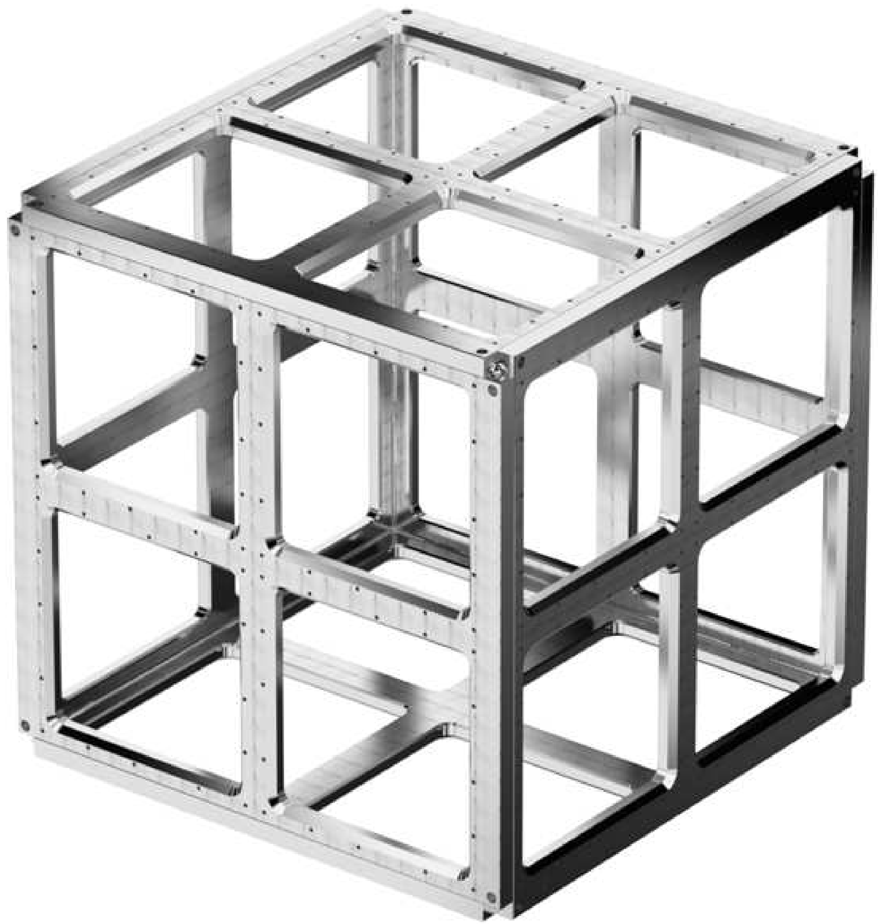
A short introduction to relativistic elasticity

José Natário

(Instituto Superior Técnico, Lisbon)

Lisbon, July 2026

Rigid bodies and Ehrenfest's paradox

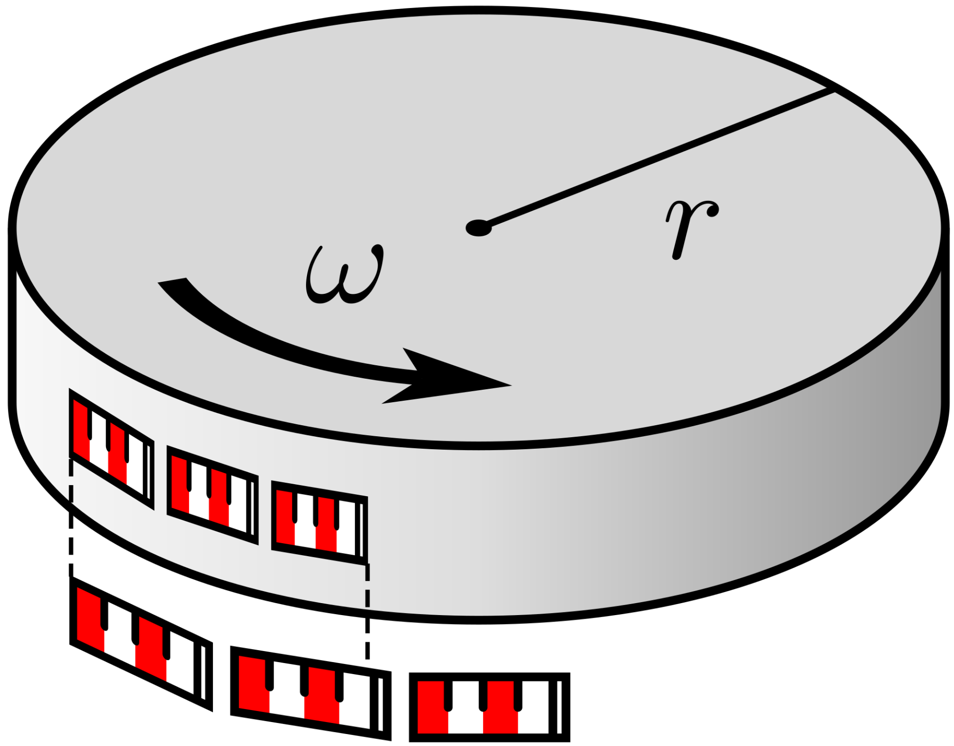


- Rigid body: distance between any two points at any given instant remains constant.

- So no rigid bodies in relativity.

- Physically, it takes time for one end of the body to receive information about forces acting on the other end.

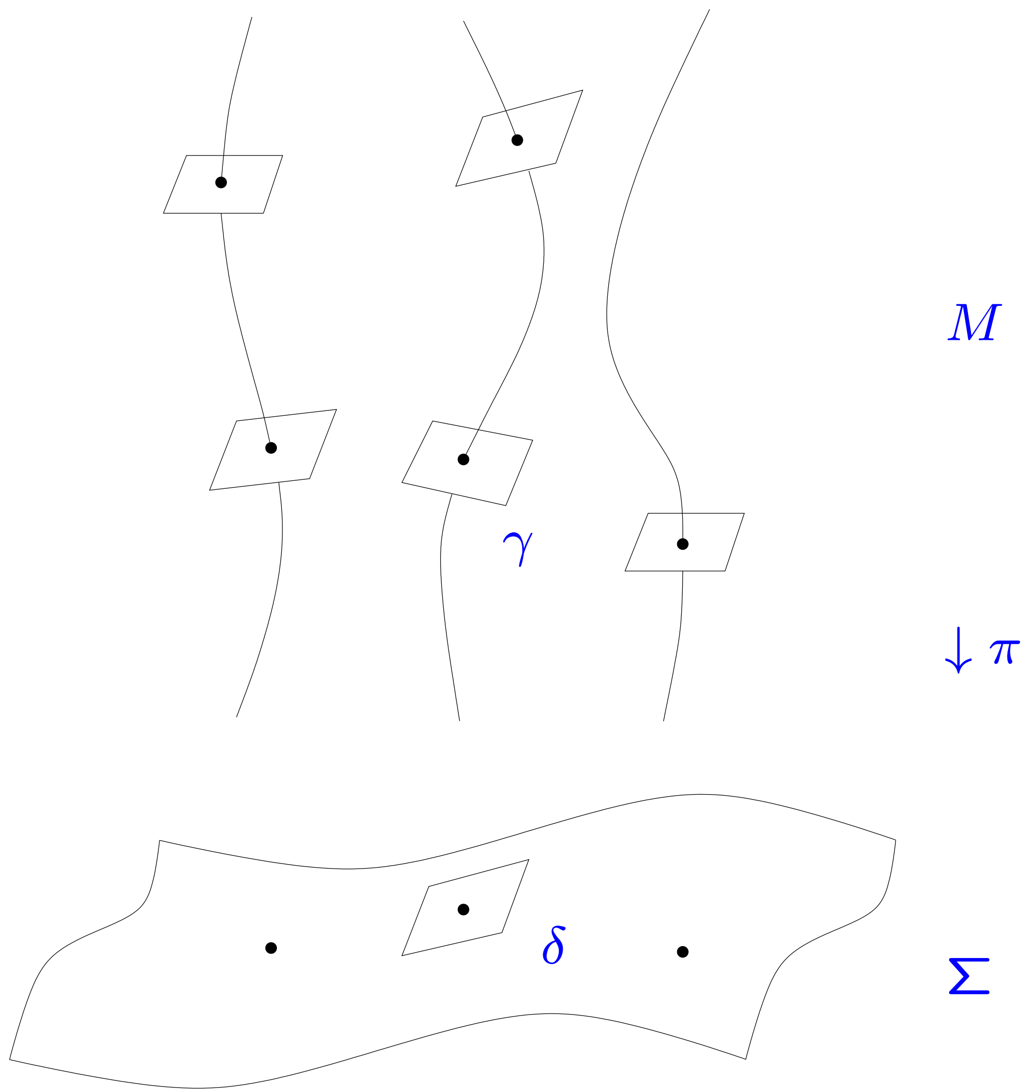
- Ehrenfest's paradox:



- So no undeformable bodies in relativity.

Relativistic elasticity

- An elastic medium in general relativity is described by:
  - A spacetime (M, g);
  - A Riemannian 3-manifold (Σ, δ) (reference configuration);
  - A projection map $\pi : M \to \Sigma$ whose level sets are timelike curves (the worldlines of the medium particles).


M
γ
$\downarrow \pi$
δ
Σ

- If we choose local coordinates $(\bar{x}^1, \bar{x}^2, \bar{x}^3)$ on Σ then we can think of π as a set of three scalar fields.

- We can complete $(\bar{x}^1, \bar{x}^2, \bar{x}^3)$ into coordinates $(\bar{t}, \bar{x}^1, \bar{x}^2, \bar{x}^3)$ for (M, g) yielding the rest frame of any given worldline:

$$g = -d\bar{t}^2 + \gamma_{ij} d\bar{x}^i d\bar{x}^j \qquad \text{(at that worldline).}$$

- Note that

$$\gamma = \gamma_{ij} d\bar{x}^i d\bar{x}^j$$

is a (time-dependent) Riemannian metric on Σ, describing the local deformations of the medium along each worldline.

- We can compute the (inverse) metric γ from

$$\gamma^{ij} = g^{\mu\nu} \frac{\partial \bar{x}^i}{\partial x^\mu} \frac{\partial \bar{x}^j}{\partial x^\nu}.$$

- Choose a Lagrangian density $\mathcal{L}$ of the form $\mathcal{L} = \mathcal{L}(\bar{x}^i, \delta_{ij}, \gamma^{ij})$ for the action

$$S = \int_M \mathcal{L} \sqrt{-g}\, d^4 x.$$

The energy-momentum tensor is then

$$T_{\mu\nu} = 2\,\frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} - \mathcal{L}\, g_{\mu\nu} = 2\,\frac{\partial \mathcal{L}}{\partial \gamma^{ij}}\, \partial_\mu \bar{x}^i\, \partial_\nu \bar{x}^j - \mathcal{L}\, g_{\mu\nu}.$$

- Therefore

$$\mathcal{L} = T_{\bar{0}\bar{0}} = \rho$$

is the rest energy density.

- The choice of $\rho = \rho(\bar{x}^i, \delta_{ij}, \gamma^{ij})$ is called the elastic law.

- Isotropic materials: ρ depends only on $({n_1}^2, {n_2}^2, {n_3}^2)$, the eigenvalues of γ^{ij} with respect to δ_{ij}. Note that (n_1, n_2, n_3) are the linear number densities (that is the inverses of the strech factors) along the principal directions.

- Assume that δ_{ij} is the Kronecker delta. In particular, we are assuming that the Riemannian 3-manifold (Σ, δ) is flat.

- More convenient variables:

$$\lambda_0 = \det(\gamma^{ij}) = (n_1 n_2 n_3)^2;$$

$$\lambda_1 = \mathrm{tr}(\gamma^{ij}) = {n_1}^2 + {n_2}^2 + {n_3}^2;$$

$$\lambda_2 = \mathrm{tr}\,\mathrm{cof}(\gamma^{ij}) = (n_1 n_2)^2 + (n_2 n_3)^2 + (n_1 n_3)^2.$$

- Examples:

  - Perfect fluid: $\rho = \rho(\lambda_0)$, yielding $p = 2\lambda_0 \dfrac{d\rho}{d\lambda_0} - \rho$.

  - Dust: $\rho = \rho_0 \sqrt{\lambda_0}$, yielding $p = 0$.

- Christodoulou's "hard phase" material: $\rho = \frac{\rho_0}{2}(\lambda_0 + 1)$, yielding $p = \rho - \rho_0$.

- Stiff fluid: $\rho = A\lambda_0$, yielding $p = \rho$.

- John quasi-Hookean materials: $\rho = f(\lambda_0) + g(\lambda_0)\lambda_2$.

- Karlovini-Samuelsson quasi-Hookean materials: $\rho = f(\lambda_0) + g(\lambda_0)\lambda_1\lambda_2$.

- Stiff ultra-rigid equation of state (SUREOS): $\rho = A\lambda_2 + B$.

Example: rigid rod

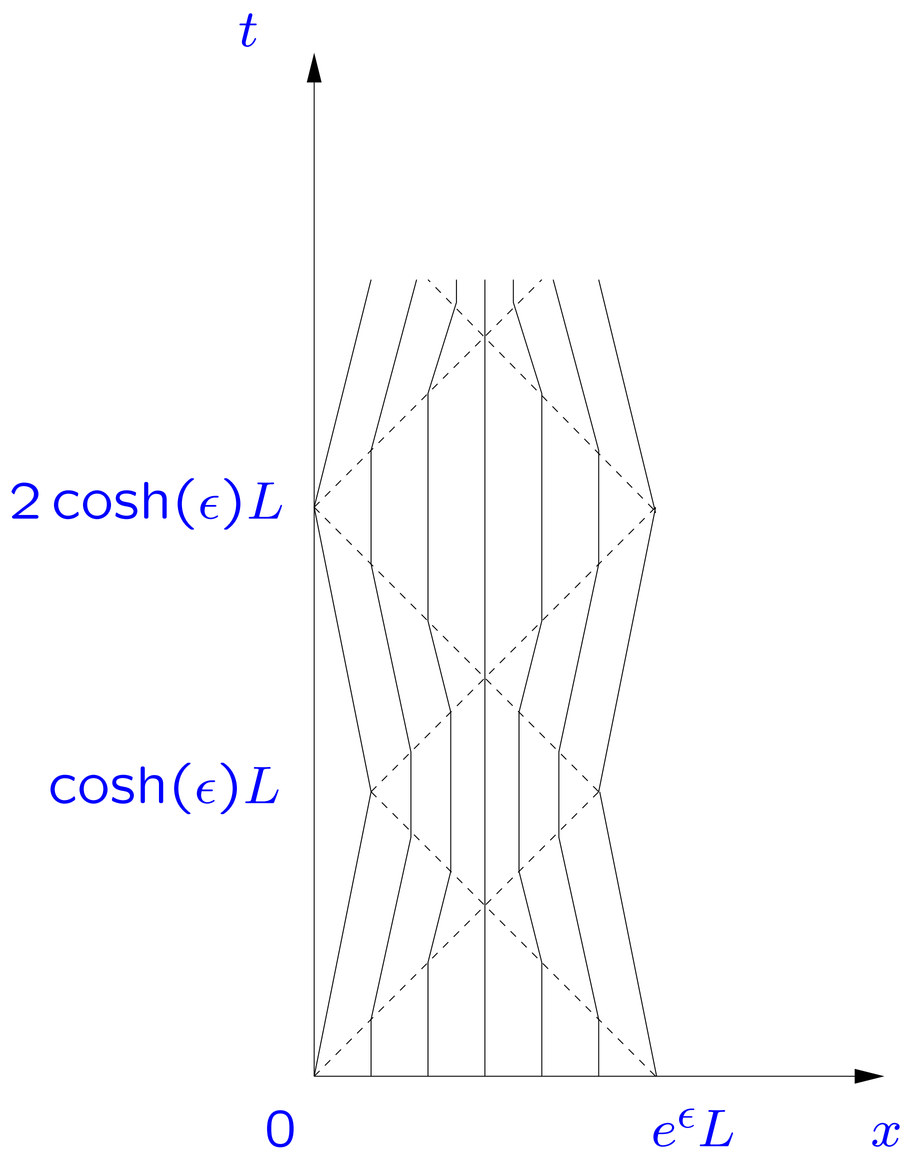
- For one-dimensional elastic bodies in a two-dimensional space-time (M, g) there is no difference between solids and fluids: the Lagrangian depends only on $\gamma^{11} = \partial_\alpha \bar{x} \partial^\alpha \bar{x}$.

- For a rigid elastic body (speed of sound = speed of light) we have $\rho = p + \rho_0$, yielding

$$\rho = \frac{\rho_0}{2}(\gamma^{11} + 1) = \frac{\rho_0}{2}(\partial_\alpha \bar{x} \partial^\alpha \bar{x} + 1).$$

- This is the Lagrangian for a massless scalar field, and so the equation of motion is just the wave equation $\Box\bar{x} = 0$, which can be exactly solved in two dimensions.

- For example, the motion of an uniformly stretched rigid rod released from rest (imposing zero pressure at the endpoints) in Minkowski's two-dimensional spacetime is as follows:



t
$2\cosh(\epsilon)L$
$\cosh(\epsilon)L$
0
$e^{\epsilon}L$
x

- Bibliography:

  - Carter and Quintana, *Foundations of general relativistic high-pressure elasticity theory*, Proc. R. Soc. Lond. A **331** (1972) 57–83.
  - Beig and Schmidt, *Relativistic Elasticity*, Class. Quant. Grav. **20** (2003) 889–904.
  - Karlovini and Samuelsson, *Elastic stars in general relativity: I. Foundations and equilibrium models*, Class. Quant. Grav. **20** (2003) 3613–3648.

Thank you!