Instituto Superior Técnico
Departamento de Matemática

SYMPLECTIC GEOMETRY - 2^{nd} Semester 2020/21

Problem Set $\sharp 4$

Due date: May 14

1. Let (M, J) be an almost complex manifold and $f : M \to \mathbb{C}$ a smooth function such that $0 \in \mathbb{C}$ is a regular value. Show that if $(\bar{\partial} f)_p = 0$ at any point $p \in N := f^{-1}(0)$, then N is a complex submanifold of (M, J).

2. Let $H : \mathbb{R}^n \to \mathcal{S}_n \equiv$ symmetric $n \times n$ matrices, be a smooth map such that $H(x)$ is non-singular for all $x \in \mathbb{R}^n$. Consider the almost complex structure J defined on $\mathbb{R}^{2n}$ by
$$J_{(x,y)} = \begin{bmatrix} 0 & -H(x)^{-1} \\ H(x) & 0 \end{bmatrix}$$
   a) Show that J is integrable if and only if there exists a smooth function $h : \mathbb{R}^n \to \mathbb{R}$ such that
$$H = \text{Hess}_x(h) \equiv \left[\frac{\partial^2 h}{\partial x_k \partial x_l} \right]_{k,l=1}^n .$$
   b) Assuming J integrable, find local holomorphic coordinates for $(\mathbb{R}^{2n}, J)$, i.e. local complex isomorphisms between $(\mathbb{R}^{2n}, J)$ e $(\mathbb{C}^n, i)$.

3. Let (M, ω) be a symplectic manifold, $J \in \mathcal{J}(M, \omega)$ an almost complex structure compatible with ω and $\langle \cdot, \cdot \rangle_{\text{J}} \equiv \omega(\cdot, J \cdot)$ the Riemannian metric associated to ω and J. Given a smooth function $h : M \to \mathbb{R}$ let $X_h, \nabla h \in \mathcal{X}(M)$ be the symplectic and Riemannian gradients of h, i.e. defined by the relation
$$\omega(X_h, \cdot) = dh(\cdot) = \langle \nabla h, \cdot \rangle_{\text{J}} .$$
   Show that
$$\nabla h = J\, X_h \quad \text{e} \quad (\nabla h) \lrcorner\, \omega = i(\bar{\partial} - \partial) h .$$

4. Let (M, J) be a complex manifold and $f : M \to \mathbb{R}$ a smooth *strictly pluri-subharmonic* function, i.e. a function such that the $(1,1)$-form $\omega_f = \frac{i}{2} \partial \bar{\partial} f$ is symplectic and compatible with J on M. Let $\nabla f \in \mathcal{X}(M)$ be the gradient of f defined with respect to the Riemannian metric on M given by $\langle \cdot, \cdot \rangle_f \equiv \omega_f(\cdot, J \cdot)$. Assuming it exists, show that the 1-parameter flow of ∇f, denoted by ϕ_t, $t \in \mathbb{R}$, is such that
$$\phi_t^*(\omega_f) = e^{4t} \omega_f .$$
   Hint: use exercise 3 to show that $\mathcal{L}_{\nabla f}\, \omega_f = 4\omega_f$.

5. Consider the Kähler manifold (S^2, ω, J), where

$$S^2 = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : (x_1)^2 + (x_2)^2 + (x_3)^2 = 1\} \subset \mathbb{R}^3$$

and $\omega \in \Omega^2(S^2)$, $J \in \mathrm{Aut}(TS^2)$ are given by

$$\omega_x(u, v) = \langle x, u \times v\rangle \quad \text{and} \quad J_x(u) = x \times u\,, \ \forall x \in S^2\,,\ \forall u, v \in T_xS^2 \subset \mathbb{R}^3$$

($\langle \cdot, \cdot\rangle$ denotes the usual $\mathbb{R}^3$ inner product and $\times$ denotes the cross product).

a) Let $\phi : S^2 \setminus (0, 0, 1) \to \mathbb{R}^2$ be the stereographic projection. Show that ϕ is a Kähler isomorphism between $(S^2 \setminus (0, 0, 1), \omega, J)$ and $(\mathbb{R}^2, \tau, J_0)$, where

$$\tau = \frac{4\, dx \wedge dy}{(x^2 + y^2 + 1)^2}$$

and J_0 is the standard complex structure on $\mathbb{R}^2$, i.e. $(\mathbb{R}^2, J_0) \cong (\mathbb{C}, i)$.

b) Build a Kähler isomorphism between (S^2, ω, J) and $(\mathbb{CP}^1, 4\omega_{\mathrm{FS}}, i)$. Conclude that $\int_{\mathbb{CP}^1} \omega_{\mathrm{FS}} = \pi$.