Instituto Superior Técnico
Departamento de Matemática

SYMPLECTIC GEOMETRY - $2^{\underline{\text{nd}}}$ Semester 2020/21
Problem Set ♯3

Due date: April 30

1. Let (M,ω) be a closed symplectic manifold, $f,g \in C^\infty(M)$ two smooth functions and $\phi_t,\psi_t \in \mathrm{Ham}(M,\omega)$ the (autonomous) Hamiltonian flows generated by X_f and X_g. Show that $\phi_t \circ \psi_t = \psi_t \circ \phi_t\,,\ \forall t \in [0,1]$, if and only of the Poisson bracket $\{f,g\} \equiv \omega(X_f, X_g) \in C^\infty(M)$ is identically zero.

2. Show that the flux : $\widetilde{\mathrm{Diff}}_0(M,\omega) \to H^1(M,\mathbb{R})$, defined by
$$\mathrm{flux}(\phi_t) = \int_0^1 [X_t \lrcorner\, \omega]\, dt\ ,$$
where $\phi_t \in \mathrm{Diff}_0(M,\omega)$ satisfies
$$\frac{d}{dt}\phi_t = X_t \circ \phi_t \qquad \text{e} \qquad \phi_0 = \mathrm{id}\,,$$
is in fact a group homomorphism.

3. On a symplectic manifold (M,ω), consider the symplectomorphism $\psi \in \mathrm{Diff}(M,\omega)$ and an isotopy $\phi_t \in \mathrm{Diff}_0(M,\omega)\,,\ t \in [0,1]$. Show that
$$\mathrm{flux}(\psi^{-1} \circ \phi_t \circ \psi) = \psi^*(\mathrm{flux}(\phi_t))\ .$$

4. Consider the symplectic manifold $(M \times M, \tilde{\omega} = -\pi_1^*(\omega) + \pi_2^*(\omega))$, where (M,ω) is a symplectic manifold and $\pi_k : M \times M \to M\ ,\ k = 1,2$, are the natural projections. Let $\iota_\Delta : M \to M \times M$ be the diagonal inclusion $\iota_\Delta(p) = (p,p)\,,\ \forall p \in M$. Given an isotopy $\phi_t \in \mathrm{Diff}_0(M,\omega)\,,\ t \in [0,1]$, show that
$$\mathrm{flux}(\phi_t) = \iota_\Delta^*(\mathrm{flux}(\mathrm{id} \times \phi_t))\ .$$

5. Consider the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, with symplectic form ω naturally induced by the standard symplectic form ω_0 on $\mathbb{R}^2$. Let $\Gamma \subset H^1(M,\mathbb{R})$ be the flux group, i.e. $\Gamma \equiv \mathrm{flux}(\pi_1(\mathrm{Diff}_0(M,\omega)))$.
   a) Show that $\Gamma = H^1(M,\mathbb{Z}) \subset H^1(M,\mathbb{R})$.
   b) Let $\phi_t \in \mathrm{Diff}(\mathbb{T}^2)\,,\ t \in [0,1]$, be an isotopy given by
   $$\phi_t(x,y) = (x + f_t(x,y), y + g_t(x,y))\,,$$
   with $f_t, g_t : \mathbb{R}^2 \to \mathbb{R}$ two smooth families of smooth periodic functions, with period 1 in each of the variables $x,y \in \mathbb{R}$, and such that $f_0 \equiv g_0 \equiv 0$. Show that if $\phi_t \in \mathrm{Ham}(\mathbb{T}^2,\omega)\,,\ \forall t \in [0,1]$, then
   $$\int_{\mathbb{T}^2} f_t\, \omega = \int_{\mathbb{T}^2} g_t\, \omega = 0\,,\ \forall t \in [0,1]\ .$$