Instituto Superior Técnico
Departamento de Matemática

SYMPLECTIC GEOMETRY - $2^{\underline{\text{nd}}}$ Semester 2020/21

Problem Set $\sharp 5$

Due date: May 28

1. Let $F : \mathbb{R}^n \to \mathcal{S}_n \equiv$ symmetric $n \times n$ matrices, a smooth function such that $F(u)$ is non-singular for all $u \in \mathbb{R}^n$. Consider a 2-form ω defined in $\mathbb{R}^{2n} = \{(u,v) : u, v \in \mathbb{R}^n\}$ by the skew-symmetric matrix

$$\omega_{(u,v)} = \begin{bmatrix} 0 & F(u) \\ -F(u) & 0 \end{bmatrix}$$

   a) Show that $d\omega = 0$ if and only if there exists a smooth function $f : \mathbb{R}^n \to \mathbb{R}$, $f = f(u)$, such that

$$F = \mathrm{Hess}_u(f) \equiv \left[\frac{\partial^2 f}{\partial u_k \partial u_l} \right]_{k,l=1}^{n,n} .$$

   b) Consider the action of $\mathbb{R}^n$ on $\mathbb{R}^{2n}$ given by

$$t \cdot (u,v) = (u, v+t)\,, \ \forall\, t, u, v \in \mathbb{R}^n .$$

   Assuming that $d\omega = 0$, show that this action of $\mathbb{R}^n$ on $(\mathbb{R}^{2n}, \omega)$ is Hamiltonian and determine a moment map $\mu : \mathbb{R}^{2n} \to \mathbb{R}^n$.

   c) Under the conditions of b), determine local symplectic coordinates for $(\mathbb{R}^{2n}, \omega)$, i.e. local symplectomorphisms between $(\mathbb{R}^{2n} = \{(u,v)\}, \omega_{(u,v)})$ and $(\mathbb{R}^{2n} = \{(x,y)\}, \omega_0 = dx \wedge dy)$.

2. Let (M, ω) be a symplectic manifold and $\phi : \mathbb{T}^m \to \mathrm{Ham}(M, \omega)$ a Hamiltonian action with moment map $\mu : M \to \mathbb{R}^m$. Let $\psi : \mathbb{T}^l \to \mathbb{T}^m$ be a homomorphism given by $\psi(\theta) = A\theta$, with $A \in \mathbb{Z}^{m \times l}$ a $(m \times l)$ matrix with entries in $\mathbb{Z}$. Show that $\phi \circ \psi : \mathbb{T}^l \to \mathrm{Ham}(M, \omega)$ is a Hamiltonian action with moment map $\nu : M \to \mathbb{R}^l$ given by $\nu = A^t \mu$.

3. Let (M, ω) be a symplectic manifold and $\phi : \mathbb{T}^m = \mathbb{T}^{m_1} \times \mathbb{T}^{m_2} \to \mathrm{Ham}(M, \omega)$ a Hamiltonian action with moment map $\mu = (\mu_1, \mu_2) : M \to \mathbb{R}^m = \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$.

   a) Show that $\mu_i : M \to \mathbb{R}^{m_i}$, $i = 1, 2$, is a moment map for the Hamiltonian action of $\mathbb{T}^{m_i}$ on (M, ω) naturally induced by ϕ.

   b) Let $Z_1 = \mu_1^{-1}(0)$, suppose that $\mathbb{T}^{m_1}$ acts freely on Z_1, and denote by (M_0, τ_0) the corresponding symplectic reduction, i.e. $M_0 = Z_1/\mathbb{T}^{m_1}$ and τ_0 is the symplectic form on M_0 characterized by $\pi^*(\tau_0) = \iota^*(\omega)$, where $\pi : Z_1 \to M_0$ and $\iota : Z_1 \to M$ denote, respectively, the natural projection and inclusion. Show that the Hamiltonian action of $\mathbb{T}^{m_2}$ on M naturally induces a Hamiltonian action of $\mathbb{T}^{m_2}$ on M_0 with a moment map characterized $\nu : M_0 \to \mathbb{R}^{m_2}$ characterized by $\nu \circ \pi = \mu_2 \circ \iota$.

4. Consider the action of $\mathbb{T}^1 = S^1$ on $(\mathbb{R}^{2(n+1)}, \omega_0)$ that, using the natural identification $\mathbb{R}^{2(n+1)} = \mathbb{C}^{n+1}$, is given by

$$\theta \cdot (z_0, \ldots, z_n) = (e^{-i\theta} z_0, \ldots, e^{-i\theta} z_n)\,, \ \forall\, z \in \mathbb{C}^{n+1},\ \theta \in \mathbb{R}\ .$$

   a) Check that this action is Hamiltonian with moment map $\mu : \mathbb{C}^{n+1} \to \mathbb{R}$ given by

$$\mu(z) = \frac{1}{2}(|z|^2 - 1)\ .$$

   Hint: use exercise 2.

   b) Show that the symplectic reduction $(M_0 = \mu^{-1}(0)/S^1, \tau_0)$ is $\mathbb{C}\mathrm{P}^n$ with the Fubini-Study symplectic form ω_{FS}.

5. Show that the action of $\mathbb{T}^n$ on $(\mathbb{C}\mathrm{P}^n, \omega_{\mathrm{FS}})$ given by

$$(\theta_1, \ldots, \theta_n) \cdot [z_0, \ldots, z_n] = [z_0, e^{-i\theta_1} z_1, \ldots, e^{-i\theta_n} z_n]\,, \ \forall\, z \in \mathbb{C}^{n+1},\ \theta \in \mathbb{R}^n\,,$$

   is Hamiltonian and determine a moment map $\mu : \mathbb{C}\mathrm{P}^n \to \mathbb{R}^n$. Characterize explicitly (i.e. without using the Atiyah-Guillemin-Sternberg Convexity Theorem) the moment polytope $\mu(\mathbb{C}\mathrm{P}^n)$.
   Hint: use exercises 3 and 4.