. \mathcal{G} contains a subgroup $G = SO(3) \times SU(2)$ extending the group of constant maps. Let $\mathbb{T} \times S^1$ be the maximal torus of G . Notice that \mathbb{T} acts on $\text{Map}(S^2, SU(2))$ by precomposition with the action of \mathbb{T} on S^2 given by rotation about the vertical axis. S^1 is seen as the subgroup of constant maps with value in the maximal torus of $SU(2)$. Let τ_1 and τ_2 be elements in each factor of G that map to generators of the Weyl group. Notice also that \mathcal{G} contains the \mathbb{T} -invariant subgroup $\Omega^2 SU(2) \subset \text{Map}(S^2, SU(2))$ consisting of maps that take the north pole to the identity element. Here and henceforth, we fix the north pole of S^2 as the basepoint.

The proof of Theorem 5.20 uses a sequence of inclusions of subgroups

$$\Omega^2 SU(2) \subset \mathcal{K} \subset \mathcal{H} \subset \mathcal{G}$$

that induces maps of classifying spaces:

$$\Omega SU(2) \longrightarrow B\mathcal{K} \longrightarrow B\mathcal{H} \longrightarrow B\mathcal{G}$$

with $B\mathcal{H}$ equivalent to the homotopy orbit space of a $\mathbb{Z}/2$ -action on $B\mathcal{K}$ and $B\mathcal{G}$ equivalent to the homotopy orbit space of a $\mathbb{Z}/2$ -action on $B\mathcal{H}$. More precisely, we have the following.

Definition 5.22

$\mathcal{H} \subset \mathcal{G}$ is defined as the subgroup

$$\mathcal{H} = \mathbb{T} \times \text{Map}(S^2, SU(2)) \subset SO(3) \times \text{Map}(S^2, SU(2)) = \mathcal{G}.$$

We define an involution on \mathcal{H} induced by conjugation with the element $\tau_1 \in \mathcal{G}$. This action is given by inversion on the \mathbb{T} factor and by the action induced on $\text{Map}(S^2, SU(2))$ via the action of $\tau_1 \in SO(3)$ by left multiplication on $S^2 = SO(3)/\mathbb{T}$. This induces an involution on $B\mathcal{H}$, which we also denote by τ_1 .

Definition 5.23

$\mathcal{K} \subset \mathcal{H}$ is defined as the subgroup

$$\mathcal{K} = (\mathbb{T} \times S^1) \times \Omega^2 SU(2) \subset (\mathbb{T} \times SU(2)) \times \Omega^2 SU(2) = \mathbb{T} \times \text{Map}(S^2, SU(2)) = \mathcal{H},$$

where $S^1 \times \Omega^2 SU(2)$ may be seen as the subspace of maps from S^2 to $SU(2)$ that map the basepoint of S^2 to S^1 . Define an involution on \mathcal{K} induced by conjugation with the element $\tau_2 \in \mathcal{K}$. This action preserves the \mathbb{T} factor, acts by inversion on the S^1 -factor, and acts by pointwise conjugation with $\tau_2 \in SU(2)$ on $\Omega^2 SU(2)$. As before, this induces an involution on $B\mathcal{K}$ denoted by τ_2 .

From the above descriptions, we can describe the homotopy type of the respective classifying spaces away from the prime 2:

$$B\mathcal{K} = E(\mathbb{T} \times S^1) \times_{\mathbb{T} \times S^1} \Omega^2 BSU(2), \tag{26}$$

$$B\mathcal{H} = E\mathbb{T} \times_{\mathbb{T}} \text{Map}(S^2, BSU(2)) \tag{27}$$

$$B\mathcal{G} = ESO(3) \times_{SO(3)} \text{Map}(S^2, BSU(2)). \tag{28}$$

It is a standard argument to identify the invariant cohomology rings

$$H^*(B\mathcal{K}, R)^{\tau_2} = H^*(B\mathcal{H}, R) \quad \text{and} \quad H^*(B\mathcal{K}, R)^{\tau_1} = H^*(B\mathcal{G}, R).$$

The action of τ_1 is subtle. To understand this action, we start by \mathbb{T} -equivariantly decomposing S^2 as a pushout of two hemispheres intersecting over the equator. From (27), we get a pullback diagram

$$\begin{array}{ccc} B\mathcal{K} & \xrightarrow{ev_N} & B\mathbb{T} \times BSU(2) \\ \downarrow ev_S & & \downarrow \\ B\mathbb{T} \times BSU(2) & \longrightarrow & E\mathbb{T} \times_{\mathbb{T}} LBSU(2) \end{array}$$

where ev_N and ev_S denote evaluation at the north and south pole. Also, the notation $LBSU(2)$ refers to the free loop space of $BSU(2)$, that is, $\text{Map}(S^1, BSU(2))$. We note that the τ_1 -action on $B\mathcal{K}$ described above has the property of swapping the corners of the pullback diagram since it interchanges the north and south pole of S^2 , and inducing an action on the equator given by inversion.

Notice that (26) shows that the cohomology of $B\mathcal{K}$ is equivalent to the equivariant cohomology of $\Omega SU(2)$. This calculation has been made in [HHH, Section 6.2].

Before we give a description of this cohomology, let us set some notation. Let $u, v \in H^2(B\mathbb{T} \times BS^1, \mathbb{Z})$ be the canonical generators corresponding to the two factors, respectively. It also follows from an easy spectral sequence argument that $H^2(B\mathcal{G}, \mathbb{Z})$ is a free \mathbb{Z} -module generated by a unique class w that restricts to the generator of $H^2(\Omega SU(2), \mathbb{Z})$. Moreover, w restricts trivially to the cohomology of $B(\mathbb{T} \times S^1)$ since the inclusion of $\mathbb{T} \times S^1 \subset \mathcal{G}$ factors through the group $SU(2) \times SU(2)$. Hence,

w restricts to the canonical generator of $(\mathbb{T} \times S^1)$ -equivariant cohomology of $\Omega SU(2)$ described in [HHH]. Therefore, we have the following.

THEOREM 5.24 (see [HHH, Section 6.2])

Let $u, v,$ and w be the classes defined above. Then the cohomology of $B\mathcal{K}$ with coefficients in the ring \mathbb{Z} is given by the following free module over the ring $\mathbb{Z}[u, v]$ on generators $f_k, g_k, k \geq 0$:

$$H^*(B\mathcal{K}, \mathbb{Z}) = \mathbb{Z}[u, v]\langle g_0, f_0, g_1, f_1, g_2, \dots \rangle, \quad \text{where } g_0 = 1,$$

where the degree of f_k is $4k + 2$, and that of g_k is $4k$. Moreover, as a ring, we may identify $H^*(B\mathcal{K}, \mathbb{Z})$ as the subring of $\mathbb{Q}[u, v, w]$, where the degree of Z is 2, and the elements f_k and g_k are identified, respectively, to the elements

$$\frac{w}{(2k + 1)!} \prod_{i=1}^k ((w + i^2 u)^2 - 4i^2 v^2), \quad \frac{w(w + k^2 u + 2kv)}{(2k)!} \prod_{i=1}^{k-1} ((w + i^2 u)^2 - 4i^2 v^2).$$

From the previous remark, and the description in [HHH], we see that the τ_2 -action has the property

$$\tau_2(u) = u, \quad \tau_2(v) = -v, \quad \tau_2(w) = w.$$

Filtering $H^*(B\mathcal{K}, R)$ by powers of v , and taking $\mathbb{Z}/2$ -invariants, we easily derive the following result.

PROPOSITION 5.25

The cohomology of $B\mathcal{H}$ with coefficients in the ring R is given by the following free module over the ring $R[u, v^2]$ on generators $b_k, a_k, k \geq 0$:

$$H^*(B\mathcal{H}, R) = R[u, v^2]\langle a_0, b_0, a_1, b_1, a_2, \dots \rangle, \quad \text{where } a_0 = 1,$$

where the degree of b_k is $4k + 2$, and that of a_k is $4k$. Moreover, as a ring, we may identify $H^*(B\mathcal{H}, R)$ as the subring of $\mathbb{Q}[u, v, w]$, where the degree of w is 2, and the elements b_k and a_k are identified, respectively, to the elements

$$\frac{w}{(2k + 1)!} \prod_{i=1}^k ((w + i^2 u)^2 - 4i^2 v^2), \quad \frac{w^2}{(2k)!} \prod_{i=1}^{k-1} ((w + i^2 u)^2 - 4i^2 v^2).$$

The hard part now is to identify the action of τ_1 on $H^*(B\mathcal{H}, R)$.

PROPOSITION 5.26

The action of τ_1 on $H^*(B\mathcal{H}, R)$ is given by

$$\tau_1(w) = w, \quad \tau_1(u) = -u, \quad \tau_1(v^2) = v^2 - uw.$$

Before we proceed with the proof of the Proposition 5.26, let us see how we may derive Theorem 5.20 for the cohomology of $B\mathcal{G}$ from this action. Observe that the following elements are invariant under τ_1 :

$$w, \quad u^2, \quad 2v^2 - uw.$$

Notice also that we have the equality

$$(w + i^2u)^2 - 4i^2v^2 = w^2 + i^4u^2 - i^22(2v^2 - uw). \tag{29}$$

It follows that all the elements a_k, b_k are invariant under τ_1 . We claim the following.

PROPOSITION 5.27

We have

$$H^*(B\mathcal{G}, R) = H^*(B\mathcal{H}, R)^{\mathbb{Z}/2} = R[u^2, 2v^2 - uw]\langle a_0, b_0, a_1, b_1, \dots \rangle.$$

Proof

Since u, w are elements of $H^*(B\mathcal{H}, R)$, we may replace the element v^2 by $2v^2 - uw$ to get

$$H^*(B\mathcal{H}, R) = R[u, 2v^2 - uw]\langle a_0, b_0, a_1, b_1, \dots \rangle.$$

The proof follows on filtering $H^*(B\mathcal{H}, R)$ by powers of a and taking invariants. \square

Over R , we can replace the generator $2v^2 - uw$ with $2(2v^2 - uw)$; hence, taking note of (29) and setting

$$x = u^2, \quad y = 2(2v^2 - uw), \quad z = w, \tag{30}$$

we obtain

$$H^*(B\mathcal{G}, R) = R[x, y]\langle a_0, b_0, a_1, b_1, a_2, \dots \rangle, \quad \text{where } a_0 = 1.$$

This completes the proof of Theorem 5.20 assuming Proposition 5.26. \square

Proof of Proposition 5.26

It is clear from the definition that the action must preserve w . It also follows from Definition 5.22 that the action reverses the sign of u . Hence, only the action on v^2

needs to be described. Notice that $v^2 = ev_N^*(\sigma)$, where $\sigma \in H^4(BSU(2), \mathbb{Z})$ is a generator. Recall that the $\mathbb{Z}/2\langle \tau_1 \rangle$ -action on the diagram defining $B\mathcal{H}$ as a pullback has the property of switching the corners. Hence, it follows that $\tau_1(v^2) = ev_S^*(\sigma)$. We reconsider the pullback

$$\begin{array}{ccc}
 B\mathcal{H} & \xrightarrow{ev_N} & B\mathbb{T} \times BSU(2) \\
 \downarrow ev_S & & \downarrow \\
 B\mathbb{T} \times BSU(2) & \longrightarrow & E\mathbb{T} \times_{\mathbb{T}} LBSU(2)
 \end{array}$$

and we consider the Serre spectral sequence for the right vertical map, seen as a fibration, whose fiber is $\Omega SU(2)$. Let α be the element in the E_2 -term representing a generator of $H^2(\Omega SU(2), \mathbb{Z})$. It is well known (see [Ki], [ABKS]) that the integral cohomology of $E\mathbb{T} \times_{\mathbb{T}} LBSU(2)$ is free in degree 2, generated by u , and trivial in degree 5. Hence the class $\pm a u$ represents σ in the cohomology of the total space $B\mathbb{T} \times BSU(2)$. Recall that the class $w \in H^2(B\mathcal{H}, R)$ is represented by α in the Serre spectral sequence for the left vertical fibration (up to an indeterminacy given by a multiple of u). Hence, by choosing a suitable sign for w , we notice that $\tau_1(v^2) = -uw$ modulo lower filtration in the Serre spectral sequence of the left vertical map, seen as a fibration. We may therefore write

$$\tau_1(v^2) = -uw + av^2 + bu^2.$$

Applying τ_1 again to this equation tells us that $a = 1, b = 0$. □

We can now prove Theorem 1.3.

Proof of Theorem 1.3

The canonical projection map

$$B\phi : B\mathcal{G} \longrightarrow B\text{FDiff}$$

is an isomorphism on cohomology with $\mathbb{Z}[1/2]$ coefficients. Comparing the definition of the classes x, y , and z in the rational cohomology of $B\mathcal{G}$ given in (30) with that of X, Y , and T in Definition 5.10, we see that

$$B\phi^*(X) = x, \quad B\phi^*(Y) = y, \quad B\phi^*(T) = z. \tag{31}$$

Henceforth we identify these graded rings by the map $B\phi^*$.

Recall that we have an inclusion map

$$BK(2k) \xrightarrow{\psi_{2k}} B\text{FDiff}$$

whose effect on rational cohomology was described in Proposition 5.15. We recall that

$$\psi_{2k}^*(X) = X_{2k}, \quad \psi_{2k}^*(Y) = k^2 X_{2k} + A_{2k}^2, \quad \psi_{2k}^*(T) = kA_{2k},$$

where we are using the notation established in (21). Consider the classes a_k, b_{k-1} in the cohomology of $B\text{FDiff}$ defined in Theorem 5.20. Then

$$\psi_{2k}^*(b_{k-1}) = A_{2k} \prod_{i=1}^{k-1} (A_{2k}^2 - i^2 X_{2k}) = e_{2k}, \quad \psi_{2k}^*(a_k) = \frac{A_{2k} e_{2k}}{2},$$

where e_{2k} denotes the Euler class calculated in Lemma 5.7. Hence, $B\psi_{2k}^*$ maps the submodule $R[x, y]\langle b_{k-1}, a_k \rangle$ isomorphically onto the ideal generated by the Euler class e_{2k} . Moreover, it is also clear that the classes b_i and a_j map to zero if $i \geq k$ and $j > k$. It now follows by induction using Theorem 5.5 and Proposition 5.8 that the kernel of the map

$$H^*(B\text{FDiff}, R) \rightarrow H^*(BG_l, R)$$

is the submodule generated over $R[x, y]$ by the elements b_i, a_j , where $i \geq l$ and $j > l$. Furthermore, one has the following identification:

$$H^*(BG_\lambda, \mathbb{Q}) = \mathbb{Q}[x, y]\langle a_0, b_0, a_1, \dots, a_l \rangle = \frac{\mathbb{Q}[x, y, z]}{\langle z \prod_{i=1}^l (z^2 + i^4 x - i^2 y) \rangle},$$

and so we may identify $H^*(BG_\lambda, R) = R[x, y]\langle a_0, b_0, a_1, \dots, a_l \rangle$ naturally as a subring of the above quotient. This completes the proof of Theorem 1.3. □

Appendices

A. $\bar{\partial}$ -operators and the derivative of the Nijenhuis tensor

In this appendix, we recall some standard almost complex geometry facts and prove a formula for a certain derivative of the Nijenhuis tensor (Corollary A.9). This formula, needed in Section 2, is elementary and probably well known to experts, but we were unable to find it in the literature.

Let (M, J) be an almost complex manifold. As usual, we identify TM with the $+i$ eigenspace of the action of J on $TM \otimes \mathbb{C}$, and we consider the decomposition of $\Omega(M)$, the space of complex-valued differential forms on M , according to (p, q) -type.

In particular, $\Omega^1(M) = \Omega_j^{1,0}(M) \oplus \Omega_j^{0,1}(M)$ and $\Omega^2(M) = \Omega_j^{2,0}(M) \oplus \Omega_j^{0,2}(M) \oplus \Omega_j^{1,1}(M)$.

For a complex-valued function $f \in \Omega^0(M)$, one defines $\bar{\partial}f \in \Omega_j^{0,1}(M)$ as

$$\bar{\partial}f = (df)^{0,1}.$$

Similarly, given $\alpha \in \Omega_j^{0,1}(M)$, one defines $\bar{\partial}\alpha \in \Omega_j^{0,2}(M)$ as

$$\bar{\partial}\alpha = (d\alpha)^{0,2}.$$

One easily checks that this $\bar{\partial}$ -operator, $\bar{\partial} : \Omega_j^{0,1}(M) \rightarrow \Omega_j^{0,2}(M)$, satisfies the Leibnitz rule

$$\bar{\partial}(f\alpha) = (\bar{\partial}f) \wedge \alpha + f\bar{\partial}\alpha, \quad \forall f \in \Omega^0(M), \alpha \in \Omega_j^{0,1}(M).$$

One can also define an appropriate $\bar{\partial}$ -operator on $\Omega(TM)$, the space of TM -valued differential forms on M . This operator has a particularly simple expression on $\Omega^0(TM)$ involving the Nijenhuis tensor (cf. [G, (2.6.3), (2.7.1)]).

Definition A.1

Given an almost complex manifold (M, J) , the Nijenhuis tensor N_J is defined by

$$N_J(X, Y) = [JX, JY] - J[X, JY] - J[JX, Y] - [X, Y], \quad \forall X, Y \in \Omega^0(TM).$$

Note that $N_J \in \Omega_j^{0,2}(TM) = \Omega_j^{0,2}(M) \otimes \Omega^0(TM)$, that is,

$$N_J(JX, Y) = N_J(X, JY) = -JN_J(X, Y), \quad \forall X, Y \in \Omega^0(TM).$$

Definition A.2

Define the operator $\bar{\partial} : \Omega^0(TM) \rightarrow \Omega^{0,1}(TM)$, $Y \mapsto \bar{\partial}Y$ by

$$(\bar{\partial}Y)(X) \equiv \bar{\partial}_X Y \equiv \frac{1}{2} \left\{ [X, Y] + J[JX, Y] + \frac{1}{2} N_J(X, Y) \right\}, \quad \forall X, Y \in \Omega^0(TM).$$

One easily checks that $\bar{\partial}Y \in \Omega^{0,1}(TM)$ is indeed well defined by this formula, that is, $(\bar{\partial}Y)(X)$ is a tensor in X and $(\bar{\partial}Y)(JX) = -J(\bar{\partial}Y)(X)$.

The following proposition shows that important properties of an integrable $\bar{\partial}$ -operator are still valid in this nonintegrable context.

PROPOSITION A.3

The operator $\bar{\partial} : \Omega^0(TM) \rightarrow \Omega^{0,1}(TM)$ has the following properties:

- (i) $\bar{\partial}(f \cdot Y) = (\bar{\partial}f) \otimes Y + f \cdot (\bar{\partial}Y)$ for any function $f \in \Omega^0(M)$;
- (ii) $\bar{\partial}_X(JY) = J\bar{\partial}_X Y$.

Proof

Property (i) follows from the following calculation:

$$\begin{aligned}
 2\bar{\partial}(fY)(X) &= [X, fY] + J[JX, fY] + \frac{1}{2}N_J(X, fY) \\
 &= (X \cdot f)Y + J[(JX) \cdot f]Y + 2f\bar{\partial}_X(Y) \\
 &= [(df)(X) + i(df)(JX)]Y + 2f\bar{\partial}_X(Y) \\
 &= [2(\bar{\partial}f)(X)]Y + 2f\bar{\partial}_X(Y).
 \end{aligned}$$

To prove property (ii), note that

$$\begin{aligned}
 2[\bar{\partial}_X Y + J\bar{\partial}_X(JY)] &= [X, Y] + J[JX, Y] + \frac{1}{2}N_J(X, Y) \\
 &\quad + J[X, JY] - [JX, JY] + \frac{1}{2}JN_J(X, JY) \\
 &= [X, Y] + J[JX, Y] + J[X, JY] - [JX, JY] \\
 &\quad + \frac{1}{2}(N_J(X, Y) - J^2N_J(X, Y)) \\
 &= -N_J(X, Y) + N_J(X, Y) \\
 &= 0.
 \end{aligned}$$

□

The following proposition is needed in Section 2.

PROPOSITION A.4

On an almost complex manifold (M, J) , the Lie derivative of J with respect to a vector field $Y \in \Omega^0(TM)$ is given by

$$\mathcal{L}_Y J = (2J)(\bar{\partial}Y) + \frac{1}{2}J(Y \lrcorner N_J) \in \Omega_j^{0,1}(TM).$$

Proof

We have that

$$\begin{aligned}
 (\mathcal{L}_Y J)(X) &= [Y, JX] - J[Y, X] = J[X, Y] - [JX, Y] \\
 &= (2J)\frac{1}{2}\left([X, Y] + J[JX, Y] + \frac{1}{2}(N_J(X, Y) - N_J(X, Y))\right) \\
 &= (2J)(\bar{\partial}_X Y) + \frac{1}{2}JN_J(Y, X).
 \end{aligned}$$

□

As we now see, the $\bar{\partial}$ -operator on $\Omega_j^{0,1}(TM)$ can be identified with an appropriate derivative of the Nijenhuis tensor.

Definition A.5

Define the operator $\bar{\partial} : \Omega_j^{0,1}(TM) \rightarrow \Omega_j^{0,2}(TM)$, $A \mapsto \bar{\partial}A$ as the unique linear operator that is given on elements of the form $A = \alpha \otimes Z \in \Omega_j^{0,1}(M) \otimes \Omega^0(TM) = \Omega_j^{0,1}(TM)$ by

$$\bar{\partial}A = \bar{\partial}(\alpha \otimes Z) = \bar{\partial}\alpha \otimes Z - \alpha \wedge \bar{\partial}Z.$$

Let \mathcal{J} denote the space of almost complex structures on the manifold M . The Nijenhuis tensor can be seen as a map

$$N : \mathcal{J} \rightarrow \Omega^2(TM)$$

with derivative

$$dN : T\mathcal{J} \rightarrow \Omega^2(TM).$$

Given $J \in \mathcal{J}$, we have that

$$T_J\mathcal{J} = \{A \in \text{Aut}(TM) : AJ + JA = 0\} \cong \Omega_j^{0,1}(TM),$$

which means that

$$dN_J : \Omega_j^{0,1}(TM) \rightarrow \Omega^2(TM).$$

The following lemma follows from a standard calculation.

LEMMA A.6

Given $J \in \mathcal{J}$ and $A \in \Omega_j^{0,1}(TM)$, we have that $dN_J(A) \in \Omega^2(TM)$ is given by

$$\begin{aligned} dN_J(A)(X, Y) &= [AX, JY] + [JX, AY] - J([X, AY] + [AX, Y]) \\ &\quad - A([X, JY] + [JX, Y]) \end{aligned}$$

for any $X, Y \in \Omega^0(TM)$.

PROPOSITION A.7

Given $J \in \mathcal{J}$ and $A \in \Omega_j^{0,1}(TM)$, we have that $dN_J(A) \in \Omega^2(TM) = \Omega_j^{2,0}(TM) \oplus \Omega_j^{0,2}(TM) \oplus \Omega_j^{1,1}(TM)$ can be decomposed as

$$dN_J(A) = (dN_J(A))^{0,2} + (dN_J(A))^{2,0} + (dN_J(A))^{1,1},$$

where

$$(dN_J(A))^{0,2}(X, Y) = (-2J)(\bar{\partial}A)(X, Y),$$

$$(dN_J(A))^{2,0}(X, Y) = \frac{1}{2}JA(N_J(X, Y)),$$

and

$$(dN_J(A))^{1,1}(X, Y) = \frac{1}{2}J[N_J(X, AY) + N_J(AX, Y)],$$

for any $X, Y \in \Omega^0(TM)$.

Proof

All the expressions have the right (p, q) -type and the $\bar{\partial}$ -operator defined by $J(dN_J - (dN_J)^{2,0} - (dN_J)^{1,1})/2$ has the characterizing property of Definition A.5. \square

Remark A.8

When $J \in \mathcal{I} \subset \mathcal{J}$ is an integrable complex structure, Proposition A.7 tells us that

$$dN_J = (dN_J)^{0,2} = (-2J)\bar{\partial}.$$

Let $\Omega^{0,2}(TM)$ denote the vector bundle over \mathcal{J} whose fiber over a point $J \in \mathcal{J}$ is given by

$$\Omega^{0,2}(TM)|_J = \Omega_J^{0,2}(TM).$$

Since $\Omega^{0,2}(TM)$ is a canonical summand of the trivial bundle $\Omega^2(TM) \times \mathcal{J}$ over \mathcal{J} , it carries a natural connection ∇ defined by projection:

$$\nabla \cdot = (d\cdot)^{0,2}.$$

The Nijenhuis tensor N can be seen as a natural section of this vector bundle:

$$N : \mathcal{J} \rightarrow \Omega^{0,2}(TM).$$

Proposition A.7 immediately implies the following generalization of Remark A.8.

COROLLARY A.9

For any $J \in \mathcal{J}$, we have that

$$\nabla N_J = (dN_J)^{0,2} = (-2J)\bar{\partial}.$$

B. A commutation relation for Kähler manifolds

Our goal here is to prove that on a Kähler manifold (M, J, ω) , with Riemannian metric given by $g(\cdot, \cdot) = \omega(\cdot, J\cdot)$, the diagram

$$\begin{array}{ccc} X \in \Omega^0(TM) & \xrightarrow{\bar{\partial}} & \Omega^{0,1}(TM) \ni \Gamma \\ \downarrow & & \downarrow \\ \alpha_X \in \Omega^{0,1}(M) & \xrightarrow{\bar{\partial}} & \Omega^{0,2}(M) \ni \alpha_\Gamma \end{array}$$

commutes, where

$$\alpha_X(Y) = g(X, Y) - i\omega(X, Y)$$

and

$$\alpha_\Gamma(Y, Z) = [g(\Gamma(Y), Z) - g(\Gamma(Z), Y)] - i[\omega(\Gamma(Y), Z) - \omega(\Gamma(Z), Y)].$$

In other words, we need to show that

$$\bar{\partial}(\alpha_X) - \alpha_{\bar{\partial}X} = 0, \quad \forall X \in \Omega^0(TM). \quad (32)$$

LEMMA B.1

The map

$$\begin{aligned} \Omega^0(TM) &\rightarrow \Omega_j^{0,2}(M), \\ X &\mapsto \bar{\partial}(\alpha_X) - \alpha_{\bar{\partial}X} \end{aligned}$$

is a tensor.

Proof

As usual, it suffices to show that the given map is $\Omega^0(M)$ -linear. For any function $f \in \Omega^0(M)$ and vector field $X \in \Omega^0(TM)$, we have

$$\bar{\partial}(\alpha_{fX}) = \bar{\partial}(f\alpha_X) = (\bar{\partial}f) \wedge \alpha_X + f\bar{\partial}\alpha_X,$$

while

$$\bar{\partial}(fX) = (\bar{\partial}f) \otimes X + f\bar{\partial}X \Rightarrow \alpha_{\bar{\partial}(fX)} = \alpha_{(\bar{\partial}f) \otimes X} + f\alpha_{\bar{\partial}X}.$$

Since

$$\begin{aligned}
 \alpha_{(\bar{\partial}f)\otimes X}(Y, Z) &= [g((\bar{\partial}f)(Y)X, Z) - g((\bar{\partial}f)(Z)X, Y)] \\
 &\quad - i[\omega((\bar{\partial}f)(Y)X, Z) - \omega((\bar{\partial}f)(Z)X, Y)] \\
 &= (\bar{\partial}f)(Y)[g(X, Z) - i\omega(X, Z)] \\
 &\quad - (\bar{\partial}f)(Z)[g(X, Y) - i\omega(X, Y)] \\
 &= (\bar{\partial}f)(Y)\alpha_X(Z) - (\bar{\partial}f)(Z)\alpha_X(Y) \\
 &= (\bar{\partial}f \wedge \alpha_X)(Y, Z),
 \end{aligned}$$

we conclude that

$$\begin{aligned}
 \bar{\partial}(\alpha_{fX}) - \alpha_{\bar{\partial}(fX)} &= (\bar{\partial}f \wedge \alpha_X + f\bar{\partial}\alpha_X) - (\bar{\partial}f \wedge \alpha_X + f\alpha_{\bar{\partial}X}) \\
 &= f(\bar{\partial}(\alpha_X) - \alpha_{\bar{\partial}X}),
 \end{aligned}$$

which proves the lemma. □

Lemma B.1 implies that it suffices to prove (32) at an arbitrary point. Since a Kähler manifold behaves up to first order at a point as flat \mathbb{C}^n , where (32) clearly holds, we can conclude that (32) is true on any Kähler manifold.

C. The A_∞ -action of G_λ on the tubular neighborhoods

In this section, we fill in some details in the proof of Theorem 5.5.

We need to use the notion of homotopy (co)limits. We have already used two examples of homotopy colimits: the homotopy pushout (14) and the homotopy orbits of a group action (12). The homotopy colimit of a sequence of maps is also familiar: it is the infinite mapping telescope. A friendly and elegant reference for homotopy limits and colimits is [HV] (see also [BK] and [V]).

As usual, we regard posets as categories with exactly one arrow between two objects a and b when $a \leq b$.

If X is a space, we denote by \mathcal{P}_X the poset of subspaces of X ordered by *reverse* inclusion.

Let G be a topological group. We denote by \mathcal{K}_G the partially ordered set of compact subspaces of G ordered by inclusion. There is a canonical map

$$A_G = \text{hocolim}_{K \in \mathcal{K}_G} K \xrightarrow{\phi} G$$

which is a weak homotopy equivalence.

There is a strictly associative multiplication on A_G induced by the functor

$$\begin{aligned} \mathcal{K}_G \times \mathcal{K}_G &\rightarrow \mathcal{K}_G, \\ (K, L) &\mapsto KL \end{aligned}$$

and the natural isomorphism (see [HV, Proposition 3.1(4)])

$$A_G \times A_G = \operatorname{hocolim}_{(K,L) \in \mathcal{K}_G \times \mathcal{K}_G} K \times L.$$

Note that ϕ is a strictly multiplicative weak equivalence.

Consider the set

$$\mathcal{K}S_G = \{(K_n, \dots, K_1) \mid n \geq 0, K_i \subset G \text{ compact and } \neq \emptyset\}$$

(when $n = 0$, we mean the empty word) with the partial order defined by

$$(K_n, \dots, K_1) \leq (H_m, \dots, H_1) \quad \text{if } n \leq m \text{ and } K_i \subset H_i.$$

Given sequences $S, T \in \mathcal{K}S_G$, we write $S * T$ for their concatenation.

Definition C.1

Let G be a topological group, let X be a G -space, and let $U \subset X$ be a subspace (not necessarily G -invariant). A *near action* of G on U consists of a functor

$$\begin{aligned} \mathcal{K}S_G &\rightarrow \mathcal{P}_U, \\ (K_n, \dots, K_1) &\mapsto U_{(K_n, \dots, K_1)} \end{aligned}$$

such that

- (i) $U_\emptyset = U$ (where \emptyset denotes the empty sequence);
- (ii) for each $S \in \mathcal{K}$, the inclusion $U_S \rightarrow U$ is a weak equivalence;
- (iii) given $K \in \mathcal{K}_G$ and $T \in \mathcal{K}S_G$, the restriction of the G -action

$$K \times U_{(K)*T} \rightarrow X$$

has its image contained in U_T .

A *near G -equivariant map* is a natural transformation of functors commuting strictly with the action of the compact subsets.

LEMMA C.2

A near action of G on U induces a canonical action of A_G on $T(U) = \operatorname{holim}_{S \in \mathcal{K}S_G} U_S$.

Proof

There is an obvious action $K \times T(U) \rightarrow T(U)$ for each compact set, and this extends canonically to the required action. \square

Note that there is a canonical homotopy equivalence

$$T(U) \xrightarrow{\pi} U$$

induced by the inclusion of the empty sequence in $\mathcal{K}S_G$.

Remark C.3

An A_∞ -action (see [S]) of a topological monoid G on a space X is a map $G \times X \rightarrow X$ which is “associative up to all higher homotopies” with respect to the multiplication on G . One way of giving such an A_∞ -action is to give an actual action of a topological monoid G' on a space X' together with weak equivalences $G \rightarrow G'$ and $X \rightarrow X'$. One can then perform homotopy meaningful constructions (such as homotopy orbits) replacing G, X with G', X' .

Setting $G' = A_G$ and $U' = T(U)$, Lemma C.2 shows that a near action of G on U gives rise to an A_∞ -action of G on U .

Definition C.4

The *homotopy orbit space* $\overline{U_{hG}}$ of a near action of G on U is the realization of the semisimplicial space

$$n \mapsto A_G^n \times T(U).$$

A G -space U has a trivial near G -action where $U_S = U$ for all $S \in \mathcal{K}S_G$, and unless we specify otherwise we always give G -spaces this near G -action.

LEMMA C.5

If U is G -invariant, there is a weak equivalence

$$\overline{U_{hG}} \rightarrow U_{hG}.$$

Proof

For each compact set, there is a canonical homotopy making the diagram

$$\begin{array}{ccc} K \times T(U) & \longrightarrow & K \times U \\ \downarrow & & \downarrow \\ T(U) & \longrightarrow & U \end{array}$$

and this yields a homotopy coherent weak equivalence between the two semisimplicial spaces in question. The result follows by taking realization. □

We can now elaborate on the proof of Theorem 5.5. With reference to the notation in the statement of that theorem, in the remainder of this section we write K for the isometry group $K(m)$, W for the representation of K on the normal to the corresponding stratum (described in Theorem 4.2), V for the stratum $V_{\ell+1}$, NV for its tubular neighborhood, X for $\mathcal{F}_{\lambda+1}$, and G for $G_{\lambda+1}$.

We fix a K -invariant metric on W , and given a continuous function $\epsilon : G \rightarrow \mathbb{R}_+$, we write

$$G \times_K (W \setminus 0)_\epsilon = \{g \cdot w \mid g \in G, 0 < |w| < \epsilon(g)\}.$$

PROPOSITION C.6

There is a continuous function $\epsilon : G \rightarrow \mathbb{R}_+$ such that

$$\begin{array}{ccc} G \times_K (W \setminus 0)_\epsilon & \xhookrightarrow{\psi} & NV \setminus V \\ \downarrow & & \downarrow \pi \\ G/K & \xhookrightarrow{\quad} & V \end{array}$$

commutes in the (weak) homotopy category. Moreover, ψ is a weak equivalence.

Proof

The function ϵ exists by continuity of the action. The slice theorem for the action of the symplectomorphism group G on the space of compatible almost complex structures* together with the uniqueness of tubular neighborhoods give, for each right K -invariant compact subset $L \subset G$, a homeomorphism ψ_L , homotopic to the inclusion, such that the diagram

$$\begin{array}{ccc} L \times_K (V \setminus 0) & \xhookrightarrow{\psi_L} & \pi^{-1}(L/K) \\ & \searrow & \swarrow \pi \\ & L/K & \end{array}$$

commutes. The uniqueness of tubular neighborhoods implies that if $L \subset L'$, $\psi_{L'|(L \times_K (V \setminus 0))}$ is homotopic over L/K to ψ_L . The result follows. \square

*The construction of the slice for the action of the diffeomorphism group on the space of metrics in [E] works in this case.

Remark C.7

The crucial point in the proof of Proposition C.6 is the existence of a slice for the action. One can apply the arguments in this section whenever this is the case.

The subspace

$$G \times_K (W \setminus 0)_\epsilon \subset NV \setminus V$$

can be endowed with a near G -action by choosing for each sequence $S \in \mathcal{K}S_G$ a continuous function $\epsilon_S: G \rightarrow \mathbb{R}_+$ with $\epsilon_\emptyset = \epsilon$ in such a way that for each compact subset $L \subset G$,

$$L \cdot (G \times_K (W \setminus 0)_{\epsilon_{(L)*S}}) \subset G \times_K (W \setminus 0)_{\epsilon_S}.$$

Giving a G -space U the trivial near G -action, we have a pushout diagram of near G -spaces and near G -equivariant maps

$$G/K \longleftarrow G \times_K (W \setminus 0)_\epsilon \longrightarrow (X \setminus V).$$

Writing P for the homotopy pushout, there is an obvious near G -action on P together with a near G -equivariant map

$$P \rightarrow X$$

which is clearly a weak equivalence. Since the canonical map

$$\text{hocolim} (T(G/K) \leftarrow T(G \times_K (W \setminus 0)_\epsilon) \rightarrow T(X \setminus V)) \rightarrow T(P)$$

is a weak equivalence, applying A_G homotopy orbits (in the sense of Definition C.4) and Lemma C.5 we get

$$\text{hocolim} (BK \leftarrow (W \setminus 0)_{hK} \rightarrow (X \setminus V)_{hG}) \simeq X_{hG}, \tag{A.1}$$

as required.

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