Warp Drive

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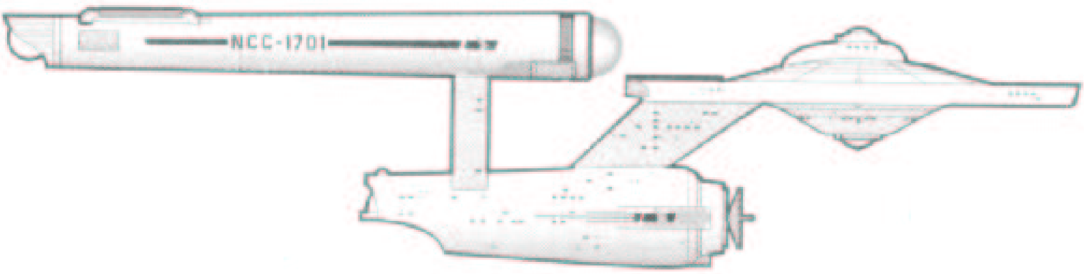


Special Relativity

Two basic ingredients:

1. **Inertial frames:** spacetime is $\mathbb{R}^4$ with coordinates (t, x, y, z); each slice $\{t = \text{constant}\}$ is an Euclidean 3-space; free particle motions are staight lines (law of inertia).
2. **Speed of light** is 1 for all inertial observers.

Poincaré Transformations: Coordinate transformation between different inertial observers. Must

1. Preserve straight lines $\Rightarrow$ affine (Exercise: prove that any bijection $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ preserving straight lines is an affine function).
2. Linear part (**Lorentz transformations**) must preserve cone $t^2 = x^2 + y^2 + z^2$.

Lorentz trasformations are **isometries** of the metric

$$g = -dt \otimes dt + dx \otimes dx + dy \otimes dy + dz \otimes dz$$

Spacetime is the **pseudo-Riemannian manifold** $(\mathbb{R}^4, g)$ (**Minkowski space**).

Vectors can be

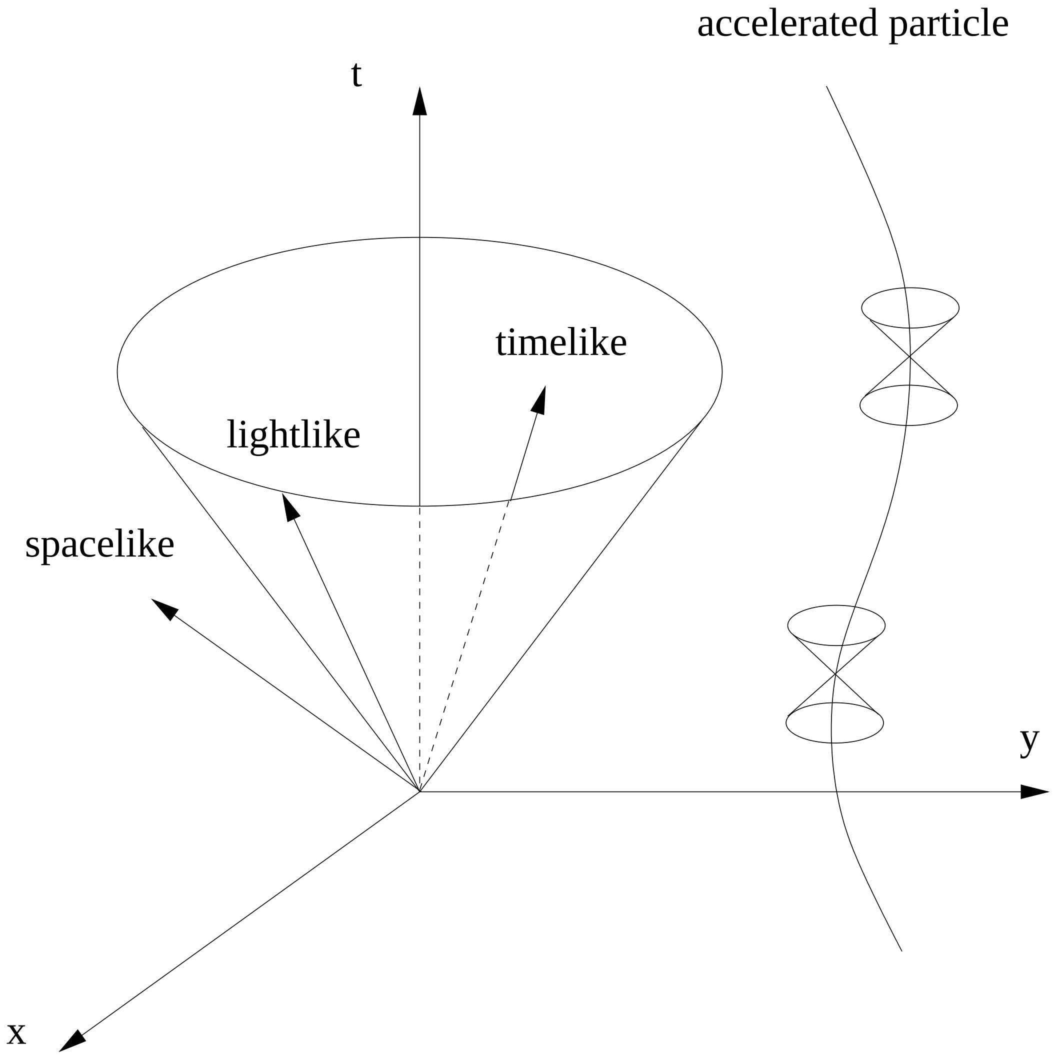
1. **Timelike**: $g(v, v) < 0$;
2. **Lightlike**: $g(v, v) = 0$;
3. **Spacelike**: $g(v, v) > 0$.

So can curves, geodesics.

It is not possible to order events along a spacelike curve $\Rightarrow$ **particles cannot have spacelike trajectories** (nothing can move faster than light). Length = elapsed **proper time**.

Free particles follow **timelike geodesics**.

Light rays follow **lightlike geodesics**.


accelerated particle
t
timelike
lightlike
spacelike
y
x

General Relativity

Particle in gravitational potential ϕ:

$$m_{\text{inertial}} \frac{d^2\mathbf{x}}{dt^2} = -m_{\text{gravitational}} \nabla \phi$$

$m_{\text{inertial}} = m_{\text{gravitational}} \Rightarrow$ free-falling particles follow same trajectories in spacetime (**Galileo**).

To describe gravity, take **curved** (M, g), with $\dim M = 4$ and g with signature $(- + ++)$.

Matter in Special Relativity is decribed by **energy momentum tensor** T. Equations of motion are $\mathrm{div}(T) = 0$.

Einstein tensor: $G = R - \frac{1}{2}\mathrm{tr}(R)g$
($R =$ Ricci tensor). Bianchi identity $\Rightarrow \mathrm{div}(G) = 0$.

Einstein equation: $G = 8\pi T$

Initial value formulation

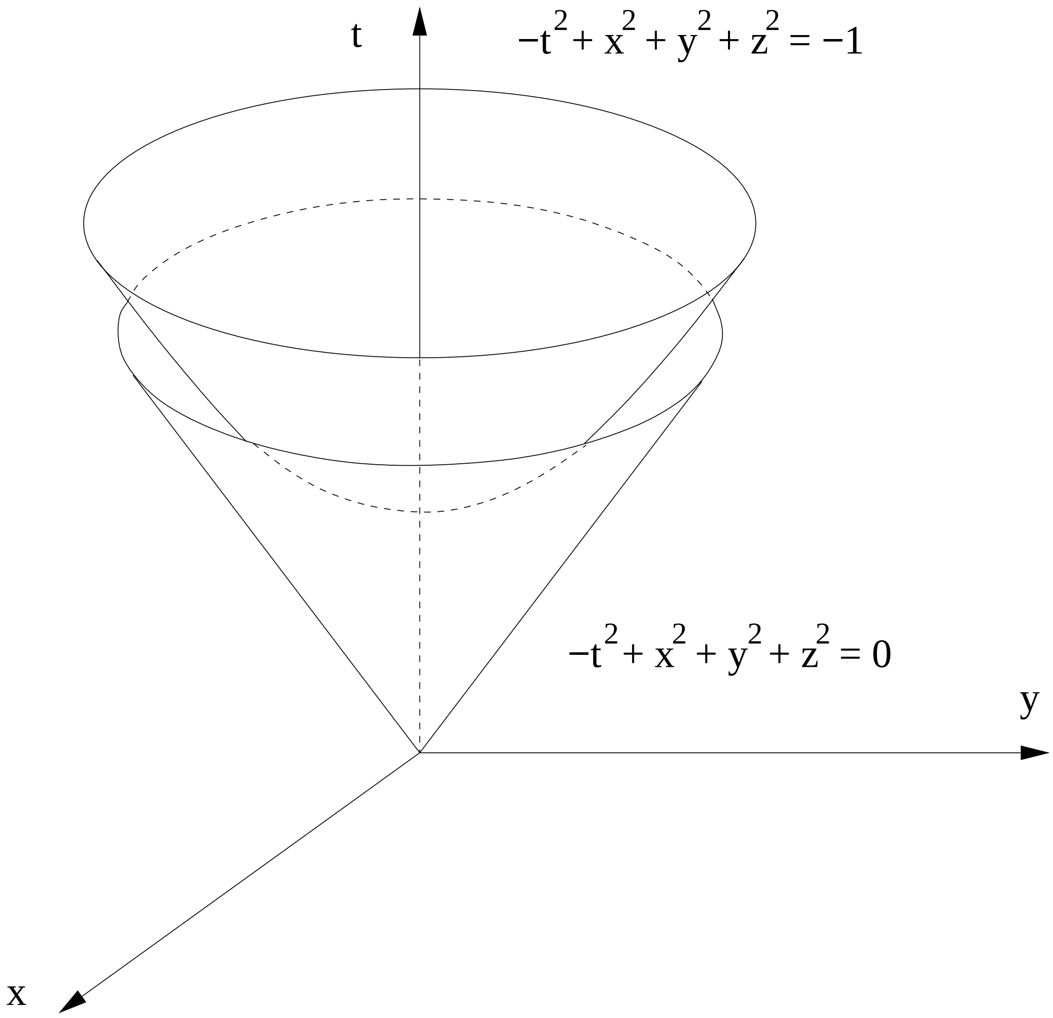
Take hypersurface $\Sigma \subset M$. If the normal vector to Σ is timelike, g induces a **Riemannian** metric on Σ.

Example: Each leaf of the hyperboloid

$$\Sigma = \{(t, x, y, z) : -t^2 + x^2 + y^2 + z^2 = -1\}$$

in Minkowski space is isometric to the hyperbolic 3-space. This can be used to show that the group of (proper) Lorentz transformations is the group of (orientation-preserving) isometries of the hyperbolic 3-space (i.e., $PSL(2, \mathbb{C})$).

For each point in Σ, take the geodesic whose initial condition is the unit normal vector. Flowing for a length t along each geodesic we get a hypersurface Σ_t diffeomorphic to $\Sigma = \Sigma_0$.


t
$-t^2 + x^2 + y^2 + z^2 = -1$
$-t^2 + x^2 + y^2 + z^2 = 0$
y
x

Local coordinates $\{x^1, x^2, x^3\}$ in Σ extend to local coordinates $\{t, x^1, x^2, x^3\}$ in M, and

$$g = -dt \otimes dt + \sum_{i,j=1}^{3} \gamma_{ij}(t) dx^i \otimes dx^j$$

where $\gamma(t)$ must be Riemannian. Therefore the Einstein equations yield an **evolution problem for a Riemannian metric $\gamma(t)$ on a 3-manifold Σ.**

Interpretation: The Riemannian manifold $(\Sigma, \gamma(t))$ describes how the family of free-falling observers whose trajectories are the geodesics (**Eulerian observers**) measure their relative positions evolve in their common proper time t.

Light rays move with speed 1 in $(\Sigma, \gamma(t))$. Slowly moving massive particles follow energy-minimizing curves (geodesics if γ constant).

Defining the **extrinsic curvature**

$$K(t) = \frac{1}{2}\frac{d\gamma}{dt} = \frac{1}{2}\pounds_{\frac{\partial}{\partial t}} g$$

the evolution equations are

$$\begin{aligned}
&\frac{d}{dt}\mathrm{tr}(K) + \mathrm{tr}(K^2) = \dots \\
&d(\mathrm{tr}(K)) - \mathrm{div}(K) = \dots \\
&\frac{dK}{dt} - 2K^2 + \mathrm{tr}(K)K + R = \dots
\end{aligned}$$

$\mathrm{tr}(K)$ measures the expansion of the volume element:

$$\pounds_{\frac{\partial}{\partial t}} dV_3 = (\mathrm{tr}(K)) dV_3$$

Strong energy condition $\Rightarrow$

$$\frac{d}{dt}\mathrm{tr}(K) + \mathrm{tr}(K^2) \leq 0$$

$(\mathrm{tr}A)^2 \leq n\mathrm{tr}(A^2)$ for A $n \times n$ symmetric $\Rightarrow$

$$\frac{d}{dt}\mathrm{tr}(K) + \frac{1}{3}(\mathrm{tr}(K))^2 \leq 0$$

$\Rightarrow$ $\mathrm{tr}(K)$ blows up. This could be either a **coordinate singularity** or a **geometric singularity** (meaning geodesic incompleteness). This is a very primitive version of the Penrose-Hawking singularity theorems.

Big Bang model

$\gamma(t)$ **homogeneous** and **isotropic** $\Rightarrow \gamma(t) = a(t)\gamma_0$, with γ_0 either

1. **Euclidean:**

   $$\gamma_0 = dr \otimes dr + r^2(d\theta \otimes d\theta + \sin^2\theta d\varphi \otimes d\varphi).$$

2. **Spherical:**

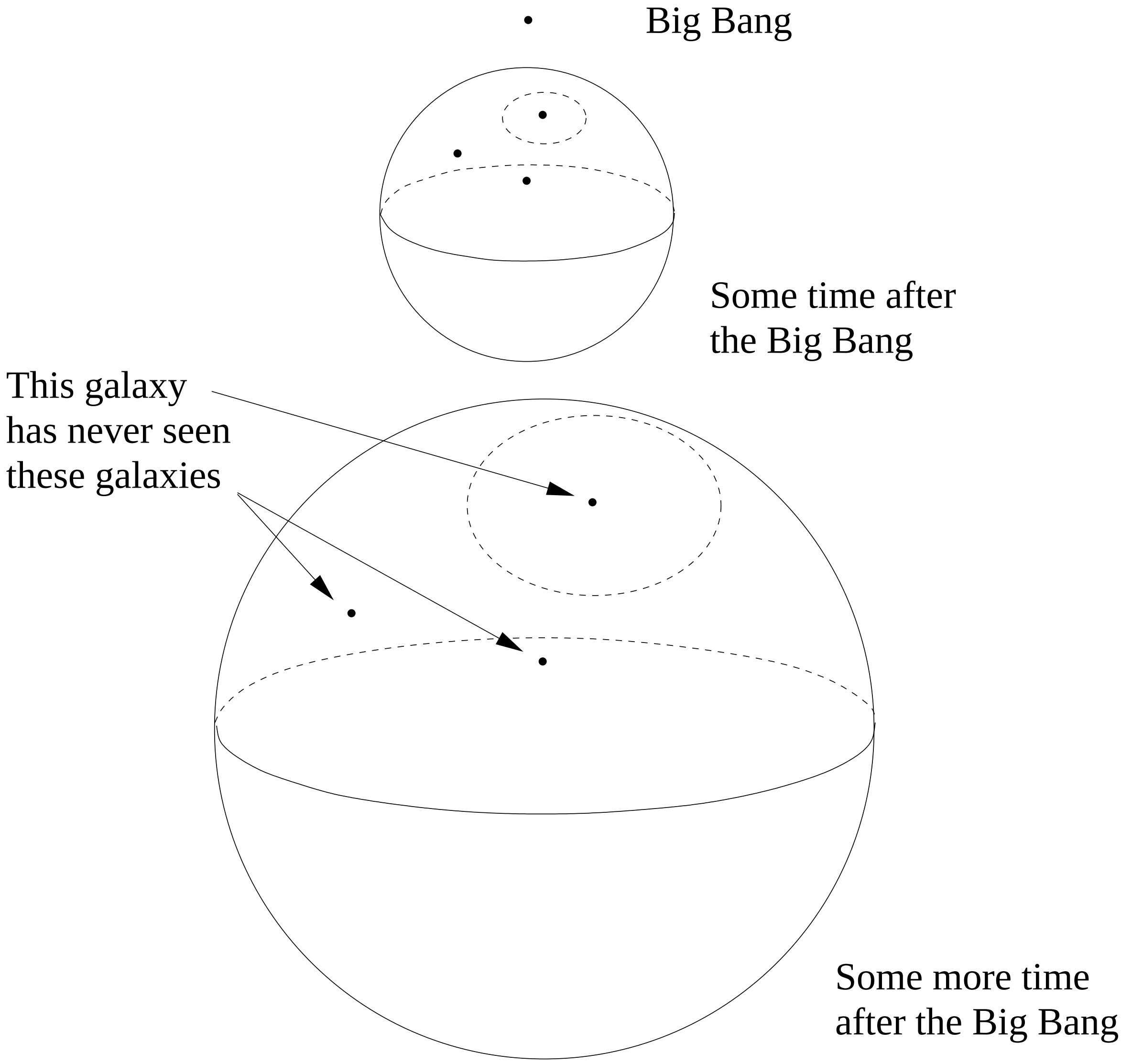
   $$\gamma_0 = dr \otimes dr + \sin^2 r(d\theta \otimes d\theta + \sin^2\theta d\varphi \otimes d\varphi).$$

3. **Hyperbolic:**

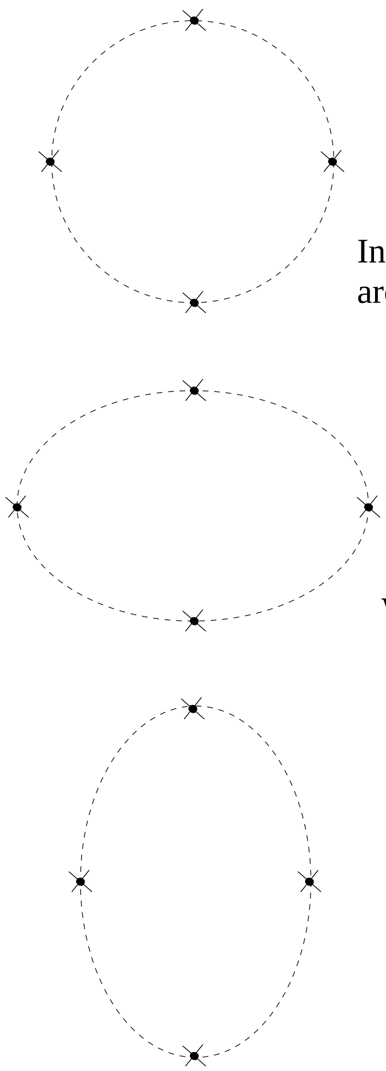
   $$\gamma_0 = dr \otimes dr + \sinh^2 r(d\theta \otimes d\theta + \sin^2\theta d\varphi \otimes d\varphi).$$

Evolution equations yield second order ODE for the **radius of the Universe** $a(t)$. The Big Bang corresponds to $a(t) = 0$. It did not occur in a particular point: there is no “center of the Universe”.

Galaxies may be moving faster than light with respect to each other (meaning that the distance between them is increasing faster than 1). This relates to the question of why the night sky is dark (even if the Universe has infinite volume)


Big Bang
Some time after
the Big Bang
This galaxy
has never seen
these galaxies
Some more time
after the Big Bang

Gravitational waves



Initially satellites
are at these distances

Wave hits

Warp drive spacetimes

Take $M = \mathbb{R}^4$ with coodinates (t, x^1, x^2, x^3), $\mathbf{X}$ a time-dependent vector field in $\mathbb{R}^3$ and

$$g = -dt \otimes dt + \sum_{i=1}^{3} (dx^i - X^i dt) \otimes (dx^i - X^i dt).$$

The slices $\{t = \text{constant}\}$ are just $(\mathbb{R}^3, \gamma)$ with

$$\gamma = \sum_{i=1}^{3} dx^i \otimes dx^i$$

(Euclidean 3-space!), but normal unit is

$$n = \frac{\partial}{\partial t} + \mathbf{X}$$

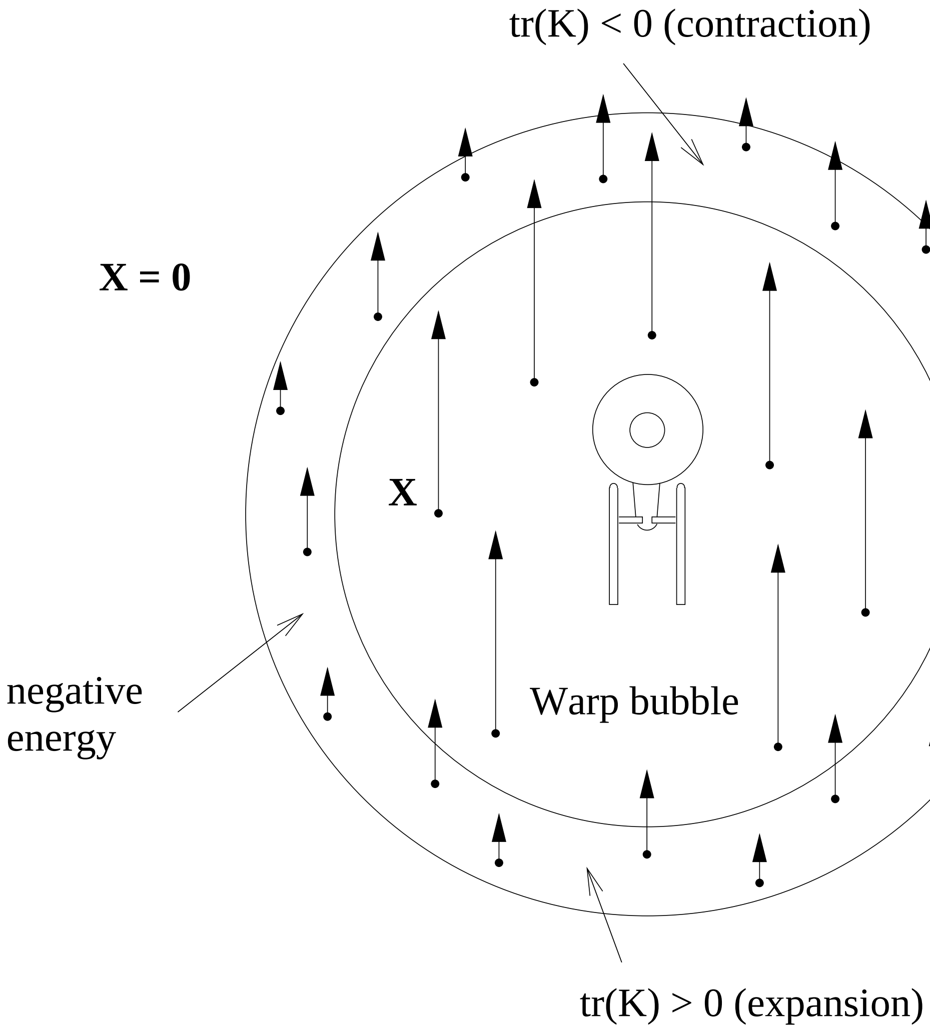
$\Rightarrow$ (free-falling) Eulerian observers move around in Euclidean 3-space following the (time-dependent) flow of $\mathbf{X}$. Extrinsic curvature is

$$K = \frac{1}{2} \pounds_n g = \frac{1}{2} \pounds_{\mathbf{X}} \gamma = \frac{1}{2} (D\mathbf{X} + D\mathbf{X}^t)$$

(in particular $\text{tr}(K) = \text{div}(\mathbf{X})$). Consequently, there is only curvature (hence matter, gravitational forces) where $\mathbf{X}$ is **not** a Killing vector field for the Euclidean metric.

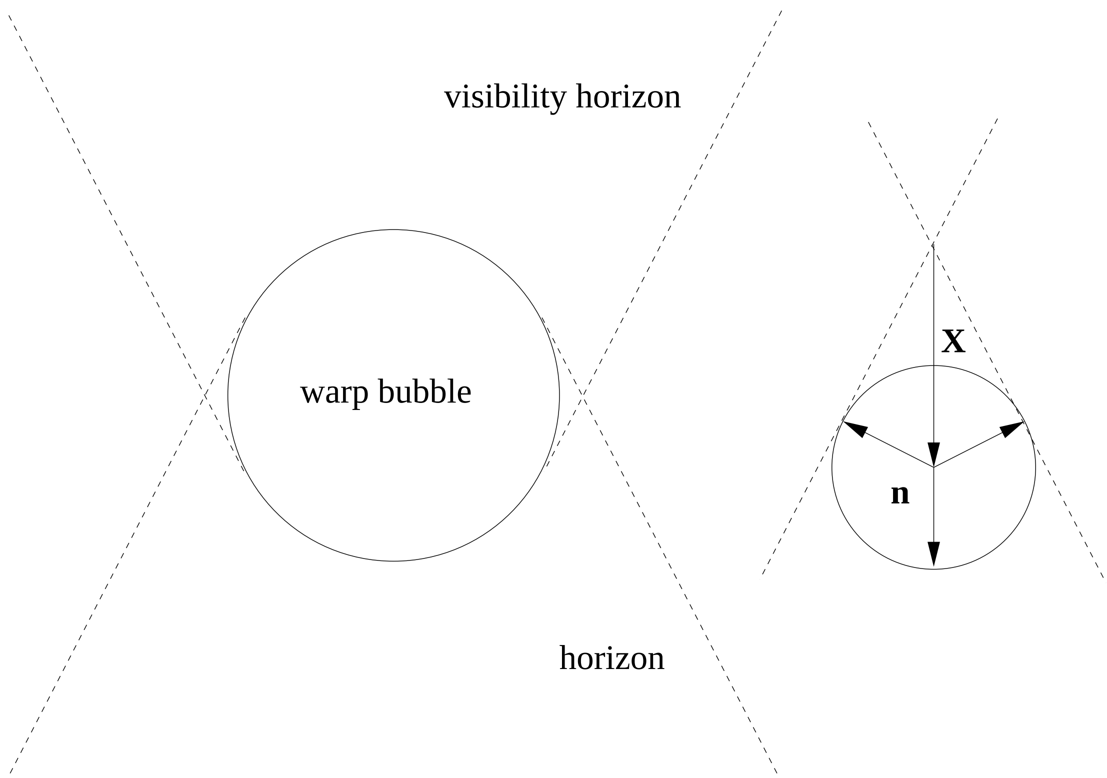
This metric is a solution of the Einstein equations for **some** matter field ($T = \frac{G}{8\pi}$). However it must violate strong energy condition (because $\mathrm{tr}(K)$ does not explode).

Alcubierre (1994):

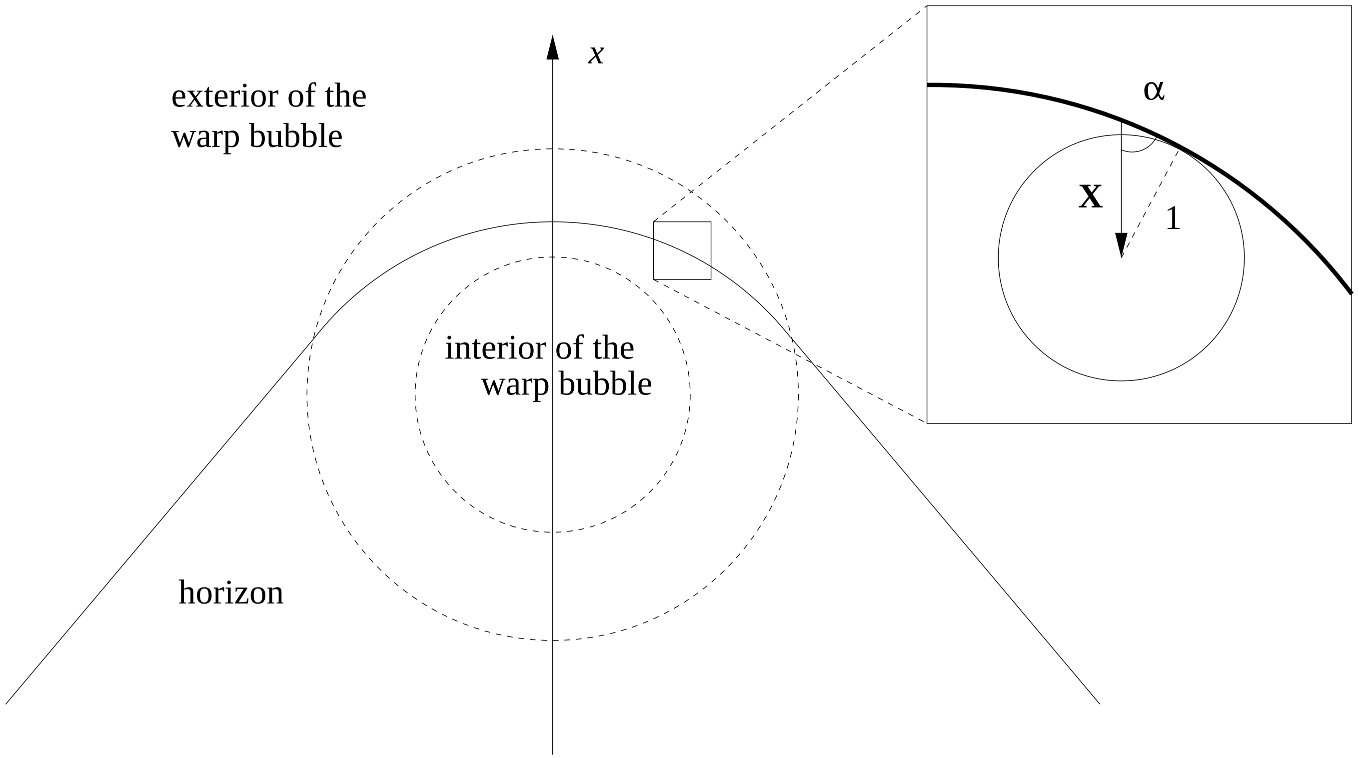


But contraction/expansion are not necessary: simply find **divergenceless** vector field $\mathbf{X}$ constant inside the warp bubble and zero outside.

Can assume warp bubble stationary. If $\|\mathbf{X}\| > 1$ there appear horizons (much like a supersonic Mach cone):



This is a problem:


x
exterior of the
warp bubble
interior of the
warp bubble
horizon
α
x
1