Instituto Superior Técnico
Departamento de Matemática

SYMPLECTIC GEOMETRY - $2^{\underline{\text{nd}}}$ Semester 2020/21
Problem Set ♯1
Due date: March 19

1. Let (V, ω) be a symplectic vector space. Show that any codimension 1 subspace $S \subset V$ is coisotropic.

2. (a) Let E be a real vector space. Show that $E \oplus E^*$ has a canonical symplectic structure ω_0 determined by $\omega_0(u \oplus \alpha, v \oplus \beta) = \beta(u) - \alpha(v)$.
   (b) Let L be a Lagrangian subspace of a symplectic vector space (V, ω). Show that there exists a symplectic linear map $\psi : (V, \omega) \to (L \oplus L^*, \omega_0)$ such that $\psi(u) = u \oplus 0\,,\ \forall u \in L$.

3. Let (V, ω) be a symplectic vector space, $J \in \mathcal{J}(V, \omega)$ a complex structure compatible with ω and $g_J(\cdot, \cdot) = \omega(\cdot, J\cdot)$ the associated inner product. Show that a subspace $L \subset V$ is Lagrangian iff $J(L) = L^\perp \equiv$ orthogonal complement of L with respect to g_J. Conclude that L is Lagrangian iff $J(L)$ is Lagrangian.

4. Let V be a real vector space of dimension $2n$ and $J : V \to V$ a complex structure Show that the space of symplectic forms on V that are compatible with J is convex.

5. Consider the symplectic manifold (S^2, ω), where
$$S^2 = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\} \subset \mathbb{R}^3$$
and $\omega \in \Omega^2(S^2)$ is given by
$$\omega_x(u, v) = \langle x, u \times v \rangle\,,\ \forall x \in S^2\,,\ \forall u, v \in T_xS^2 = \{x\}^\perp \subset \mathbb{R}^3$$
($\langle \cdot, \cdot \rangle$ denotes the usual inner product on $\mathbb{R}^3$ and $\times$ denotes the cross product). In other words, ω is the area form on S^2 induced by the Euclidean metric on $\mathbb{R}^3$.
   (a) Show that on $S^2 \setminus \{(0, 0, \pm 1)\}$, the symplectic form ω is given in polar cylindrical coordinates (θ, x_3), with $0 \le \theta < 2\pi$ and $-1 < x_3 < 1$, by $\omega = d\theta \wedge dx_3$.
   Note: this shows that the horizontal projection of the cylinder to the sphere is area preserving, a well known fact since Archimedes.
   (b) Use the previous result to show that the Hamiltonian flow generated by the function $h : S^2 \to \mathbb{R}$, $h(x) = x_3$, consists of rotations of S^2 around its vertical axis.

6. (a) Let L be a smooth manifold. Any diffeomorphism $f : L \to L$ induces naturally a diffeomorphism $F : T^*L \to T^*L$ by the formula
$$F(x, \alpha) = (f(x), ((df)_x^{-1})^* \alpha)\ .$$
   Show that F is a symplectomorphism of T^*L, i.e. $F^*\omega_{\text{can}} = \omega_{\text{can}}$.
   (b) Let Y be a vector field on L, $f_t : L \to L$ the flow generated by Y, $F_t : T^*L \to T^*L$ the 1-parameter group of symplectomorphisms induced by f_t, and X the vector field on T^*L that generates F_t. Show that $X = X_h$ is the Hamiltonian vector field of the function $h : T^*L \to \mathbb{R}$ given by $h(x, \alpha) = \alpha_x(Y(x))$.