Instituto Superior Técnico
Departamento de Matemática

SYMPLECTIC GEOMETRY - 2^{nd} Semester 2020/21

Problem Set ♯2

Due date: April 09

1. Let (M,ω) be a symplectic manifold, $\psi : M \to M$ a symplectomorphism and $h : M \to \mathbb{R}$ a smooth function. Show that the Hamiltonian vector fields X_h and $X_{h\circ\psi^{-1}}$ satisfy the relation
$$X_{h\circ\psi^{-1}} = \psi_* X_h \ .$$

2. Let L be a smooth manifold and T^*L its cotangent bundle equipped with the canonical symplectic form $\omega_{\text{can}} = -d\lambda_{\text{can}}$. Let also $h : L \to \mathbb{R}$ be a smooth function and $\psi_h : T^*L \to T^*L$ a diffeomorphism defined by
$$\psi_h(x,\alpha) = (x, \alpha + dh_x)\,,\ \forall x \in L\,,\ \alpha \in T^*_x L\ .$$
Show that
$$\psi_h^*(\lambda_{\text{can}}) = \lambda_{\text{can}} + \pi^*(dh)\,,$$
where $\pi : T^*L \to L$ is the natural projection $\pi(x,\alpha) = x$. Conclude that ψ_h is a symplectomprphism of $(T^*L, \omega_{\text{can}})$.

3. Let M be a closed, connected, smooth manifold of dimension m, and let $\sigma_0, \sigma_1 \in \Omega^m(M)$ be two volume forms. Show that if $[\sigma_0] = [\sigma_1]$ in $H^m_{\text{dR}}(M)$, i.e. $\int_M \sigma_0 = \int_M \sigma_1$, then there exists a diffeomorphism $\phi : M \to M$ such that $\phi^*(\sigma_1) = \sigma_0$.

4. Consider the torus $\mathbb{T}^4 = \mathbb{R}^4/\mathbb{Z}^4 \cong S^1 \times S^1 \times S^1 \times S^1$, with symplectic form ω naturally induced by the standard symplectic form ω_0 on $\mathbb{R}^4$. Give examples of symplectic, isotropic, coisotropic and Lagrangian submanifolds of $(\mathbb{T}^4, \omega)$.

5. Let (M,ω) be a symplectic manifold, $h : M \to \mathbb{R}$ a smooth function with 0 as a regular value and $Q \subset M$ a codimension 1 (hence coisotropic) submanifold defined by $Q = h^{-1}(0)$. Show that the Hamiltonian vector field X_h satisfies
$$(X_h)_q \in (T_qQ)^\omega\ ,\ \forall q \in Q\ .$$

6. Let (M,ω) be a symplectic manifold and $Q \subset M$ a coisotropic submanifold. Show that the distribution $(TQ)^\omega \subset TQ$ is isotropic and integrable.