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INSTITUTO SUPERIOR TÉCNICO

Cosmic No-Hair for Spherically Symmetric Black Hole Spacetimes

Pedro Fontoura Correia de Oliveira

Supervisor:	**Doctor José António Maciel Natário**
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Thesis approved in public session to obtain the PhD Degree in Mathematics

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Abstract

It is generally expected that, for late cosmological times, spacetimes with a positive cosmological constant will asymptotically approach the de Sitter solution. This expectation is substantiated in the celebrated Cosmic No-Hair Conjecture, to which this thesis is devoted. Physically, it leads to the isotropization and homogenization of a spacetime undergoing an accelerating expansion, and provides important insights into the evolution of our own Universe.

To study this conjecture, we start by analyzing the decay of solutions of the wave equation in some expanding cosmological spacetimes, namely in flat Friedmann-Lemaître-Robertson-Walker (FLRW) models and in the cosmological region of the Reissner-Nordström-de Sitter (RNdS) solution, as a proxy to the full system including the Einstein equations. By introducing a partial energy and using an iteration scheme, we find that, for initial data with finite higher order energies, the decay rate of the time derivative is faster than previously existing estimates. For models undergoing accelerated expansion, our decay rate appears to be (almost) sharp.

We then examine in detail the geometry and dynamics of the cosmological region arising in spherically symmetric black hole solutions of the Einstein-Maxwell-scalar field system with a positive cosmological constant. More precisely, we solve, for such a system, a characteristic initial value problem with data emulating a dynamic cosmological horizon. Our assumptions are fairly weak, in that we only require that the data approach that of a subextremal Reissner-Nordström-de Sitter black hole, without imposing any rate of decay. We then show that the radius (of symmetry) blows up along any null ray parallel to the cosmological horizon ("near" i^+), in such a way that $r = +\infty$ is, in an appropriate sense, a spacelike hypersurface. We also prove a version of the Cosmic No-Hair Conjecture by showing that, in the past of any causal curve reaching infinity, both the metric and the Riemann curvature tensor asymptote those of a de Sitter spacetime. Finally, we discuss conditions under which all the previous results can be globalized.

Keywords

General Relativity, cosmological constant, wave equation in curved spacetime, Cauchy problem, Cosmic No-Hair.

Resumo

De um modo geral espera-se que, para tempos cosmológicos tardios, espaços-tempos com uma constante cosmológica positiva se aproximem assintoticamente da solução de de Sitter, expectativa essa substanciada na célebre Conjetura de Cosmic No-Hair, que vamos estudar nesta tese. Fisicamente, este fenómeno leva à isotropização e homogeneização de um espaço-tempo sujeito a uma expansão acelerada e permite tirar ilações importantes acerca da evolução do nosso próprio Universo.

Para estudar esta conjetura, começamos por analisar o decaimento de soluções da equação de onda em alguns espaços-tempos cosmológicos em expansão, nomeadamente em modelos de Friedmann-Lemaître-Robertson-Walker (FLRW) planos e na região cosmológica da solução de Reissner-Nordström-de Sitter (RNdS), enquanto substitutos para o sistema completo que inclui as equações de Einstein. Introduzindo uma energia parcial e utilizando um esquema iterativo, descobrimos que, para dados iniciais com energias de ordens superiores finitas, a taxa de decaimento da derivada temporal é mais rápida do que a das estimativas previamente existentes. Para modelos sob expansão acelerada, a nossa taxa de decaimento parece ser (quase) ótima.

De seguida examinamos em detalhe a geometria e dinâmica da região cosmológica que surge em soluções esfericamente simétricas do sistema Einstein-Maxwell-campo escalar com uma constante cosmológica positiva que contenham buracos negros. Mais precisamente, resolvemos, para tal sistema, um problema de valor inicial característico com dados que imitam um horizonte cosmológico dinâmico. As nossas hipóteses são razoavelmente fracas, na medida em que assumimos apenas que os dados se aproximam dos de um buraco negro de Reissner-Nordström-de Sitter sub-extremo, sem impor nenhuma taxa de decaimento. Mostramos então que o raio (de simetria) explode ao longo de qualquer geodésica nula paralela ao horizonte cosmológico ("perto" de i^+) de tal maneira que $r = +\infty$ é, num sentido apropriado, uma hiper-superfície do tipo espaço. Provamos também uma versão da Conjetura de Cosmic No-Hair mostrando que, no passado de qualquer curva causal que atinja o infinito, tanto a métrica como o tensor de curvatura de Riemann se aproximam assintoticamente dos de um espaço-tempo de de Sitter. Finalmente, discutimos condições sob as quais todos os resultados anteriores podem ser globalizados.

Palavras-Chave

Relatividade Geral, constante cosmológica, equação de onda em espaço-tempo curvo, problema de Cauchy, Cosmic No-Hair.

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List of Symbols

$\mathcal{M}$	spacetime manifold
$\mathcal{Q}$	space of group orbits $\mathcal{M}/SO(3)$
$\mathcal{C}$, $\mathcal{CH}$	cosmological horizon
$\mathscr{I}^+$	future null infinity
i^+	future timelike infinity
$J^{\pm}(p)$	causal future/past of p
$g_{\mu\nu}$	Lorentzian metric components
$d\Omega^2$, $\mathring{g}$	metric of unit round sphere
r	radius function
$\dot{}$	derivative with respect to t
$'$	either: a) derivative with respect to r; or b) derivative of a function of a single variable
ϕ	real massless scalar field
$F_{\mu\nu}$	Maxwell tensor components
$T_{\mu\nu}$	stress-energy tensor components
$R_{\mu\nu}$	Ricci tensor components
$R^{\alpha}{}_{\beta\gamma\delta}$	Riemann tensor components
Λ	cosmological constant
ϖ	Hawking mass
μ	mass aspect function, $\partial^\alpha r\,\partial_\alpha r \;=:\; 1-\mu \;=\; 1-\frac{2\varpi}{r}+\frac{e^2}{r^2}-\frac{\Lambda}{3}r^2$, where e is the electric charge
(u,v)	double-null coordinates
$\nu :=$	$\partial_u r$
$\lambda :=$	$\partial_v r$
$\zeta :=$	$r\partial_u\phi$
$\theta :=$	$r\partial_v\phi$
C	generic positive constant
δ, ϵ	(small) generic positive constant

1

Introduction

Contents

This thesis aims to present the results of the research work carried out during the course of my doctoral studies in Mathematical Relativity. For this purpose, we introduce in this first chapter the problem in which we are interested. Then, in Chapters 2 and 3 we will describe in detail the contents of articles on which this thesis is based – [17] and [16], respectively. Lastly, in Chapter 4 we will discuss some prospective avenues for future research.

1.1 General Relativity

General Relativity, originally developed and first published in 1915 by Albert Einstein, is the canonical geometric theory describing gravitation as the effects of matter and energy on the curvature of the spacetime manifold, the main concepts of which can be found on any standard textbook on Riemannian and Lorentzian geometry and on the mathematical techniques of relativity (to this effect, see for instance [21, 22, 31, 47, 61]). It is extremely well supported by an impressive amount of empirical evidence, ranging from the so-called classical tests – namely the perihelion precession of Mercury's orbit around the Sun, Arthur Eddington's measurement of the deflection of light rays by the Sun, and the gravitational redshift of light (verified for example by the Pound, Rebka, and Snider tests, as well as by satellite time discrepancies) – to the recent detection of gravitational waves by the LIGO Scientific Collaboration and the Virgo Collaboration [2].

1.1.1 Einstein, Maxwell and wave equations

The Einstein field equations are the cornerstone of the theory of general relativity. Independently found by both Einstein and Hilbert, the former motivated by physical considerations, the latter deriving them from a variational principle, they describe the geometry of spacetime as a function of the distribution of matter or energy, whose information is stored within the stress-energy tensor T.

In coordinates (x^0, x^1, x^2, x^3), where $x^0 \equiv t$ can be thought as being the time coordinate and (x^1, x^2, x^3) the spatial coordinates (Cartesian or spherical, for example), these equations take the form

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} , \tag{1.1}$$

where $G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2}R\, g_{\mu\nu}$ represents the Einstein tensor and Λ is the cosmological constant. It is useful to choose units such that $c = 4\pi G = 1$, where c is the speed of light in a vacuum and G the universal gravitational constant.

Einstein's equations need to be complemented with matter-energy equations in order to obtain a closed system. In this thesis we consider the contributions of both an electromagnetic field and a real massless scalar field, which are encoded in the expression

$$T_{\mu\nu} = \partial_\mu\phi\, \partial_\nu\phi - \frac{1}{2}\partial_\alpha\phi\, \partial^\alpha\phi\, g_{\mu\nu} + F_{\mu\alpha}F^{\alpha}{}_{\nu} - \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}g_{\mu\nu} , \tag{1.2}$$

with F being the respective electromagnetic field 2-tensor, defined by $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$, for a given electromagnetic 4-potential A, and ϕ being the scalar field.

In addition, by using the Hodge star notation, the source-free Maxwell's equations in curved spacetime, which describe the electromagnetic interaction in the context of Lorentzian geometry, can be written in a very compact form:

$$dF = d \star F = 0\,. \tag{1.3}$$

Similarly, the wave equation for a real massless scalar field ϕ is

$$\Box_g \phi = 0\,, \tag{1.4}$$

where $\Box_g := \nabla^\alpha \nabla_\alpha$. This wave equation can in turn be derived from the divergencelessness of the stress-energy tensor,

$$\nabla^\mu T_{\mu\nu} = 0\,. \tag{1.5}$$

Equations (1.1), (1.2), (1.3), and (1.4) thus comprise the setting of the Einstein-Maxwell-scalar field problem we ultimately wish to solve [16]. In order to study this system, though, one can start by using a simplified proxy consisting of solving the linear wave equation in a fixed background metric, that is, in an electrovacuum solution, which is obtained from solving just the source-free Einstein-Maxwell problem [17].

1.2 Cosmic No-Hair

The expression *cosmic no-hair*[1] (CNH) designates the phenomenon through which the cosmological region of a generic solution of the Einstein field equations with a positive cosmological constant dynamically evolves in such a way that, for late cosmological times, it asymptotically approaches the de Sitter metric.

In 1998, cosmological observations [49, 52] measuring the redshift and apparent brightness of type Ia supernovae (considered as "standard candles" due to their known constant intrinsic luminosity) revealed that our Universe is expanding at an accelerating rate. Now, adding a positive cosmological constant to the Einstein field equations provides that which is arguably the simplest way to model cosmological spacetimes undergoing an accelerating expansion. Such expansion has remarkable effects in the global causal structure of solutions, as well as their asymptotic behavior. In fact, it is widely expected that, generically, for late cosmological times, the expansion will completely dominate the dynamics, thus damping all inhomogeneities and anisotropies in the process, in such a

[1]As the name indicates, this in a way resembles, and was inspired by, the analogous family of conjectures concerning black holes, which state that the Kerr-Newman(-de Sitter) solution is an attractor for generic electrovacuum black hole exteriors in asymptotically flat (or asymptotically de Sitter) spacetimes. From the point of view of late-time observers, the only information that eventually survives the dynamical evolution of such a spacetime is the black hole's mass, electric charge and angular momentum. This is, in fact, the idea wherein lies the origin of the phrase coined by Jacob Bekenstein and popularized by John Archibald Wheeler: "black holes have no hair".

way that, in the end, only the information contained in the cosmological constant will persist; given these heuristics, it is then natural to predict the asymptotic approach to a de Sitter solution, the simplest and most symmetric solution of the field equations with a positive cosmological constant, an expectation which was, early on, substantiated in the celebrated *Cosmic No-Hair Conjecture*, initially proposed by Gary Gibbons and Stephen Hawking in 1976 [30].

This, in turn, specifically implies the isotropization and homogenization of the spacetime, which of course is itself the main physical motivation compelling us to consider the relevance of cosmic no-hair since, once again, and as is well known, cosmological observations reveal that not only is our Universe undergoing an accelerated expansion rate, for which the simplest model is the existence of a positive cosmological constant, but also that, at present, it is approximately isotropic and homogeneous when looked at from a very large scale perspective – despite the fact that it was not always so.

Nevertheless, and as usual in general relativity, obtaining a precise statement for this conjecture is one its biggest challenges. Right from the start one has to deal with the fact that the Nariai solution (see Appendix E) provides a simple counterexample for all the claims above (as do the other constant radius solutions arising from Birkhoff's theorem and which are listed in Appendix D); but since this solution is widely believed to be unstable [5] (compare with [27]), cosmic no-hair is still expected to hold generically. It turns out that the standard formulation of the conjecture does not differ much from the imprecise picture provided in the previous paragraphs.

A natural, but hopeless, attempt to capture the idea of cosmic no-hair is to say that it holds provided that generic solutions contain foliations of a neighborhood of (future null) infinity $\mathscr{I}^+$ along which they approach de Sitter uniformly. Although tempting, it fails to capture some subtleties created by *cosmic silence*. To better understand these let us for a moment consider the linear wave equation, in a fixed de Sitter[2] background, as a proxy to the Einstein equations. In this analogy, the previous formulation of cosmic no-hair would require (generic) solutions to decay to a constant at $\mathscr{I}^+$. However, it is well known that this is not the case [17, 51, 56]. To get an idea of why, note that cosmic silence is illustrated by the following construction: given any two inextendible causal curves γ_i, $i = 1, 2$, parameterized by proper time t and reaching $\mathscr{I}^+$ as $t \to \infty$, there is "a late enough" ($t \gg 1$) Cauchy surface Σ_t such that the sets $\mathcal{D}_i = J^-(\gamma_i) \cap J^+(\Sigma_t)$ are disjoint. Using the fact that the wave equation admits constant solutions, we prescribe a different constant at each $\mathcal{D}_i$, and then construct initial data on Σ_t by gluing the data induced on $J^-(\gamma_i) \cap \Sigma_t$ by the fixed constants; finally, we solve the wave equation for such data. In the end we obtain a solution which is not constant on $\mathscr{I}^+$! This shows that the previous "uniform approach" to de Sitter is too strong a requirement. A similar conclusion can be obtained for the full Einstein equations by considering the freedom we have to prescribe data at $\mathscr{I}^+$ for the Friedrich conformal field equations [25, 26, 36].

1.2.1 Overview of results to date

In 1983, Robert Wald [60] proved that spatially homogeneous solutions obeying certain energy conditions and with a positive cosmological constant eventually isotropize, that is, that the cosmic no-hair conjecture holds in this setting, though future global existence was not established and so further work is needed for specific cases.

[2] Or in Reissner-Nordström-de Sitter [17, 56] or Kerr-de Sitter [56].

Then, in 1986, Helmut Friedrich [25] proved the stability of de Sitter space, meaning that small perturbations in initial data corresponding to an expanding solution give rise to maximal globally hyperbolic developments that are future causally complete and asymptotically de Sitter, and which is essentially cosmic no-hair for solutions starting already close to the de Sitter metric.

As we have seen in the previous section, and as is typical to general relativity, it is not easy to obtain a precise statement of such a conjecture. The reasons for this are threefold, for, one the one hand, it has to be sufficiently generic – otherwise it would not be as interesting –, and on the other, it must be able to avoid some immediate counterexamples; besides, the prima facie obvious possibility one might be tempted to try – that cosmic no-hair holds if generic solutions contain foliations of a neighborhood of null future infinity along which they approach de Sitter asymptotically – fails. Hence, the standard formulation of cosmic no-hair is basically just the informal description to the effect that one should expect the expansion to dominate the dynamics for late cosmological times, damping all inhomogeneities and anisotropies in the process, so that the dynamical solution become undistinguishable from the de Sitter metric for all practical purposes. It is in fact sufficiently vague so that a considerable amount of relevant work concerning the global structure of solutions of the Einstein equations with a positive cosmological constant immediately fits the picture.

Some notable examples of such work, corresponding to research which has been carried out for several matter models and under various symmetry assumptions, include, for example, by Costa-Alho-Natário [12], Costa [10], Schlue [57], Rendall [51], Tchapnda-Rendall [6], Tchapnda-Noutchegueme [59], Dafermos-Rendall [18], Rodnianski-Speck [55], Speck [58], Ringström [53], Lübbe-Kroon [44], Oliynyk [48], and Friedrich [26]. In addition, there have also been some numerical simulations corroborating the heuristics described above (and hence our expectations) including, for instance, Nakao-Nakamura-Oohara-Maeda [46], Kitada-Maeda [34], and Beyer [5].

A precise formulation of a cosmic no-hair statement, properly taking into account the previous issues, was obtained only more recently by Andréasson and Ringström [3] (a formulation which in particular has helped us to obtain an appropriate statement for our own theorem). The fundamental new insight is to relax the previous requirement and instead say that cosmic no-hair holds if for each future inextendible causal curve γ and $\mathcal{D}_t = J^-(\gamma) \cap J^+(\Sigma_t)$, defined as before, $(\mathcal{D}_t, g)$ approaches (in a precise sense[3]) de Sitter as $t \to \infty$. In other words, every such observer will see the spacetime structure around him approaching that of a de Sitter spacetime, although, in general, this will not happen in a uniform way for all observers. Their result focuses on $\mathbb{T}^3$-Gowdy symmetry and uses the Einstein-Vlasov matter model, which is normally useful to emulate perfect fluids, in the sense that it behaves like radiation near the singularity and like dust in the expanding direction. We expect this form of cosmic no-hair to be valid for most, if not all, of the models considered in the references given above (see also Radermacher [50] for results concerning scalar fields in similar symmetric settings).

1.3 Summary of contributions

This section provides an overview of the original work carried out in this thesis, and which will be explained in detail in Chapters 2 and 3.

[3]See point 4 of Theorem 9 for the formulation used here.

1.3.1 Decay of solutions of the wave equation in expanding cosmological spacetimes

Before moving on to the study of the Einstein-Maxwell-scalar field problem, with our final goal being to establish a cosmic no-hair theorem in such a setting, it is useful to first look at the linear problem obtained by decoupling the wave equation for the scalar field from the rest of the system. In other words, we start by fixing the spacetime metric and then proceed to study how the scalar field behaves in this situation. This is, of course, interesting in itself, though our final motivation is to obtain the decay estimates for the scalar field which we may then compare with the respective rates obtained for the full nonlinear problem we shall look at later, and so this analysis provides us with important insights to such effect. This is the topic of chapter 2, which in turn is based on the work developed in article [17].

Specifically, our purpose is to obtain the exact decay rates for solutions of the wave equation in expanding cosmological spacetimes. We are particularly interested in the decay rates of time derivatives, because they characterize the asymptotics of solutions in spacetimes with a spacelike null infinity $\mathscr{I}^+$. Moreover, if one regards the wave equation as a proxy for the Einstein equations, then time derivatives are akin to second fundamental forms of time slices, and their decay rates may be useful in formulating and proving cosmic no-hair theorems (in the spirit of [3]).

A physical argument about which decay to expect can be made by considering an expanding Friedmann-Lemaître-Robertson-Walker (FLRW) model with flat n-dimensional spatial sections of radius $a(t)$. On one hand, the energy density of a solution ϕ of the wave equation is of the order of $(\partial_t\phi)^2$. On the other hand, if the wavelength of the particles associated with ϕ follows the expansion, then it is proportional to $a(t)$, and so their energy varies as $a(t)^{-1}$. Therefore, the energy density should behave like $a(t)^{-n-1}$, and we would expect $\partial_t\phi$ to decay as $a(t)^{-\frac{n+1}{2}}$. We shall see that, in reality, things are more complicated: this decay rate only seems to hold for spacetimes which are expanding sufficiently slowly, and, in particular, do not have a spacelike $\mathscr{I}^+$.

There are few results in the literature about this problem. Klainerman and Sarnak [35] gave the explicit solution for the wave equation in FLRW models corresponding to dust matter, zero cosmological constant and 3-dimensional flat or hyperbolic spatial sections. This was used in [1] to show that, in the flat case, solutions with initial data of compact support decay as t^{-1}, that is, as $a(t)^{-\frac{3}{2}}$ (but without decay estimates for the time derivative). The same problem was studied further in [28, 29], including L^p-L^q decay estimates and paramatrices. The wave equation in the de Sitter spacetime with flat 3-dimensional spatial sections was analyzed by Rendall [51]; he proved that the time derivative decays at least as e^{-Ht} (with $H = \sqrt{\Lambda/3}$ being the Hubble constant, where $\Lambda > 0$ is the cosmological constant), that is, as $a(t)^{-1}$, and conjectured a decay of order e^{-2Ht}, that is, $a(t)^{-2}$. This was also the decay found in [11] for spherical waves when approaching i^+. This problem was studied further in [62], including L^p-L^q decay estimates. Recently, an extensive study of systems of linear wave equations on various cosmological backgrounds was presented in the monograph [54].

A second important class of cosmological spacetimes is given by the Reissner-Nordström-de Sitter (RNdS) solution (Schwarzschild-de Sitter being a particular case). The behavior of linear waves in the static region of these solutions has been studied (in the mathematics literature) in the work of Dafermos-

Rodnianski [20], Bony-Häfner [7], Melrose-Sá Baretto-Vasy [45] and Dyatlov [23, 24]. Schlue [56] studied the wave equation in the cosmological region of the Schwarzschild-de Sitter (also Kerr-de Sitter) solution. He obtained a decay of at least $\frac{1}{r^2}$ for the r derivative[4] of solutions of the wave equation as $r \to +\infty$.

By introducing a partial energy and using an iteration scheme, we find that, for initial data with finite higher order energies, the decay rate of the time derivative is faster than previously existing estimates. For models undergoing accelerated expansion, our decay rate appears to be (almost) sharp, judging from the Fourier mode analysis shown in Appendix A for FLRW and in Appendix B for RNdS.

1.3.2 Cosmic no-hair in spherically symmetric black hole spacetimes

We now turn our attention to the Einstein-Maxwell-scalar field problem and to its respective cosmic no-hair statement. This is the focus of chapter 3, which follows the work presented in article [16].

In the context of cosmic no-hair, either by symmetry assumptions or smallness conditions, all the existing (non-linear) results exclude the existence of black holes a priori. The main goal of Chapter 3 is to help bridge this gap. Nonetheless, there exist some notable partial exceptions: [18] deals with the Einstein-Vlasov system under various symmetry assumptions and provides a very general, mostly qualitative, description of the global structure of the correspond-

[4] Note that r is a time coordinate in the cosmological region.

ing solutions. In particular, for spherically symmetric solutions, it is shown there that the Penrose diagram of the *cosmological region*[5], if non-empty, is bounded to the future by an acausal curve where the radius of symmetry is infinite (assuming an appropriate non-extremality condition); this is in line with the expectations of cosmic no-hair but does not provide enough quantitative information to show that the geometry is approaching that of de Sitter. We also mention the recent work by Schlue [57], where he takes a remarkable step towards the proof of the (non-linear) stability of the cosmological region of Schwarzschild-de Sitter for the (vacuum) Einstein equations, without any symmetry assumptions.

With all of these considerations in mind, our final aim is to provide the first (to the best of our knowledge) complete realization of cosmic no-hair in the context of subextremal black hole spacetimes. Here we will follow the spirit, although not the letter, of the Andréasson-Ringström formulation. A distinct feature, for instance, will come from the fact that we will be considering black hole spacetimes, in which not all future inextendible causal curves have to reach infinity.

Specifically, we analyze in detail the geometry and dynamics of the cosmological region arising in spherically symmetric black hole solutions of the Einstein-Maxwell-scalar field system with a positive cosmological constant. More precisely, we solve, for such a system, a characteristic initial value problem with data emulating a dynamic cosmological horizon. Our assumptions are fairly weak, in that we only require that the data approach that of a subextremal Reissner-Nordström-de Sitter black hole, without imposing any rate of decay. We then show that the radius function blows up along any null ray parallel to the cosmological horizon ("near" i^+), in such a way that $r = +\infty$ is, in an appropriate

[5]See Chapter 3 for the clarification of this terminology.

sense, a spacelike hypersurface. We then prove a version of the Cosmic No-Hair Conjecture by showing that in the past of any causal curve reaching infinity both the metric and the Riemann curvature tensor asymptote those of a de Sitter spacetime.

2

Decay of solutions of the wave equation in expanding cosmological spacetimes

Contents

As mentioned in the Introduction, the purpose of this chapter, as laid out in paper [17], is to obtain the exact decay rates for solutions of the wave equation in expanding cosmological spacetimes, particularly the decay rates of time derivatives, for they characterize the asymptotics of solutions in spacetimes with a spacelike null infinity $\mathscr{I}^+$. Additionally, taking the wave equation as a proxy for the Einstein equations, then time derivatives are akin to second fundamental forms of time slices, and their respective decay rates may prove useful in formulating and proving cosmic no-hair theorems, as we will later see in Chapter 3.

In particular, we consider two types of expanding spacetimes: Friedmann-Lemaître-Robertson-Walker (FLRW) models and the Reissner-Nordström-de Sitter (RNdS) solution. For each kind of spacetime, we introduce a partial energy functional and, following an iteration scheme, we find that, for initial data with finite higher order energies, the decay rate of an appropriate norm of the time derivative of the scalar field and, consequently – due to Sobolev's embedding theorem –, the decay rate of the scalar field's time derivative itself are faster than the ones found in previously existing estimates. For models undergoing accelerated expansion, and having in mind the Fourier mode analysis performed in Appendix A for FLRW and in Appendix B and [51] for RNdS, our decay rate appears to be almost sharp.

2.0.1 Main results

Our main result in the FLRW setting is the following:

Theorem 1. *Consider an expanding FLRW model with flat n-dimensional spatial sections ($n \geq 2$), given by $I \times \mathbb{R}^n$ ($I \subset \mathbb{R}$ being an open an interval) with the* *metric*

$$g = -dt^2 + a^2(t)\left(\left(dx^1\right)^2 + \ldots + (dx^n)^2\right), \tag{2.1}$$

where $a(t)$ satisfies $\dot{a}(t) \geq 0$ for $t \geq t_0$. Let ϕ be a smooth solution of the Cauchy problem

$$\begin{cases} \Box_g \phi = 0 \\ \phi(t_0, x) = \phi_0(x) \\ \partial_t \phi(t_0, x) = \phi_1(x) \end{cases}, \tag{2.2}$$

and suppose that there exists $k > \frac{n}{2} + 2$ such that

$$\|\phi_0\|_{H^k(\mathbb{R}^n)} < +\infty \quad \text{and} \quad \|\phi_1\|_{H^{k-1}(\mathbb{R}^n)} < +\infty\,. \tag{2.3}$$

Then, given $\delta > 0$, we have, for all $t \geq t_0$,

$$\|\partial_t \phi(t, \cdot)\|_{L^\infty(\mathbb{R}^n)} \lesssim a(t)^{-2+\varepsilon+\delta}, \tag{2.4}$$

where $\varepsilon > 0$ is any positive number such that

$$\int_{t_0}^{+\infty} a(t)^{-\varepsilon} dt < +\infty\,. \tag{2.5}$$

Remark 2. *For de Sitter's spacetime, in particular, we have $a(t) = e^{Ht}$ (with $H = \sqrt{2\Lambda/(n(n-1))}$ being the Hubble constant, where $\Lambda > 0$ is the cosmological constant), and so ε can be chosen arbitrarily small. Therefore, for any $\delta > 0$ we have*

$$|\partial_t \phi| \lesssim a(t)^{-2+\delta} = e^{-(2-\delta)Ht}, \tag{2.6}$$

in agreement with Rendall's conjecture (up to the small quantity $\delta > 0$). Note that this does not agree with the naïve physical expectation in Subsection 1.3.1, except, by coincidence, for $n = 3$. We show in Appendix A that this decay rate is (almost) sharp.

Remark 3. *For $a(t) = t^p$ we have $\varepsilon = \frac{1+\delta}{p}$ for any $\delta > 0$, and so Theorem 1 gives*

$$|\partial_t \phi| \lesssim a(t)^{-2+\frac{1+\delta}{p}} = t^{-(2p-1-\delta)} \,. \tag{2.7}$$

Again, this does not conform to the naïve physical expectation. We show in Appendix A that this decay rate is (almost) sharp for $p > 1$, that is, for the case where the expansion is accelerating, and there exists a spacelike $\mathscr{I}^+$. For $p < 1$, that is, for the case where the expansion is decelerating, the decay rate given by Theorem 1 is poor, and in fact the exponent $-2 + \frac{1+\delta}{p}$ can be easily improved to -1. However, the mode calculations in Appendix A strongly suggest that in this case the relevant exponent should be the one coming from the naïve physical argument, namely $-\frac{n+1}{2}$.

Remark 4. *It will be clear from the proof that this result can easily be generalized to expanding FLRW models whose spatial sections have different geometries and/or topologies. In particular, it is true for compact spatial sections, showing that the decay mechanism is the cosmological expansion, and not dispersion.*

Our main result in the RNdS setting reads as follows:

Theorem 5. *Consider an $(n+1)$-dimensional sub-extremal Reissner-Nordström-de Sitter solution ($n \geq 3$), given by the metric*

$$g = -\left(r^2 + \frac{2M}{r^{n-2}} - \frac{e^2}{r^{n-1}} - 1\right)^{-1} dr^2 + \left(r^2 + \frac{2M}{r^{n-2}} - \frac{e^2}{r^{n-1}} - 1\right) dt^2 + r^2 d\Omega^2 \,, \tag{2.8}$$

where $d\Omega^2$ represents the metric of the unit $(n-1)$-dimensional sphere S^{n-1}, the constants M and e are proportional to the mass and charge of the black holes, and we have set the cosmological constant equal to $\frac{1}{2}n(n-1)$ by an appropriate *choice of units. Let ϕ be a smooth solution of the wave equation*

$$\Box_g \phi = 0\,, \tag{2.9}$$

and suppose that there exists $k > \frac{n}{2} + 2$ such that

$$\|\phi\|_{H^k(\mathcal{CH}_1^+)} < +\infty \quad \text{and} \quad \|\phi\|_{H^k(\mathcal{CH}_2^+)} < +\infty\,, \tag{2.10}$$

where $\mathcal{CH}_1^+ \cong \mathcal{CH}_2^+ \cong \mathbb{R} \times S^{n-1}$ are the two connected components of the future cosmological horizon, parameterized by the flow parameter of the global Killing vector field $\frac{\partial}{\partial t}$. Then, given $\delta > 0$, we have, for all $r \geq r_1$,

$$\|\partial_r \phi(r_1, \cdot)\|_{L^\infty(\mathbb{R}\times S^{n-1})} \lesssim {r_1}^{-3+\delta}\,. \tag{2.11}$$

Remark 6. *This is the decay rate one would expect from Rendall's conjecture, since for free-falling observers in the cosmological region one has $r(\tau) \sim e^\tau \sim a(\tau)$, where τ is the proper time and $a(\tau)$ the radius of a comparable FLRW model, so that $\partial_r\phi \sim \partial_\tau\phi/\partial_\tau r \sim e^{-2\tau}/e^{-\tau}$. The hypotheses in* (2.10) *can be recovered from the (higher dimensional version of the) analysis of the static region in [20].*

Remark 7. *In Appendix B we briefly sketch a Fourier mode analysis for RNdS in three spatial dimensions (in the same spirit as the one we did for the FLRW case in Appendix A), which is, again, essentially in agreement with the decay rate we are obtaining here.*

Remark 8. *It is interesting to contrast the behavior of solutions of the wave equation and the conformally invariant wave equation, which can be easily expressed in terms of solutions of the wave equation in the Minkowski spacetime. To this effect, see Appendix C.*

2.1 Decay in FLRW: Proof of Theorem 1

In this section we present the proof of Theorem 1. For the reader's convenience we break it up into elementary steps.

2.1.1 Wave equation in FLRW

Consider an expanding FLRW model with flat n-dimensional spatial sections, given by the metric

$$g = -dt^2 + a^2(t)\left(\left(dx^1\right)^2 + \ldots + (dx^n)^2\right) \,, \tag{2.12}$$

with $\dot{a}(t) \geq 0$ for $t \geq t_0$. The wave equation in this background,

$$\Box_g \phi = 0 \Leftrightarrow \partial_\mu\left(\sqrt{-g}\,\partial^\mu \phi\right) = 0 \Leftrightarrow \partial_\mu\left(a^n \partial^\mu \phi\right) = 0\,, \tag{2.13}$$

can be written as

$$-\ddot{\phi} - \frac{n\dot{a}}{a}\dot{\phi} + \frac{1}{a^2}\delta^{ij}\partial_i\partial_j\phi = 0\,, \tag{2.14}$$

where the dot denotes differentiation with respect to t and the latin indices i and j run from 1 to n.

2.1.2 Energy

Recall that the energy-momentum tensor for the wave equation is

$$T_{\mu\nu} = \partial_\mu \phi \, \partial_\nu \phi - \frac{1}{2} \partial_\alpha \phi \, \partial^\alpha \phi \, g_{\mu\nu} \,, \tag{2.15}$$

so that

$$T_{00} = \frac{1}{2} \left(\dot{\phi}^2 + a^{-2} \delta^{ij} \partial_i \phi \partial_j \phi \right) . \tag{2.16}$$

Choosing the multiplier vector field

$$X = a^{2-n} \frac{\partial}{\partial t} \,, \tag{2.17}$$

we form the current

$$J_\mu = T_{\mu\nu} X^\nu \tag{2.18}$$

and obtain the energy

$$E(t) = \int_{\{t\} \times \mathbb{R}^n} J_\mu N^\mu = \int_{\mathbb{R}^n} a^2 T_{00} d^n x = \int_{\mathbb{R}^n} \frac{1}{2} \left(a^2 \dot{\phi}^2 + \delta^{ij} \partial_i \phi \partial_j \phi \right) d^n x \tag{2.19}$$

(where $N = \frac{\partial}{\partial t}$ is the future unit normal). The deformation tensor associated with the multiplier X is

$$\Pi = \frac{1}{2} \mathcal{L}_X g = -dt \mathcal{L}_X dt + \dot{a} a^{3-n} \delta_{ij} dx^i dx^j \,. \tag{2.20}$$

Noting that

$$\mathcal{L}_X dt = d(\iota(X) dt) = d \left(a^{2-n} \right) = (2-n) \dot{a} a^{1-n} dt \,, \tag{2.21}$$

we obtain

$$\Pi = (n-2) \dot{a} a^{1-n} dt^2 + \dot{a} a^{3-n} \delta_{ij} dx^i dx^j \,. \tag{2.22}$$

Therefore the bulk term is

$$\begin{aligned}\nabla_\mu J^\mu = T^{\mu\nu}\Pi_{\mu\nu} &= (n-2)\dot{a}a^{1-n}\dot{\phi}^2 + \frac{n-2}{2}\dot{a}a^{1-n}\partial_\alpha\phi\,\partial^\alpha\phi \\ &\quad + \dot{a}a^{-1-n}\delta^{ij}\partial_i\phi\partial_j\phi - \frac{n}{2}\dot{a}a^{1-n}\partial_\alpha\phi\,\partial^\alpha\phi \\ &= (n-1)\dot{a}a^{1-n}\dot{\phi}^2 \geq 0\,.\end{aligned} \tag{2.23}$$

For each $R > 0$ define the set

$$\mathcal{B} = \left\{(t_0, x^1, \ldots, x^n) \in I \times \mathbb{R} : \delta_{ij}x^i x^j \leq R^2\right\}\,. \tag{2.24}$$

Applying the divergence theorem to the current J on the region

$$\mathcal{R} = D^+(\mathcal{B}) \cap \{t \leq t_1\} \tag{2.25}$$

(see Figure (2.1)), noticing that the flux across the future null boundaries is non-positive, and letting $R \to +\infty$, we obtain

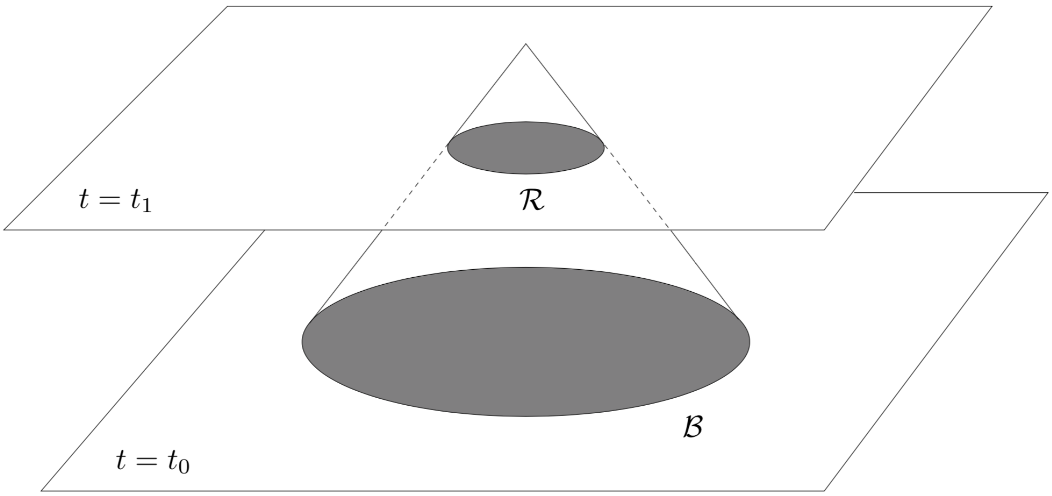

Figure 2.1: Region $\mathcal{R}$ and its boundary.

$$E(t_1) \leq E(t_0) < +\infty \tag{2.26}$$

(by (2.3), since $k > 0$). We deduce from this inequality that, for all $t \geq t_0$,

$$\int_{\mathbb{R}^n} \dot{\phi}^2 \, d^n x \lesssim \frac{1}{a^2} \tag{2.27}$$

and

$$\int_{\mathbb{R}^n} \delta^{ij} \partial_i \phi \partial_j \phi \, d^n x \lesssim 1 \,. \tag{2.28}$$

2.1.3 Partial energy

Let us define the partial energy

$$F(t) = \int_{\mathbb{R}^n} \dot{\phi}^2 \, d^n x \,. \tag{2.29}$$

We already know from (2.27) that $F \lesssim a^{-2}$, but we want a better estimate. Differentiating this partial energy and using the wave equation (2.14) gives

$$\begin{aligned} \dot{F} &= \int_{\mathbb{R}^n} 2\dot{\phi}\ddot{\phi} \, d^n x = 2 \int_{\mathbb{R}^n} \left(-\frac{n\dot{a}}{a}\dot{\phi}^2 + \frac{1}{a^2}\dot{\phi}\delta^{ij}\partial_i\partial_j\phi \right) d^n x \\ &= -\frac{2n\dot{a}}{a}F + \frac{2}{a^2}\int_{\mathbb{R}^n} \dot{\phi}\delta^{ij}\partial_i\partial_j\phi \, d^n x \,. \end{aligned} \tag{2.30}$$

Therefore,

$$a^{2n}\dot{F} + 2na^{2n-1}\dot{a}F = 2a^{2n-2}\int_{\mathbb{R}^n} \dot{\phi}\delta^{ij}\partial_i\partial_j\phi \, d^n x \,, \tag{2.31}$$

and so, by integrating from t_0 to t_1,

$$F(t_1) = \frac{{a_0}^{2n}}{a(t_1)^{2n}}F_0 + \frac{2}{a(t_1)^{2n}}\int_{t_0}^{t_1} a^{2n-2}\int_{\mathbb{R}^n} \dot{\phi}\delta^{ij}\partial_i\partial_j\phi \, d^n x dt \,, \tag{2.32}$$

where $a_0 = a(t_0)$ and $F_0 = F(t_0)$. From the Cauchy-Schwarz inequality we have

$$F(t_1) \lesssim \frac{1}{a(t_1)^{2n}} + \frac{1}{a(t_1)^{2n}} \int_{t_0}^{t_1} a^{2n-2} \left(\int_{\mathbb{R}^n} \dot{\phi}^2 \, d^n x \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} (\delta^{ij} \partial_i \partial_j \phi)^2 \, d^n x \right)^{\frac{1}{2}} dt \,. \tag{2.33}$$

Since each partial derivative $\partial_i \phi$ is also a solution of the wave equation, and $k \geq 2$ in (2.3), we have from (2.28), applied to the partial derivatives $\partial_i \phi$, that the last integral above is bounded, whence

$$F(t_1) \lesssim \frac{1}{a(t_1)^{2n}} + \frac{1}{a(t_1)^{2n}} \int_{t_0}^{t_1} a^{2n-2} F^{\frac{1}{2}} dt \,. \tag{2.34}$$

2.1.4 Iteration

Let $\varepsilon > 0$ be such that

$$\int_{t_0}^{+\infty} a^{-\varepsilon} dt < +\infty \,, \tag{2.35}$$

and define

$$x_k = \frac{2^{k+2} - 2}{2^k} - \varepsilon \frac{2^{k+1} - 2}{2^k} \,. \tag{2.36}$$

We will prove by induction that

$$F \lesssim a^{-x_k} \tag{2.37}$$

for all $k \in \mathbb{N}_0$. If $k = 0$ then this is just $F \lesssim a^{-2}$, which we already had from the energy estimate. Assuming that it is true for a given $k \in \mathbb{N}_0$, we have from

(2.34) that

$$\begin{aligned} F(t_1) &\lesssim \frac{1}{a(t_1)^{2n}} + \frac{1}{a(t_1)^{2n}} \int_{t_0}^{t_1} a^{2n-2} a^{-\frac{x_k}{2}} \, dt \\ &\lesssim \frac{1}{a(t_1)^{2n}} + \frac{a(t_1)^{2n-2-\frac{x_k}{2}+\varepsilon}}{a(t_1)^{2n}} \int_{t_0}^{t_1} a^{-\varepsilon} dt \\ &\lesssim a(t_1)^{-\frac{x_k}{2}-2+\varepsilon} \end{aligned} \tag{2.38}$$

(where we used $x_k < 4$, so that the exponent inside the first integral is positive). Since

$$\begin{aligned} \frac{x_k}{2} + 2 - \varepsilon &= \frac{2^{k+2}-2}{2^{k+1}} - \varepsilon \frac{2^{k+1}-2}{2^{k+1}} + 2 - \varepsilon \\ &= \frac{2^{k+3}-2}{2^{k+1}} - \varepsilon \frac{2^{k+2}-2}{2^{k+1}} = x_{k+1} \, , \end{aligned} \tag{2.39}$$

we have established (2.37).

Note that, because

$$\lim_{k \to +\infty} x_k = 4 - 2\varepsilon \, , \tag{2.40}$$

we have in fact shown that

$$F \lesssim a^{-4+2\varepsilon+2\delta} \tag{2.41}$$

for any $\delta > 0$. In other words,

$$\|\dot{\phi}\|_{L^2(\mathbb{R}^n)} \lesssim a^{-2+\varepsilon+\delta} \, . \tag{2.42}$$

Since any partial derivative $\partial_{i_1} \cdots \partial_{i_k} \phi$ is also a solution of the wave equation, and since (2.3) holds, we have

$$\|\dot{\phi}\|_{H^k(\mathbb{R}^n)} \lesssim a^{-2+\varepsilon+\delta} \tag{2.43}$$

for some $k > \frac{n}{2}$ (recall that we need one extra derivative to obtain estimate (2.34)). Therefore, Sobolev's embedding theorem gives

$$|\dot{\phi}| \lesssim a^{-2+\varepsilon+\delta} \,. \tag{2.44}$$

2.2 Decay in RNdS: Proof of Theorem 5

In this section we present the proof of Theorem 5. For the reader's convenience we break it up into elementary steps.

2.2.1 Reissner-Nordström-de Sitter metric

The Reissner-Nordström-de Sitter metric is a solution of the Einstein-Maxwell equations with positive cosmological constant, representing a pair of antipodal charged black holes in a spherical universe undergoing accelerated expansion. It is given in $n+1$ dimensions by the metric

$$g = -V^{-1}dr^2 + V dt^2 + r^2 d\Omega^2 \,, \tag{2.45}$$

where

$$V = r^2 + \frac{2M}{r^{n-2}} - \frac{e^2}{r^{n-1}} - 1 \,, \tag{2.46}$$

and where $d\Omega^2$ is the unit round metric on S^{n-1}. The constants M and e are proportional to the mass and charge of the black holes, and we have set the cosmological constant equal to $\frac{1}{2}n(n-1)$ by an appropriate choice of units.

In the cosmological region, corresponding to $r > r_c$, we have $V > 0$, and the hypersurfaces of constant r are spacelike cylinders with future-pointing unit normal

$$N = V^{\frac{1}{2}} \frac{\partial}{\partial r} \tag{2.47}$$

and volume element

$$dV_n = V^{\frac{1}{2}} r^{n-1} dt d\Omega \,. \tag{2.48}$$

2.2.2 Energy

Assume that ϕ is a solution of the wave equation. Recall once again that the energy-momentum tensor associated to ϕ is

$$T_{\mu\nu} = \partial_\mu \phi \, \partial_\nu \phi - \frac{1}{2} \partial_\alpha \phi \, \partial^\alpha \phi \, g_{\mu\nu} \,. \tag{2.49}$$

Therefore we have

$$\begin{aligned} T(N,N) &= (N \cdot \phi)^2 + \frac{1}{2} \left[-V \phi'^2 + V^{-1} \dot{\phi}^2 + \frac{1}{r^2} |\mathring{\nabla} \phi|^2 \right] \\ &= \frac{1}{2} \left[V \phi'^2 + V^{-1} \dot{\phi}^2 + \frac{1}{r^2} |\mathring{\nabla} \phi|^2 \right] \,, \end{aligned} \tag{2.50}$$

where $\phi' = \frac{\partial \phi}{\partial r}$, $\dot{\phi} = \frac{\partial \phi}{\partial t}$, $\mathring{\nabla} \phi$ is the gradient of ϕ seen as a function on S^{n-1} and $|\mathring{\nabla} \phi|^2$ is its squared norm (both taken with respect to the unit round metric).

Choosing the multiplier

$$X = \frac{V^{\frac{1}{2}}}{r^{n-1}} N = \frac{V}{r^{n-1}} \frac{\partial}{\partial r} \,, \tag{2.51}$$

we form the current

$$J_\mu = T_{\mu\nu} X^\nu \tag{2.52}$$

and obtain the energy

$$E(r) = \int_{\mathbb{R}\times S^{n-1}} T(X,N)dV_n = \int_{\mathbb{R}\times S^{n-1}} \frac{1}{2}\left[V^2\phi'^2 + \dot{\phi}^2 + \frac{V}{r^2}|\mathring{\nabla}\phi|^2\right] dt d\Omega\,. \tag{2.53}$$

This energy is related to the one used by Schlue in [56], but differs (essentially) by a factor of r, so that no rescaling is needed at $\mathscr{I}^+$. We will show in Subsection 2.2.6 that hypotheses (2.10) imply that

$$\|\phi\|_{H^k(\{r=r_0\})} < +\infty \tag{2.54}$$

for $r_0 > r_c$ and $k > \frac{n}{2} + 2 > 0$. In particular, $E(r_0) < +\infty$ for any $r_0 > r_c$.

The deformation tensor associated to the multiplier X is

$$\Pi = \frac{1}{2}\mathcal{L}_X g = -V^{-1}dr\mathcal{L}_X dr + \frac{V'}{2Vr^{n-1}}dr^2 + \frac{VV'}{2r^{n-1}}dt^2 + \frac{V}{r^{n-2}}d\Omega^2\,. \tag{2.55}$$

Noting that

$$\mathcal{L}_X dr = d(\iota(X)dr) = d\left(\frac{V}{r^{n-1}}\right) = \left(\frac{V'}{r^{n-1}} - \frac{(n-1)V}{r^n}\right)dr\,, \tag{2.56}$$

we obtain

$$\Pi = \frac{V'}{2r^{n-1}}\left(-V^{-1}dr^2 + Vdt^2\right) + \frac{n-1}{r^n}dr^2 + \frac{V}{r^{n-2}}d\Omega^2\,. \tag{2.57}$$

If we write

$$\Pi = \Pi^{(1)} + \Pi^{(2)} + \Pi^{(3)} \tag{2.58}$$

for each of the three terms above, we have

$$T^{\mu\nu}\Pi^{(1)}_{\mu\nu} = \frac{V'}{2r^{n-1}}\left[-V\phi'^2 + V^{-1}\dot{\phi}^2 - \frac{2}{2}\left(-V\phi'^2 + V^{-1}\dot{\phi}^2 + \frac{1}{r^2}|\mathring{\nabla}\phi|^2\right)\right]$$
$$= -\frac{V'}{2r^{n+1}}|\mathring{\nabla}\phi|^2 \tag{2.59}$$

for the first term,

$$T^{\mu\nu}\Pi^{(2)}_{\mu\nu} = \frac{n-1}{r^n}\left[V^2\phi'^2 + \frac{1}{2}V\left(-V\phi'^2 + V^{-1}\dot{\phi}^2 + \frac{1}{r^2}|\mathring{\nabla}\phi|^2\right)\right]$$
$$= \frac{n-1}{2r^n}\left(V^2\phi'^2 + \dot{\phi}^2 + \frac{V}{r^2}|\mathring{\nabla}\phi|^2\right) \tag{2.60}$$

for the second term, and

$$T^{\mu\nu}\Pi^{(3)}_{\mu\nu} = \frac{V}{r^n}\left[\frac{1}{r^2}|\mathring{\nabla}\phi|^2 - \frac{n-1}{2}\left(-V\phi'^2 + V^{-1}\dot{\phi}^2 + \frac{1}{r^2}|\mathring{\nabla}\phi|^2\right)\right]$$
$$= \frac{V}{r^{n+2}}|\mathring{\nabla}\phi|^2 + \frac{n-1}{2r^n}\left(V^2\phi'^2 - \dot{\phi}^2 - \frac{V}{r^2}|\mathring{\nabla}\phi|^2\right). \tag{2.61}$$

for the third term. The full bulk term is therefore

$$\nabla_\mu J^\mu = T^{\mu\nu}\Pi_{\mu\nu} = \frac{(n-1)V^2}{r^n}\phi'^2 + \left(\frac{V}{r^{n+2}} - \frac{V'}{2r^{n+1}}\right)|\mathring{\nabla}\phi|^2\,. \tag{2.62}$$

Now,

$$\frac{V}{r^{n+2}} - \frac{V'}{2r^{n+1}} = \frac{1}{2r^{n-1}}\left(\frac{2V}{r^3} - \frac{V'}{r^2}\right) = -\frac{1}{2r^{n-1}}\left(\frac{V}{r^2}\right)'$$
$$= -\frac{1}{r^{n+2}}\left(1 - \frac{nM}{r^{n-2}} + \frac{(n+1)e^2}{2r^{n-1}}\right) \geq -\frac{C}{r^{n+2}} \tag{2.63}$$

on the cosmological region $r > r_c$, and so

$$T^{\mu\nu}\Pi_{\mu\nu} \geq -\frac{C}{r^{n+2}}|\mathring{\nabla}\phi|^2\,. \tag{2.64}$$

For each $T > 0$ define the set

$$\mathcal{C} = \{r = r_0\} \cap \{-T \leq t \leq T\}\,. \tag{2.65}$$

Applying the divergence theorem to the current J on the region

$$\mathcal{S} = D^+(\mathcal{C}) \cap \{r \leq r_1\} \tag{2.66}$$

(see Figure (2.2)), noticing that the flux across the future null boundaries is non-positive, and letting $T \to +\infty$, we obtain

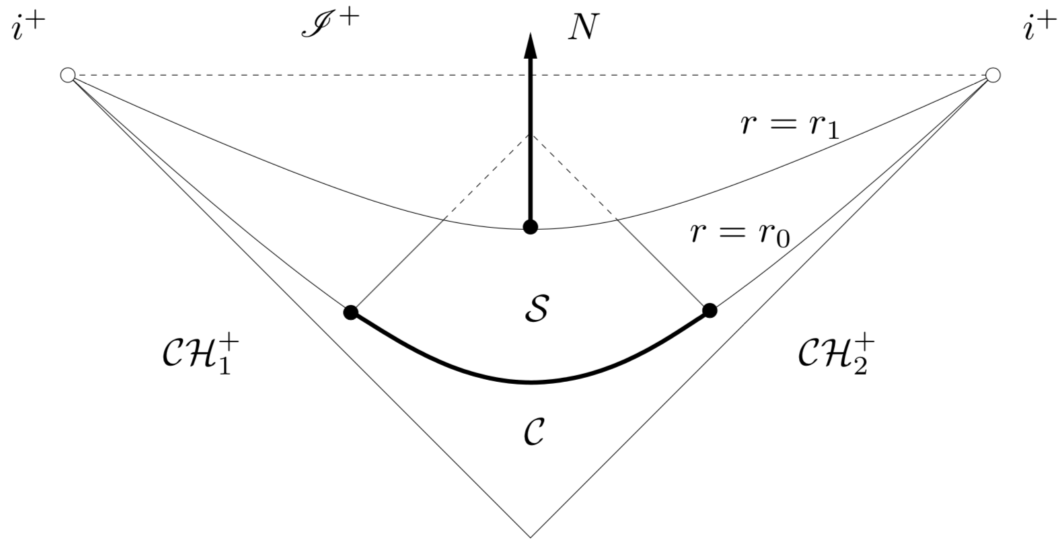

Figure 2.2: Region $\mathcal{S}$ and its boundary.

$$E(r_0) - E(r_1) \geq -\int_{r_0}^{r_1} \int_{\mathbb{R}\times S^{n-1}} \frac{C}{r^{n+2}} |\mathring{\nabla}\phi|^2 r^{n-1} dt d\Omega dr\,. \tag{2.67}$$

Notice that, for $r \geq r_0$,

$$\int_{\mathbb{R}\times S^{n-1}} |\mathring{\nabla}\phi|^2 dt d\Omega \leq \frac{2r^2}{V} E(r) \leq C(r_0) E(r)\,, \tag{2.68}$$

since $\frac{2r^2}{V}$ is positive for $r > r_c$ and

$$\lim_{r\to\infty} \frac{2r^2}{V} = 2\,. \tag{2.69}$$

Substituting in the previous inequality yields

$$E(r_1) \leq E(r_0) + \int_{r_0}^{r_1} \frac{C(r_0)}{r^3} E(r) dr\,. \tag{2.70}$$

From Grönwall's inequality we finally obtain

$$E(r_1) \leq E(r_0) \exp\left(\int_{r_0}^{r_1} \frac{C(r_0)}{r^3} dr\right) \leq C(r_0)E(r_0)\,. \tag{2.71}$$

In particular, we have, for all $r \geq r_0$:

$$\int_{\mathbb{R}\times S^{n-1}} V^2 \phi'^2 dt d\Omega \lesssim 1\,; \tag{2.72}$$

$$\int_{\mathbb{R}\times S^{n-1}} \dot{\phi}^2 dt d\Omega \lesssim 1\,; \tag{2.73}$$

$$\int_{\mathbb{R}\times S^{n-1}} \frac{V}{r^2} |\mathring{\nabla\!\!\!/}\phi|^2 dt d\Omega \lesssim 1\,. \tag{2.74}$$

2.2.3 Wave equation

The wave equation in the RNdS background,

$$\Box_g \phi = 0 \Leftrightarrow \partial_\mu \left(\sqrt{-g}\, \partial^\mu \phi\right) = 0 \Leftrightarrow \partial_\mu \left(r^{n-1} \left(\mathring{g\!\!\!/}\right)^{\frac{1}{2}} \partial^\mu \phi\right) = 0 \tag{2.75}$$

(where $\mathring{\not{g}}$ is the determinant of the unit round sphere metric), can be written as

$$\begin{aligned}
&-\left(r^{n-1}V\phi'\right)'+r^{n-1}V^{-1}\ddot{\phi}+r^{n-3}\mathring{\not{\Delta}}\phi=0\Leftrightarrow\\
&-(V\phi')'-\frac{n-1}{r}V\phi'+V^{-1}\ddot{\phi}+\frac{1}{r^2}\mathring{\not{\Delta}}\phi=0\,,
\end{aligned}\tag{2.76}$$

where $\mathring{\not{\Delta}}\phi$ is the Laplacian of ϕ seen as a function on S^{n-1} (taken with respect to the unit round metric).

2.2.4 Partial energy

Let us define the partial energy

$$F(r)=\int_{\mathbb{R}\times S^{n-1}}V^2\phi'^2dtd\Omega\,. \tag{2.77}$$

We already know from (2.72) that $F\lesssim 1$, but we want a better estimate. Differentiating this partial energy and using the wave equation gives

$$\begin{aligned}
F'&=2\int_{\mathbb{R}\times S^{n-1}}V\phi'(V\phi')'dtd\Omega\\
&=2\int_{\mathbb{R}\times S^{n-1}}V\phi'\left(-\frac{n-1}{r}V\phi'+V^{-1}\ddot{\phi}+\frac{1}{r^2}\mathring{\not{\Delta}}\phi\right)dtd\Omega\\
&=-\frac{2n-2}{r}F+2\int_{\mathbb{R}\times S^{n-1}}V\phi'\left(V^{-1}\ddot{\phi}+\frac{1}{r^2}\mathring{\not{\Delta}}\,\phi\right)dtd\Omega\,.
\end{aligned}\tag{2.78}$$

Noting that

$$F'+\frac{2n-2}{r}F=\frac{1}{r^{2n-2}}\left(r^{2n-2}F\right)'\,, \tag{2.79}$$

we can integrate (2.78) to obtain

$$
\begin{aligned}
F(r_1) &= \frac{{r_0}^{2n-2}}{{r_1}^{2n-2}}F(r_0) + \frac{2}{{r_1}^{2n-2}}\int_{r_0}^{r_1}\int_{\mathbb{R}\times S^{n-1}} r^{2n-2}V\phi'\left(V^{-1}\ddot{\phi} + \frac{1}{r^2}\mathring{\Delta}\phi\right) dtd\Omega dr \\
&\leq \frac{{r_0}^{2n-2}}{{r_1}^{2n-2}}F(r_0) + \frac{2}{{r_1}^{2n-2}}\int_{r_0}^{r_1} r^{2n-4}F^{\frac{1}{2}}\left(\int_{\mathbb{R}\times S^{n-1}} 2\left(\frac{r^4}{V^2}\ddot{\phi}^2 + \left(\mathring{\Delta}\phi\right)^2\right) dtd\Omega\right)^{\frac{1}{2}} dr\,,
\end{aligned}
\tag{2.80}
$$

where we used Cauchy-Schwarz and Young's inequalities in the last step.

Recall that S^{n-1} admits $\frac{n(n-1)}{2}$ independent Killing vector fields, given by

$$
L_{ij} = x^i\frac{\partial}{\partial x^j} - x^j\frac{\partial}{\partial x^i} \tag{2.81}
$$

for $i < j$ (under the usual embedding $S^{n-1} \subset \mathbb{R}^n$), and moreover that

$$
\mathring{\Delta}\phi = \sum_{i<j} L_{ij}\cdot(L_{ij}\cdot\phi)\,. \tag{2.82}
$$

Since $\frac{\partial}{\partial t}$ and L_{ij} are Killing vector fields, $\dot{\phi}$ and $L_{ij}\cdot\phi$ are also solutions of the wave equation, and, because $k \geq 2$ in (2.10), they satisfy the bounds (2.73) and (2.74). Using

$$
(L_{ij}\cdot(L_{ij}\cdot\phi))^2 \leq (\mathring{\nabla}(L_{ij}\cdot\phi))^2\,, \tag{2.83}
$$

we see that the last integral in (2.80) is bounded, whence

$$
F(r_1) \lesssim \frac{1}{{r_1}^{2n-2}} + \frac{1}{{r_1}^{2n-2}}\int_{r_0}^{r_1} r^{2n-4}F^{\frac{1}{2}}dr\,. \tag{2.84}
$$

2.2.5 Iteration

Define

$$x_k = \frac{2^{k+1} - 2}{2^k} \,. \tag{2.85}$$

We will prove by induction that

$$F(r_1) \lesssim {r_1}^{-x_k} \tag{2.86}$$

for all $k \in \mathbb{N}_0$. If $k = 0$ then this is just $F \lesssim 1$, which we already had from the energy estimate. Assuming that it is true for a given $k \in \mathbb{N}_0$, we have from (2.84) that

$$\begin{aligned} F(r_1) &\lesssim \frac{1}{{r_1}^{2n-2}} + \frac{1}{{r_1}^{2n-2}} \int_{r_0}^{r_1} r^{2n-4} r^{-\frac{x_k}{2}} \, dr \lesssim \frac{1}{{r_1}^{2n-2}} + \frac{{r_1}^{2n-3-\frac{x_k}{2}}}{{r_1}^{2n-2}} \\ &\lesssim {r_1}^{-\frac{x_k}{2}-1} \end{aligned} \tag{2.87}$$

(where we used $x_k < 2$, so that the exponent $2n - 3 - \frac{x_k}{2}$ is positive). Since

$$\frac{x_k}{2} + 1 = \frac{2^{k+1} - 2}{2^{k+1}} + 1 = \frac{2^{k+2} - 2}{2^{k+1}} = x_{k+1} \,, \tag{2.88}$$

we have established (2.86).

Note that, since

$$\lim_{k \to +\infty} x_k = 2 \,, \tag{2.89}$$

we have shown that

$$F(r_1) \lesssim {r_1}^{-2+2\delta} \tag{2.90}$$

for any $\delta > 0$. Noticing that

$$V(r_1) \sim {r_1}^2 \,, \tag{2.91}$$

we see that, in fact,

$$\|\phi'\|_{L^2(\mathbb{R}\times S^{n-1})} \lesssim {r_1}^{-3+\delta}\,. \tag{2.92}$$

Since we obtain solutions of the wave equation by acting on ϕ with any finite sequence of Killing vector fields $\frac{\partial}{\partial t}$ and L_{ij}, and since (2.10) holds, we have

$$\|\phi'\|_{H^k(\mathbb{R}\times S^{n-1})} \lesssim {r_1}^{-3+\delta} \tag{2.93}$$

for some $k > \frac{n}{2}$ (recall that we need one extra derivative to obtain estimate (2.84)). Therefore Sobolev's embedding theorem gives[1]

$$|\phi'| \lesssim {r_1}^{-3+\delta}\,. \tag{2.94}$$

2.2.6 Weak redshift estimates

We now obtain the condition that must be satisfied at the cosmological horizon so that the energy $E(r)$, corresponding to the multiplier X, is finite at $r = r_0$. We start by writing the Reissner-Nordström-de Sitter metric (2.45) as

$$\begin{aligned} g &= V\left(-V^{-2}dr^2 + dt^2\right) + r^2 d\Omega^2 \\ &= -V\left(V^{-1}dr + dt\right)\left(V^{-1}dr - dt\right) + r^2 d\Omega^2 \\ &= -V du\left(-du + 2V^{-1}dr\right) + r^2 d\Omega^2 \\ &= V du^2 - 2du dr + r^2 d\Omega^2\,, \end{aligned} \tag{2.95}$$

[1]Sobolev's embedding theorem holds for any complete Riemannian manifold with positive injectivity radius and bounded sectional curvature, see for instance [4].

where the coordinate u is defined as

$$u = t + \int \frac{dr}{V} \,. \tag{2.96}$$

The first diagonal block for the matrix of the metric in the coordinates (u, r) satisfies

$$\det \begin{pmatrix} V & -1 \\ -1 & 0 \end{pmatrix} = -1 \,, \tag{2.97}$$

and so this coordinate system extends across the cosmological horizon $r = r_c$, where $V = 0$. Note that the hypersurfaces of constant u are null and transverse to the cosmological horizon, and so only one of the branches of the cosmological horizon (connecting the bifurcation sphere to future null infinity $\mathscr{I}^+$) is covered by the coordinates (u, r); to cover the other branch, corresponding to $u = -\infty$, one has to introduce new coordinates (v, r), defined by

$$v = -t + \int \frac{dr}{V} \,, \tag{2.98}$$

and repeat the same construction (see Figure 2.3).

At any rate, the Killing vector field

$$K = \frac{\partial}{\partial u} = \frac{\partial}{\partial t} \tag{2.99}$$

is well-defined across the (first branch of the) cosmological horizon, and is null on the cosmological horizon (although the coordinate t is not defined there). Moreover, the vector field

$$Y = \left(\frac{\partial}{\partial r} \right)_u \tag{2.100}$$

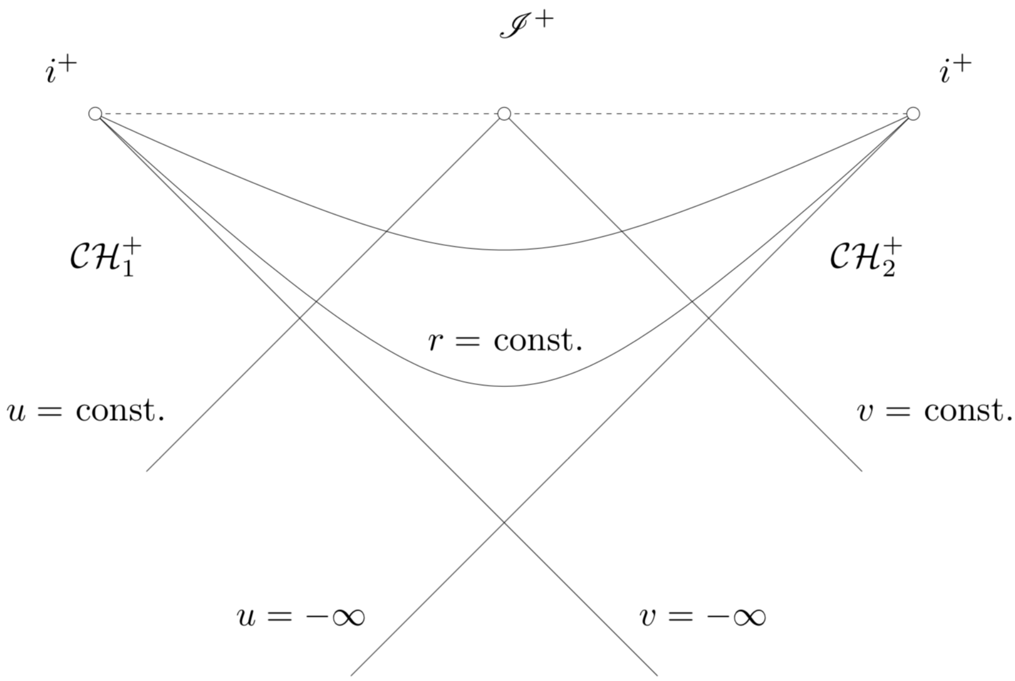

Figure 2.3: Coordinates u and v. Note that in this diagram t increases from right to left.

is null and transverse to the cosmological horizon. From the identities

$$du(Y) = 0 \qquad \text{and} \qquad Y \cdot r = 1 \tag{2.101}$$

one easily obtains

$$Y = \frac{\partial}{\partial r} - \frac{1}{V}\frac{\partial}{\partial t} \tag{2.102}$$

on the cosmological region. Finally, to find the expression for the multiplier vector field X in the coordinates (u, r), we start by computing

$$N = -\frac{\operatorname{grad} r}{|\operatorname{grad} r|}\,. \tag{2.103}$$

Given that

$$\begin{pmatrix} V & -1 \\ -1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & -1 \\ -1 & -V \end{pmatrix}, \tag{2.104}$$

we have

$$\langle \operatorname{grad} r, \operatorname{grad} r \rangle = \langle dr, dr \rangle = -V, \tag{2.105}$$

and so N is the vector associated to the covector $-V^{-\frac{1}{2}}dr$, that is,

$$N = V^{-\frac{1}{2}} \left(\frac{\partial}{\partial u} + V \frac{\partial}{\partial r} \right) . \tag{2.106}$$

Consequently,

$$X = \frac{V^{\frac{1}{2}}}{r^{n-1}} N = \frac{1}{r^{n-1}} \left(\frac{\partial}{\partial u} + V \frac{\partial}{\partial r} \right) \tag{2.107}$$

is well-defined across the cosmological horizon.

Note that the energy

$$E(r) = \int_{\mathbb{R}\times S^{n-1}} T(X,N) dV_n = \int_{\mathbb{R}\times S^{n-1}} \frac{1}{2} \left[V^2 \phi'^2 + \dot{\phi}^2 + \frac{V}{r^2} |\mathring{\nabla} \phi|^2 \right] dt d\Omega \tag{2.108}$$

approaches

$$E(r_c) = \frac{1}{2} \int_{\mathbb{R}\times S^{n-1}} (K \cdot \phi)^2 du d\Omega + \frac{1}{2} \int_{\mathbb{R}\times S^{n-1}} (K \cdot \phi)^2 dv d\Omega \tag{2.109}$$

as $r \to r_c$, where each of the two integrals above refers to a different branch of the cosmological horizon. It is therefore clear that $E(r)$ loses control of transverse and angular derivatives as $r \to r_c$. To cincumvent this problem, we define a new energy by adding the vector field Y to the original multiplier X:

$$\mathcal{E}(r) = E(r) + \int_{\mathbb{R}\times S^{n-1}} T(Y,N) dV_n \, . \tag{2.110}$$

Now,

$$T(Y,N) = T\left(\frac{\partial}{\partial r}, N\right) - \frac{1}{V}T\left(\frac{\partial}{\partial t}, N\right)$$
$$= V^{-\frac{1}{2}}\left[T(N,N) - T\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial r}\right)\right], \tag{2.111}$$

and so

$$\mathcal{E}(r) = E(r) + \int_{\mathbb{R}\times S^{n-1}} \frac{1}{2}\left[V\phi'^2 + V^{-1}\dot{\phi}^2 + \frac{1}{r^2}|\mathring{\nabla}\phi|^2 - 2\dot{\phi}\phi'\right] r^{n-1}dtd\Omega$$
$$= E(r) + \int_{\mathbb{R}\times S^{n-1}} \frac{1}{2}\left[V\left(\phi' - \frac{1}{V}\dot{\phi}\right)^2 + \frac{1}{r^2}|\mathring{\nabla}\phi|^2\right] r^{n-1}dtd\Omega$$
$$= E(r) + \int_{\mathbb{R}\times S^{n-1}} \frac{1}{2}\left[V\left(Y\cdot\phi\right)^2 + \frac{1}{r^2}|\mathring{\nabla}\phi|^2\right] r^{n-1}dtd\Omega\,. \tag{2.112}$$

Note that

$$\mathcal{E}(r_c) = E(r_c) + \frac{{r_c}^{n-3}}{2}\int_{\mathbb{R}\times S^{n-1}} |\mathring{\nabla}\phi|^2 dud\Omega + \frac{{r_c}^{n-3}}{2}\int_{\mathbb{R}\times S^{n-1}} |\mathring{\nabla}\phi|^2 dvd\Omega\,, \tag{2.113}$$

and so the new energy retains some control of the angular derivatives as $r \to r_c$. Note that this energy is a weaker version of the Dafermos-Rodnianski redshift energy, which also controls transverse derivatives. Nevertheless, our (simpler) construction suffices to show that $E(r_0)$ is finite from hypotheses (2.10).

To compute the deformation tensor associated to the multiplier Y we note that

$$\mathcal{L}_{\frac{\partial}{\partial r}} g = V^{-2}V'dr^2 + V'dt^2 + 2rd\Omega^2 \tag{2.114}$$

and

$$\mathcal{L}_{-\frac{1}{V}\frac{\partial}{\partial t}} g = 2V dt \mathcal{L}_{-\frac{1}{V}\frac{\partial}{\partial t}} dt = 2V dt d\left(-\frac{1}{V}\right) = 2V^{-1}V'dtdr\,. \tag{2.115}$$

Therefore, the deformation tensor is

$$\begin{aligned}\Xi = \frac{1}{2}\mathcal{L}_Y g &= \frac{1}{2}V^{-2}V'dr^2 + \frac{1}{2}V'dt^2 + rd\Omega^2 + V^{-1}V'dtdr \\ &= \frac{1}{2}V'\left(dt + V^{-1}dr\right)^2 + rd\Omega^2 = \frac{1}{2}V'du^2 + rd\Omega^2\,. \end{aligned} \tag{2.116}$$

Noticing that

$$du = -g(Y, \cdot)\,, \tag{2.117}$$

we have

$$T^{\mu\nu}\Xi_{\mu\nu} = \frac{1}{2}V'(Y\cdot\phi)^2 + \frac{1}{r^3}|\mathring{\nabla}\phi|^2 - \frac{n-1}{2r}\langle d\phi, d\phi\rangle\ . \tag{2.118}$$

Since

$$\langle d\phi, d\phi\rangle = -2(K\cdot\phi)(Y\cdot\phi) - V(Y\cdot\phi)^2 + \frac{1}{r^2}|\mathring{\nabla}\phi|^2\,, \tag{2.119}$$

we finally obtain

$$T^{\mu\nu}\Xi_{\mu\nu} = \left(\frac{V'}{2} + \frac{(n-1)V}{2r}\right)(Y\cdot\phi)^2 + \frac{n-1}{r}(K\cdot\phi)(Y\cdot\phi) - \frac{n-3}{2r^3}|\mathring{\nabla}\phi|^2\,. \tag{2.120}$$

Since

$$\frac{V'}{2}(Y\cdot\phi)^2 + \frac{n-1}{r}(K\cdot\phi)(Y\cdot\phi) = \frac{V'}{2}\left[(Y\cdot\phi) + \frac{n-1}{rV'}(K\cdot\phi)\right]^2 - \frac{(n-1)^2}{2r^2V'}(K\cdot\phi)^2\,, \tag{2.121}$$

and using the fact that $V'(r) > 0$ for $r \geq r_c$ (global redshift), we have

$$T^{\mu\nu}\Xi_{\mu\nu} \geq -\frac{(n-1)^2}{2r^2V'}(K\cdot\phi)^2 - \frac{n-3}{2r^3}|\mathring{\nabla}\phi|^2\,, \tag{2.122}$$

and so

$$T^{\mu\nu}\Pi_{\mu\nu} + T^{\mu\nu}\Xi_{\mu\nu} \geq -C(K\cdot\phi)^2 - C|\mathring{\nabla}\phi|^2 \tag{2.123}$$

for $r_c < r < r_0$.

Given $r_c < r_1 < r_0$ and $T > 0$, define the set

$$\mathcal{D} = \{r = r_1\} \cap \{-T \leq t \leq T\} \,. \tag{2.124}$$

Applying the divergence theorem on the region

$$\mathcal{T} = D^+(\mathcal{D}) \cap \{r \leq r_0\}, \tag{2.125}$$

noticing that the flux across the future null boundaries is non-positive, and letting $T \to +\infty$, we obtain

$$\mathcal{E}(r_1) - \mathcal{E}(r_0) \geq -\int_{r_1}^{r_0} \int_{\mathbb{R}\times S^{n-1}} C \left((K \cdot \phi)^2 + |\mathring{\nabla}\phi|^2 \right) r^{n-1} dt d\Omega dr \,. \tag{2.126}$$

Since

$$\begin{aligned} \mathcal{E}(r) = & \int_{\mathbb{R}\times S^{n-1}} \frac{1}{2} \left[V^2 \phi'^2 + (K \cdot \phi)^2 + \frac{V}{r^2} |\mathring{\nabla}\phi|^2 \right] dt d\Omega \\ & + \int_{\mathbb{R}\times S^{n-1}} \frac{1}{2} \left[V \left(Y \cdot \phi \right)^2 + \frac{1}{r^2} |\mathring{\nabla}\phi|^2 \right] r^{n-1} dt d\Omega \,, \end{aligned} \tag{2.127}$$

we have, for $r_c < r_1 < r_0$,

$$\mathcal{E}(r_1) - \mathcal{E}(r_0) \geq -\int_{r_1}^{r_0} C\mathcal{E}(r) dr \,, \tag{2.128}$$

that is,

$$\mathcal{E}(r_0) \leq \mathcal{E}(r_1) + \int_{r_1}^{r_0} C\mathcal{E}(r) dr \,. \tag{2.129}$$

From Grönwall's inequality we have

$$\mathcal{E}(r_0) \leq \mathcal{E}(r_1) \exp \left(\int_{r_1}^{r_0} C dr \right) \leq C(r_0) \mathcal{E}(r_1) \,. \tag{2.130}$$

Letting $r_1 \to r_c$, we finally obtain

$$E(r_0) \leq \mathcal{E}(r_0) \lesssim \mathcal{E}(r_c) \lesssim \|\phi\|_{H^1(\mathcal{CH}_1^+)} + \|\phi\|_{H^1(\mathcal{CH}_2^+)} < +\infty\,. \tag{2.131}$$

Commuting with the Killing vector fields $\frac{\partial}{\partial t}$ and L_{ij}, we see that hypotheses (2.10) imply that

$$\|\phi\|_{H^k(\{r=r_0\})} \lesssim \|\phi\|_{H^k(\mathcal{CH}_1^+)} + \|\phi\|_{H^k(\mathcal{CH}_2^+)} < +\infty \tag{2.132}$$

for some $k > \frac{n}{2} + 2$.

3

Cosmic no-hair in spherically symmetric black hole spacetimes

Contents

As stated in the Introduction, the aim of this chapter – and ultimately of this thesis – is to provide the first (to the best of our knowledge) realization of cosmic no-hair, in the spirit of the Andréasson-Ringström formulation, in the context of subextremal black hole spacetimes and in spherical symmetry. This is the content of paper [16].

With this goal in mind we take as reference solutions the subextremal elements of the Reissner-Nordström-de Sitter (RNdS) family[1] (see Subsection 3.1.2 for more details). These model static, spherically symmetric and electromagnetically charged black holes in an expanding universe. They are solutions of the Einstein-Maxwell equations parameterized by mass M, charge e and cosmological constant Λ. The global causal structure of these black hole spacetimes is completely captured by the Penrose diagram in Figure 3.1 and is considerably different from the one of their asymptotically flat ($\Lambda = 0$) counterparts. The first noticeable difference corresponds to the existence of a periodic string of causally disconnected isometric regions. Among these regions, the only ones with no analogue in the $\Lambda = 0$ case are the cosmological regions[2] (regions V and VI in Figure 3.1), which, in the topology of the plane, are bounded by cosmological horizons $\mathcal{C}^{\pm}$, future and past null infinity $\mathscr{I}^{\pm}$, and a collection of points $i^{\pm}$. Since all of its connected components are isometric, we can focus on only one of them; for convenience, we will consider a future component, i.e., one whose boundary contains a component of $\mathscr{I}^{+}$ (region V in Figure 3.1).

We will start our evolution from a dynamical cosmological horizon, whose properties generalize those of the future Killing cosmological horizon $\mathcal{C}_A^{+}$ of our reference RNdS solutions; by imposing appropriate data on a transverse null

[1] To set the convention, whenever we refer to an RNdS solution $(\mathcal{M}, g)$ we mean the maximal domain of dependence $D(\Sigma) = \mathcal{M}$ of a complete Cauchy hypersurface $\Sigma = \mathbb{R} \times \mathbb{S}^2$.

[2] It is also standard to refer to these as expanding regions, but we prefer the designation "cosmological", since the local regions (regions I and III in Figure 3.1) are also expanding [8].

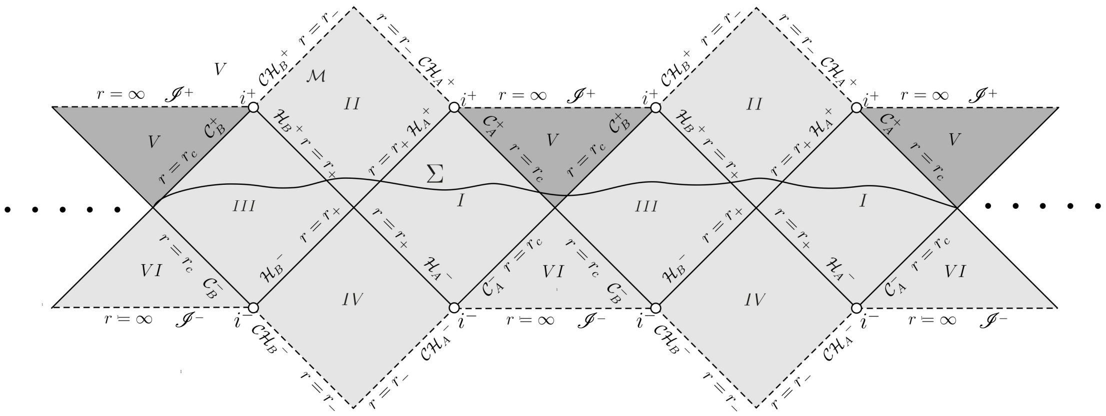

Figure 3.1: Conformal diagram of Reissner-Nordström-de Sitter spacetime.

surface to the cosmological horizon we then obtain a well-posed characteristic initial value problem. The remarkable results by Hintz and Vasy [32, 33] concerning the non-linear stability of the local region (regions I and III in Figure 3.1) of de Sitter black holes suggest[3] that our defining properties for a dynamical horizon can, in principle, be recovered from (spherically symmetric) Cauchy initial data which is close, in an appropriate sense, to RNdS data. In fact, we expect our assumptions to hold for larger classes of Cauchy data, especially since although we will require our characteristic data to approach the data of a reference RNdS black hole, we will not need to impose any specific rates of decay.

In this chapter, we will show that the future evolution of the characteristic initial data above will lead to a solution with the following properties: the radius (of symmetry) blows up along any null ray parallel to the cosmological horizon ("near" i^+). Moreover, $r = +\infty$ is, in an appropriate sense, a (differentiable) spacelike hypersurface. Finally we will also obtain a version of cosmic no-hair by showing that in the past of any causal curve reaching $r = \infty$, both the metric and the Riemann curvature tensor asymptote those of a de Sitter spacetime, at a specific rate. We will also briefly discuss conditions under which all

[3] These stability results concern the Einstein vacuum and Einstein-Maxwell equations, and not the Einstein-Maxwell-scalar field system.

the previous results can be globalized to a "complete" cosmological region.

3.0.1 Main results

Our main results can be summarized in the following:

Theorem 9. *Fix, as a reference solution, a subextremal member of the RNdS family, with parameters* (M, e, Λ) *and cosmological radius* r_c*.*

Let $(\mathcal{M}, g, F, \phi)$ *be the maximal globally hyperbolic development of smooth spherically symmetric initial data given on* $\Sigma = \mathbb{R} \times \mathbb{S}^2$*, with charge parameter* e *and cosmological constant* Λ*. Let* $\tilde{\mathcal{Q}}$ *be the projection of* $\mathcal{M}$ *to the (spherical symmetry) orbit space and assume that* $\tilde{\mathcal{Q}}$ *contains a future complete null line* $\mathcal{C}^+$*. Let* $(u, v) : \tilde{\mathcal{Q}} \to \mathbb{R}^2$ *be such that* u *is an affine parameter on* $\mathcal{C}^+ = \{v = 0\}$*,* ∂_u *and* ∂_v *are both future oriented and, in an open set of* $\tilde{\mathcal{Q}}$ *containing* $\mathcal{C}^+$*,*

$$g = -\Omega^2(u, v)dudv + r^2(u, v)\mathring{g} \ , \tag{3.1}$$

where $\mathring{g}$ *is the metric of the round 2-sphere.*

Assume also that

(i) the radius of symmetry satisfies $r(u, 0) \to r_c$*, as* $u \to \infty$ *,*

(ii) the (renormalized) Hawking mass satisfies $\varpi(u, 0) \to M$*, as* $u \to \infty$ *,*

(iii) $\partial_u r(u, 0) \geq 0$*, for all* $u \geq 0$ *,*

(iv) $\partial_v r(u, 0) > 0$*, for all* $u \geq 0$ *, and*

(v) $|\partial_u\phi(u,0)| \leq C$, *for some* $C > 0$ *and all* $u \geq 0$.

Then, there exist $U, V > 0$ *such that, in* $\mathcal{Q} = \tilde{\mathcal{Q}} \cap [U,\infty) \times [0,V)$, *we have:*

1. *Blow up of the radius function: for all* $v_1 \in (0,V)$, *there exists* $u^*(v_1) < \infty$ *such that* $r(u,v_1) \to \infty$, *as* $u \to u^*(v_1)$. *Moreover* $r(u,v) \to \infty$, *as* $(u,v) \to (u^*(v_1),v_1)$.

2. *Asymptotic behavior of the scalar field: there exists* $C > 0$ *such that*

$$\left|\frac{\partial_v\phi}{\partial_v r}\right| + \left|\frac{\partial_u\phi}{\partial_u r}\right| \leq Cr^{-2} \, . \tag{3.2}$$

3. $r = \infty$ *is spacelike: for a parameterization of the curves of constant* r *of the form* $v \mapsto (u_r(v), v)$ *there exists a constant* $C_1 > 0$ *and a function* $(0,V] \ni v \mapsto C_2(v) \in \mathbb{R}^+$, *which may blow up as* $v \to 0$, *such that*

$$-C_2(v) < u_r'(v) < -C_1 \, , \tag{3.3}$$

holds for all $r > r_c$ *and all* $v > 0$.

4. *Cosmic no-hair: Let* γ *be a causal curve along which* r *is unbounded. Let* i_{dS} *be a point in the future null infinity of the de Sitter spacetime with cosmological constant* Λ. *Let* $\{e_I\}_{I=0,1,2,3}$ *be an orthonormal frame in de Sitter defined on* $J^-(i_{dS}) \cap \{r_{dS} > r_1\}$, *for some* $r_1 > 0$; *in particular, in this frame, the de Sitter metric reads* ${}^{dS}g_{IJ} = \eta_{IJ}$. *Then, by increasing* r_1 *if necessary, there exists a diffeomorphism mapping* $J^-(\gamma) \cap \{r > r_1\}$ *to a neighborhood of* i_{dS} *in* $J^-(i_{dS})$ *such that, in the dS frame* $\{e_I\}$, *we have, for*

$r_2 \geq r_1$,

$$\sup_{J^-(\gamma)\cap\{r\geq r_2\}} |g_{IJ} - {}^{dS}g_{IJ}| \lesssim r_2^{-2} \ , \tag{3.4}$$

and

$$\sup_{J^-(\gamma)\cap\{r\geq r_2\}} |R^I_{JKL} - {}^{dS}R^I_{JKL}| \lesssim r_2^{-2} \ . \tag{3.5}$$

We will now provide extra conditions that allow the previous results to be globalized to a "complete" cosmological region:

Theorem 10. *Fix, as reference solutions, two subextremal members of the RNdS family, with parameters (M_i, e, Λ), and cosmological radius $r_{c,i}$, $i = 1, 2$. Let $(\mathcal{M}, g, F, \phi)$ and (u, v) be as in Theorem 9, but now assume that $u = 0$ is also complete and that*

(i)' $r(x, 0) \to r_{c,1}$ *and* $r(0, x) \to r_{c,2}$, *as* $x \to \infty$,

(ii)' $\varpi(x, 0) \to M_1$ *and* $\varpi(0, x) \to M_2$ *as* $x \to \infty$,

(iii)' $\partial_u r(x, 0) > 0$ *and* $\partial_v r(0, x) > 0$, *for all* $x \geq 0$,

(iv)' $\partial_v r(x, 0) > 0$ *and* $\partial_u r(0, x) > 0$, *for all* $x \geq 0$, *and*

(v)' $|\partial_u \phi(x, 0)| + |\partial_v \phi(0, x)| \leq C$, *for some* $C > 0$ *and all* $x \geq 0$.

If, moreover, either

I. there exists a sufficiently small $\epsilon > 0$, *such that for all* $x \geq 0$

$$|r(x, 0) - r_{c,1}| + |r(0, x) - r_{c,2}| + |\varpi(x, 0) - M_1| + |\varpi(0, x) - M_2| \leq \epsilon \ , \tag{3.6}$$

or

II. we assume a priori that the future boundary of $\tilde{\mathcal{Q}} \cap [0,\infty)^2$*, in* $\mathbb{R}^2$*, is of the form*

$$\mathcal{B} = \{(\infty,0)\} \cup \mathcal{N}_1 \cup \mathcal{B}_\infty \cup \mathcal{N}_2 \cup \{(0,\infty)\} \ , \tag{3.7}$$

where, for some $u_\infty, v_\infty \geq 0$*,* $\mathcal{N}_1 = \{u=\infty\} \times (0, v_\infty]$ *and* $\mathcal{N}_2 = (0, u_\infty] \times \{v=\infty\}$ *are (possibly empty) null segments, and* $\mathcal{B}_\infty$ *is an acausal curve, connecting* (∞, v_∞) *to* (u_∞, ∞)*, along which the radius* r *extends to infinity by continuity,*

then the conclusions 1–4 of Theorem 9 hold in $\mathcal{Q} = \tilde{\mathcal{Q}} \cap [0,\infty)^2$*; in particular this implies that under assumption II the segments* $\mathcal{N}_i$ *are empty.*

The proof of these theorems will be presented in Section 3.7.

Remark 11. *The a priori assumptions concerning the global structure of the future boundary of the Penrose diagram, formulated in assumption II above, were inspired by the results in [18][Theorem 1.3] concerning the Einstein-Vlasov system. Presumably, some of those results will remain valid for the matter model considered here, since they are mostly based on general properties, namely energy conditions and extension criteria, that should hold for both systems. Here we will not pursue this relation any further.*

3.1 Setting

3.1.1 The Einstein-Maxwell-scalar field system in spherical symmetry

We will consider the Einstein-Maxwell-scalar field (EMS) system in the presence of a positive cosmological constant Λ:

$$R_{\mu\nu} - \frac{1}{2}R\, g_{\mu\nu} + \Lambda g_{\mu\nu} = 2T_{\mu\nu}\,, \tag{3.8}$$

$$T_{\mu\nu} = \partial_\mu\phi\,\partial_\nu\phi - \frac{1}{2}\partial_\alpha\phi\,\partial^\alpha\phi\, g_{\mu\nu} + F_{\mu\alpha}F^{\alpha}{}_{\nu} - \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}g_{\mu\nu}\,, \tag{3.9}$$

$$dF = d\star F = 0\,, \tag{3.10}$$

$$\Box_g\phi = 0\,. \tag{3.11}$$

In this system a Lorentzian metric $g_{\mu\nu}$ is coupled to a Maxwell field $F_{\mu\nu}$ and a scalar field ϕ via the Einstein field equations (3.8), with energy-momentum tensor (3.9). To close the system we also impose the source-free Maxwell equations (3.10). Then, the scalar field satisfies, as a consequence of the previous equations, the wave equation (3.11). Note that ϕ and $F_{\mu\nu}$ only interact through the gravitational field equations; consequently, the existence of a non-vanishing electromagnetic field requires a non-trivial topology for our spacetime manifold $\mathcal{M}$.

We will work in spherical symmetry by assuming that $\mathcal{M} = \mathcal{Q}\times\mathbb{S}^2$ and by requiring the existence of double-null coordinates (u, v), on $\mathcal{Q}$, such that the spacetime metric takes the form

$$g = -\Omega^2(u,v)dudv + r^2(u,v)\mathring{g} \ , \tag{3.12}$$

with $\mathring{g}$ being the metric of the unit round sphere.

Under these conditions, Maxwell's equations decouple from the EMS system. More precisely, there exists a constant $e \in \mathbb{R}$, to which we will refer as the charge (parameter), such that

$$F = -\frac{e\,\Omega^2(u,v)}{2\,r^2(u,v)}\,du \wedge dv \ . \tag{3.13}$$

The remaining equations of the EMS system then become

$$\partial_u\partial_v r = -\frac{\Omega^2}{4r} - \frac{\partial_u r\,\partial_v r}{r} + \frac{\Omega^2 e^2}{4r^3} + \frac{\Omega^2\Lambda r}{4} \, , \tag{3.14}$$

$$\partial_u\partial_v\phi = -\,\frac{\partial_u r\,\partial_v\phi + \partial_v r\,\partial_u\phi}{r} \, , \tag{3.15}$$

$$\partial_v\partial_u \ln\Omega = -\partial_u\phi\,\partial_v\phi - \frac{\Omega^2 e^2}{2r^4} + \frac{\Omega^2}{4r^2} + \frac{\partial_u r\,\partial_v r}{r^2} \, , \tag{3.16}$$

$$\partial_u\left(\Omega^{-2}\partial_u r\right) = -r\Omega^{-2}\left(\partial_u\phi\right)^2 \ , \tag{3.17}$$

$$\partial_v\left(\Omega^{-2}\partial_v r\right) = -r\Omega^{-2}\left(\partial_v\phi\right)^2 \ . \tag{3.18}$$

It turns out to be convenient, both by its physical relevance and by its good monotonicity properties, to introduce the (renormalized) Hawking mass $\varpi = \varpi(u,v)$, defined by

$$1 - \mu := 1 - \frac{2\varpi}{r} + \frac{e^2}{r^2} - \frac{\Lambda}{3}r^2 = \partial^\alpha r \partial_\alpha r \ . \tag{3.19}$$

Later we will replace the wave equation for the conformal factor Ω (3.16) by equations (3.27) and (3.28) prescribing the gradient of ϖ.

We will use a first order formulation of the EMS system, obtained by introducing the quantities:

$$\nu := \partial_u r \,, \tag{3.20}$$

$$\lambda := \partial_v r \,, \tag{3.21}$$

$$\theta := r\partial_v \phi \,, \tag{3.22}$$

$$\zeta := r\partial_u \phi \,. \tag{3.23}$$

Note, for instance, that in terms of these new quantities (3.19) gives

$$1 - \mu = -4\Omega^{-2}\lambda\nu \,. \tag{3.24}$$

Later we will be interested in a region where $\lambda > 0$; there, we are allowed to define

$$\underline{\kappa} := -\frac{1}{4}\Omega^2\lambda^{-1} \,. \tag{3.25}$$

Our first order system is then[4]

$$\partial_u \lambda = \partial_v \nu = \underline{\kappa} \lambda \partial_r (1-\mu) \,, \tag{3.26}$$
$$\partial_u \varpi = \frac{\zeta^2}{2\underline{\kappa}} \,, \tag{3.27}$$
$$\partial_v \varpi = \frac{\theta^2}{2\lambda} (1-\mu) \,, \tag{3.28}$$
$$\partial_u \theta = -\frac{\zeta \lambda}{r} \,, \tag{3.29}$$
$$\partial_v \zeta = -\frac{\theta \nu}{r} \,, \tag{3.30}$$
$$\partial_v \underline{\kappa} = \frac{\underline{\kappa} \theta^2}{r \lambda} \,, \tag{3.31}$$
$$\nu = \underline{\kappa} (1-\mu) \,, \tag{3.32}$$

which is an overdetermined system of PDEs together with the algebraic equation (3.32) that replaces (3.25). The results in [13][Section 6] show that, under appropriate regularity conditions, this system is equivalent to the spherically symmetric EMS system, in a spacetime region where $\partial_v r = \lambda > 0$.

3.1.2 Reissner-Norström-de Sitter revisited

If the scalar field vanishes identically, then the Hawking mass is constant, $\varpi \equiv M$. Among these electrovacuum solutions[5], the Reissner-Nordström-de Sitter (RNdS) black holes, which form a 3-parameter (M, e, Λ) family of solutions, are of special relevance to us here. We will only consider the solutions of this family for which $M \geq 0$ and the function $r \mapsto (1-\mu)(M, r)$ has a positive

[4]Note that $(1-\mu)$ depends on (u, v) only through (r, ϖ). In what follows, in a slight abuse of notation, we will interchangeably regard $(1-\mu)$ as a function of either pair of variables, with the meaning being clear from the context.

[5]Note that in this setting Birkhoff's Theorem is less restrictive and allows solutions which do not belong to the RNdS family, see Appendix D and [27][Theorem 1.1].

root r_c such that

$$\partial_r(1-\mu)(M,r_c) < 0 \ , \tag{3.33}$$

and

$$\partial_r^2(1-\mu)(M,r_c) < 0 \ . \tag{3.34}$$

These conditions are enough to identify $r = r_c$ as a non-extremal[6] cosmological horizon; in the case of a vanishing charge, the condition on the second derivative is in fact a consequence of $M \geq 0$. Note that, in RNdS, $r = r_c$ is a Killing horizon [9] whose generator is the spacetime's static Killing vector field.

The form of the metric of these RNdS solutions in double-null coordinates is not particularly elucidative and will be omitted. On the other hand, their global causal structure is crucial to us and is completely described by the Penrose diagram in Figure 3.1; note that when we refer to a member of the RNdS family we are considering the solutions corresponding to the maximal (globally hyperbolic) development of appropriate Cauchy data. In this chapter we will focus on the *cosmological region* $r > r_c$, which corresponds to the causal future of the cosmological horizon $r = r_c$; more precisely, we will be interested in understanding how stable its geometry is under non-linear spherically symmetric scalar perturbations.

3.1.3 The characteristic initial value problem $\mathrm{IVP}_{\mathcal{C}^+}$

The goal of this subsection is to establish an initial value problem for the system (3.26)–(3.32) which captures the essential features of a cosmological region to the future of a dynamic (i.e. not necessarily stationary) cosmological

[6] Non-degenerate in the terminology of [9].

horizon $\mathcal{C}^+ = \{(u,v) : v = 0\}$. We will refer to this as IVP$_{\mathcal{C}^+}$.

We fix the coordinate u by setting

$$\underline{\kappa}_0(u) := \underline{\kappa}(u,0) \equiv -1 \tag{3.35}$$

and, for a fixed $V > 0$, we consider as initial data, given on

$$[0,+\infty[\, \times \{0\} \cup \{0\} \times [0,V]\,, \tag{3.36}$$

the functions

$$r_0(v) := r(0,v)\,, \tag{3.37}$$

$$\nu_0(u) := \nu(u,0)\,, \tag{3.38}$$

$$\zeta_0(u) := \zeta(u,0)\,, \tag{3.39}$$

$$\lambda_0(v) := \lambda(0,v)\,, \tag{3.40}$$

$$\varpi_0(u) := \varpi(u,0)\,, \tag{3.41}$$

$$\theta_0(v) := \theta(0,v)\,, \tag{3.42}$$

with ν_0, λ_0, θ_0 and ζ_0 continuous, and r_0 and ϖ_0 continuously differentiable.

We also impose the sign conditions

$$r_0(v) > 0\,, \tag{3.43}$$

$$\tilde{r}_0(u) > 0\,, \tag{3.44}$$

$$\lambda_0(v) > 0\,, \tag{3.45}$$

$$\nu_0(u) \geq 0\,, \tag{3.46}$$

where

$$\tilde{r}_0(u) := r_0(0) + \int_0^u \nu_0(u')\, du' \ . \tag{3.47}$$

The sign for ν_0 is motivated by the second law of black hole thermodynamics and the sign of λ_0 by the need to model a cosmological (expanding) region; the sign of $\underline{\kappa}_0$ was chosen to accommodate (3.25).

The overdetermined character of our system implies that the initial data is necessarily constrained. To deal with this we assume the following compatibility conditions:

$$r_0' = \lambda_0 \,, \tag{3.48}$$

$$\varpi_0' = -\frac{\zeta_0^2}{2} \,, \tag{3.49}$$

$$\nu_0 = -\,(1-\mu)(\varpi_0, \tilde{r}_0) \ . \tag{3.50}$$

We will also assume that our initial data approaches (without requiring any specific decay rate) a RNdS cosmological horizon along $v = 0$. More precisely we will assume the existence of constants $r_c > 0$ and $\varpi_c \geq 0$ such that

$$\lim_{u \to +\infty} \tilde{r}_0(u) = r_c < +\infty \,, \tag{3.51}$$

$$\lim_{u \to +\infty} \varpi_0(u) = \varpi_c < +\infty \,, \tag{3.52}$$

with

$$(1-\mu)(r_c, \varpi_c) = 0 \ . \tag{3.53}$$

The condition above captures the idea of approaching a RNdS horizon. To make sure that it is a cosmological horizon we impose (3.33) and (3.34), with $M = \varpi_c$. By continuity, given $\epsilon_0 > 0$, we can then shift our u coordinate in such a way that

$$|\tilde{r}_0(u) - r_c| + |\varpi_0(u) - \varpi_c| < \epsilon_0 \ \ , \text{ for all } u \geq 0 \ . \tag{3.54}$$

Then, we can choose ϵ_0 small enough such that, for all $u \geq 0$, we have

$$\partial_r(1-\mu)(\varpi_0(u), \tilde{r}_0(u)) \leq -2k < 0 \tag{3.55}$$

for some $k > 0$, which can be chosen arbitrarily close to $\frac{1}{2}\partial_r(1-\mu)(\varpi_c, r_c)$ by decreasing ϵ_0. This condition, together with (3.26), (3.35) and the fact that the range of u is unbounded imply that the line $v = 0$ is in fact a complete null geodesic (compare with [14][Section 8]).

Finally, we will assume that there exists $C_d > 0$ such that

$$|\zeta_0(u)| \leq C_d \ \ , \ \ \text{for all } \ u \geq 0 \ . \tag{3.56}$$

This finishes our characterization of $\text{IVP}_{\mathcal{C}^+}$.

Remark 12. *Note that the conditions* (3.52) *together with the constraints* (3.48) *impose restrictions on the integrability of our data: for instance,* $\varpi_0(u) \to \varpi_c$ *requires that* $\zeta_0 \in L^2([0,\infty))$.

Remark 13. *The existence of "large" classes of data satisfying all the previous conditions, which is far from clear a priori, follows by a simple adaptation of the techniques in [15][Section 3]. In alternative, one can also take the perspective of Theorem 9 where our characteristic initial data arises from the evolution of appropriate Cauchy initial data. The existence of such Cauchy data, close to RNdS data, is expected to follow from the non-linear stability results in [32, 33].*

We end this section with a simple adaptation of [13][Theorem 4.4]:

Theorem 14 (Maximal development and its domain $\mathcal{Q}$ for $\mathrm{IVP}_{\mathcal{C}^+}$)**.** *The characteristic initial value problem* $\mathrm{IVP}_{\mathcal{C}^+}$*, with initial data as described above, has a unique classical solution (in the sense that all the partial derivatives occurring in* (3.26)–(3.32) *are continuous) defined on a maximal past set* $\mathcal{Q}$ *containing a neighborhood*[7] *of* $([0,+\infty[\,\times\{0\})\cup(\{0\}\times[0,V])$ *in* $[0,+\infty[\,\times[0,V]$.

3.2 Preliminary results and an extension criterion

We start by establishing some basic sign properties, including a fundamental negative upper bound for $\partial_r(1-\mu)$ (3.63) that corresponds to a global redshift.

Lemma 15. *For the initial value problem* $\mathrm{IVP}_{\mathcal{C}^+}$*, the following conditions are satisfied in* $\mathcal{Q}$*:*

$$\lambda > 0\,, \tag{3.57}$$

$$\underline{\kappa} \leq -1\,, \tag{3.58}$$

$$\nu \geq 0\,,\ \textit{with } \nu > 0 \textit{ in } \mathcal{Q}\setminus\mathcal{C}^+\,, \tag{3.59}$$

$$r \geq r(0,0)\,, \tag{3.60}$$

$$\partial_u\varpi \leq 0\,, \tag{3.61}$$

$$\partial_v\varpi \leq 0\,, \tag{3.62}$$

$$\partial_r(1-\mu) \leq -2k\,,\ \textit{for } k>0 \textit{ as in } (3.55)\,, \tag{3.63}$$

[7] From now on, all topological statements will refer to the topology of $[0,+\infty[\times[0,V]$ induced by the standard topology of $\mathbb{R}^2$.

$$1-\mu \leq 0\,, \;\; \textit{with}\; 1-\mu<0 \;\textit{in}\; \mathcal{Q}\setminus\mathcal{C}^+\,. \tag{3.64}$$

Proof. Recall that $\lambda(0,v)>0$ and notice how the evolution equation for λ (3.26) implies that it cannot change sign (since the solution is the product of the initial value by an exponential function); that is, $\lambda>0$ in $\mathcal{Q}$. It then follows from (3.31) and the fact that $\underline{\kappa}(u,0)\equiv -1$ that $\underline{\kappa}\leq -1$. As an immediate consequence (3.27) tells us that $\partial_u\varpi\leq 0$.

Now, let $\tilde{\mathcal{Q}}=\{(u,v)\in\mathcal{Q}:\nu(u,\tilde{v})\geq 0\,,\forall\tilde{v}\in[0,v]\}$. In this subset, the signs of ν and λ immediately imply that $r\geq r(0,0)$ and, since $\nu=(1-\mu)\underline{\kappa}$, one has $1-\mu\leq 0$ (with equality if and only if $\nu=0$), and so $\partial_v\varpi\leq 0$ due to (3.28).

Furthermore,

$$\partial_\varpi\partial_r(1-\mu)=\frac{2}{r^2}>0\,, \tag{3.65}$$

and, for $v>0$,

$$\lambda>0\Rightarrow r(u,v)>r(u,0)\,. \tag{3.66}$$

By inspection of the graph of $r\mapsto 1-\mu(\varpi,r)$, we see that, for a sufficiently small ϵ_0 in (3.54), we have in $\tilde{\mathcal{Q}}$

$$\begin{aligned}\partial_r\left(1-\mu\right)\left(\varpi(u,v),r(u,v)\right)&\leq\partial_r\left(1-\mu\right)\left(\varpi(u,0),r(u,v)\right)\\&\leq\partial_r\left(1-\mu\right)\left(\varpi(u,0),r(u,0)\right)\\&\leq\;-2k\,,\end{aligned} \tag{3.67}$$

where the last inequalities follow from (3.55).

Knowing the signs of the quantities above, (3.26) implies

$$\partial_v \nu = \underline{\kappa} \lambda \partial_r (1 - \mu) > 0 \ , \tag{3.68}$$

in $\tilde{\mathcal{Q}}$. So, with the initial condition $\nu \geq 0$ on $\mathcal{C}^+$, one has $\nu > 0$ in $\tilde{\mathcal{Q}} \backslash \mathcal{C}^+$.

It now suffices to show that $\tilde{\mathcal{Q}} = \mathcal{Q}$. To this effect, define, for each $u \geq 0$, the sets $\mathcal{V}_u = \{v : (u, v) \in \mathcal{Q}\}$ and $\tilde{\mathcal{V}}_u = \left\{v : (u, v) \in \tilde{\mathcal{Q}}\right\}$. We will show that $\tilde{\mathcal{V}}_u$ is open and closed in $\mathcal{V}_u$.

Let $\{v_n\}_{n \in \mathbb{N}} \subset \tilde{\mathcal{V}}_u$ be a sequence with $v_n \to v^* \in \mathcal{V}_u$. If $\exists k \in \mathbb{N}$ such that $v_k > v^*$, then $\nu(u, v') \geq 0 \ \forall v' \leq v_k$ and, in particular, $\forall v' \leq v^*$; so, $(u, v^*) \in \tilde{\mathcal{Q}}$ and $v^* \in \tilde{\mathcal{V}}_u$. On the other hand, if there exists no such k, then there is an increasing subsequence $v_{n_m} \to v^*$; this means that $[0, v_{n_m}] \subset \tilde{\mathcal{V}}_u$, so $\bigcup_{m \in \mathbb{N}} [0, v_{n_m}] \subset \tilde{\mathcal{V}}_u$ and $[0, v^*[\ \subset \tilde{\mathcal{V}}_u$, whence, by continuity, $\nu(u, v^*) \geq 0$ and, therefore, $v^* \in \tilde{\mathcal{V}}_u$.

Suppose now that $v^* \in \tilde{\mathcal{V}}_u$, meaning $\nu(u, v') \geq 0 \ \forall v' \in [0, v^*]$; from (3.68), it follows that $\partial_v \nu(u, v^*) > 0$, and so, by continuity, $\exists \epsilon > 0$ such that $\partial_v \nu(u, v') > 0 \ \forall v' \in [v^*, v^* + \epsilon]$. This implies that $\nu(u, v') \geq 0 \ \forall v' \in [0, v^* + \epsilon]$ or, in other words, $(u, v^* + \epsilon) \in \tilde{\mathcal{Q}}$; thus, $[0, v^* + \epsilon[\ \subset \tilde{\mathcal{V}}_u$.

Since $\tilde{\mathcal{V}}_u \neq \varnothing$ (because $0 \in \tilde{\mathcal{V}}_u$) and $\mathcal{V}_u$ is connected (because $\mathcal{Q}$ is a past set), we have $\tilde{\mathcal{V}}_u = \mathcal{V}_u$ for all $u \geq 0$, whence $\tilde{\mathcal{Q}} = \mathcal{Q}$. □

Note that, in view of (3.57) and (3.59), the curves of constant r are spacelike in $\mathcal{Q} \setminus \{v = 0\}$ and can be parameterized by

$$u \mapsto (u, v_r(u)) \ , \tag{3.69}$$

or

$$v \mapsto (u_r(v), v) \ , \tag{3.70}$$

where u_r and v_r are C^1, but with domains unknown to us at the moment.

We can now easily improve some of the previous sign properties to more detailed bounds.

Lemma 16. *For the initial value problem* $\mathrm{IVP}_{\mathcal{C}^+}$*, the following estimates hold in* $\mathcal{Q}$*:*

$$-\frac{\Lambda}{6}r^3 + \frac{r}{2} + \frac{e^2}{2r} \ \leq \ \varpi \ \leq \ \varpi(0,0) \ , \tag{3.71}$$

$$1 - \frac{2\varpi(0,0)}{r} + \frac{e^2}{r^2} - \frac{\Lambda}{3}r^2 \ \leq \ 1 - \mu \ \leq \ 0 \ , \tag{3.72}$$

$$-\Lambda r + \frac{1}{r} - \frac{e^2}{r^3} \ \leq \ \partial_r(1-\mu) \ \leq \ \frac{2\varpi(0,0)}{r^2} - \frac{2e^2}{r^3} - \frac{2\Lambda}{3}r \ . \tag{3.73}$$

Proof. Since we know that $\partial_u \varpi$, $\partial_v \varpi$ and $1-\mu$ are all non-positive, we get

$$1 - \mu \leq 0 \Leftrightarrow -\frac{\Lambda}{6}r^3 + \frac{r}{2} + \frac{e^2}{2r} \leq \varpi \tag{3.74}$$

and

$$\varpi \leq \varpi(0,0) \Leftrightarrow 1 - \mu \geq 1 - \frac{2\varpi(0,0)}{r} + \frac{e^2}{r^2} - \frac{\Lambda}{3}r^2 \, . \tag{3.75}$$

Applying the bounds for ϖ to the identity

$$\partial_r(1-\mu) \ = \ \frac{2\varpi}{r^2} - \frac{2e^2}{r^3} - \frac{2\Lambda}{3}r \, , \tag{3.76}$$

leads to (3.73). $\square$

Restricting ourselves to a region with bounded radius function we can obtain further bounds:

Lemma 17. *For the initial value problem* $\mathrm{IVP}_{\mathcal{C}^+}$ *and* $R > r(0,0)$ *there exists a constant* $C_{d,R}$*, depending only on* R *and on the size of the initial data, such that, in* $\mathcal{Q} \cap \{r \leq R\}$*,*

$$\max\left\{\left|\frac{\theta}{\lambda}\right|, |\underline{\kappa}|, |\nu|, |\zeta|\right\} \leq C_{d,R}\,. \tag{3.77}$$

Proof. Integrating (3.27) in $\mathcal{Q}\cap\{r \leq R\}$, while using the bounds (3.71) and (3.60), we obtain

$$\int_0^u \frac{\zeta^2}{-\underline{\kappa}} du = 2\left(\varpi(0,v) - \varpi(u,v)\right) \leq C_{d,R}\,. \tag{3.78}$$

Then, integrating the evolution equation

$$\partial_u\left(\frac{\theta}{\lambda}\right) = -\frac{\zeta}{r} - \frac{\theta}{\lambda}\underline{\kappa}\partial_r(1-\mu)\,, \tag{3.79}$$

yields, in view of the previous and (3.63),

$$\begin{aligned}
\left|\frac{\theta}{\lambda}(u,v)\right| &\le \left|\frac{\theta}{\lambda}(0,v)\right| e^{-\int_0^u \underline{\kappa}\partial_r(1-\mu)(u',v)du'} + \int_0^u \frac{|\zeta|}{r}(u'v)e^{-\int_{u'}^u \underline{\kappa}\partial_r(1-\mu)(u'',v)du''}du' \\
&\le C_d + \frac{1}{R}\int_0^u \frac{|\zeta|}{\sqrt{-\underline{\kappa}}}\sqrt{-\underline{\kappa}}\, e^{-\int_{u'}^u \underline{\kappa}\partial_r(1-\mu)du''}du' \\
&\le C_d + \frac{1}{R}\left(\int_0^u \frac{|\zeta|^2}{-\underline{\kappa}}(u',v)du'\right)^{\frac{1}{2}}\left(\int_0^u -\underline{\kappa}\, e^{2\int_{u'}^u -\underline{\kappa}\partial_r(1-\mu)du''}du'\right)^{\frac{1}{2}} \\
&\le C_d + C_{d,R}\left(\int_0^u -\underline{\kappa}\, e^{-4k\int_{u'}^u -\underline{\kappa}du''}du'\right)^{\frac{1}{2}} \\
&= C_d + C_{d,R}\left(\left[\frac{1}{4k}\, e^{-4k\int_{u'}^u |\underline{\kappa}|du''}\right]_{u'=0}^{u'=u}\right)^{\frac{1}{2}} \\
&\le C_{d,R}\ ,
\end{aligned} \tag{3.80}$$

where C_d represents a uniform bound on the initial data.

As a consequence, we are now able to estimate $|\underline{\kappa}|$. In fact, using the evolution equation (3.31),

$$\begin{aligned}
|\underline{\kappa}(u,v)| &= \exp\left[\int_0^v \left|\frac{\theta}{\lambda}\right|^2 \frac{\lambda}{r}(u,v')\,dv'\right] \\
&\le \exp\left[C_{d,R}\int_0^v \frac{\partial_v r}{r}(u,v')\,dv'\right] \\
&= \exp\left[C_{d,R}\int_{r(u,0)}^{r(u,v)} \frac{dr'}{r'}\right] \\
&\le \left[\frac{r(u,v)}{r(0,0)}\right]^{C_{d,R}} .
\end{aligned} \tag{3.81}$$

Combining this estimate with (3.72) immediately gives the bound for $|\nu| = \nu$. Recalling that ν increases with v in $\mathcal{Q}$, we can integrate (3.30) and, using (3.56), obtain the estimate

$$
\begin{aligned}
|\zeta(u,v)| &\leq |\zeta(u,0)| + \int_0^v \left(\left|\frac{\theta}{\lambda}\right| \frac{\lambda\nu}{r}\right)(u,v')\,dv' \\
&\leq C_d + C_{d,R} \int_0^v \left(\frac{\lambda\nu}{r}\right)(u,v')\,dv' \\
&\leq C_d + C_{d,R}\,\nu(u,v) \int_0^v \frac{\lambda}{r}(u,v')\,dv' \\
&\leq C_d + C_{d,R} \log\left|\frac{r(u,v)}{r(u,0)}\right| \\
&\leq C_{d,R}\,.
\end{aligned}
\tag{3.82}
$$

□

As a result of all the estimates obtained in this section, we have uniform bounds in any region of the form $\mathcal{Q} \cap \{r \leq R\}$ for almost all quantities in terms of data and R. For example, from (3.27) and (3.52) we easily obtain

$$
|\varpi(u,v)| \leq C_{d,R}\,. \tag{3.83}
$$

The single exception is λ, for which the best bound available in this region (compare with the upcoming (3.88)), is

$$
\lambda(u,v) = \lambda(0,v) e^{\int_0^u \underline{\kappa}\partial_r(1-\mu)du'} \leq C_d e^{C_{d,R} u}\,, \tag{3.84}
$$

which follows from (3.26) and the bound for $\underline{\kappa}$. This is nonetheless enough to establish the following extension criterion, whose standard proof will be omitted.

Proposition 18. *For the initial value problem* $\mathrm{IVP}_{\mathcal{C}^+}$*, let* $p \in \overline{\mathcal{Q}} \setminus \{v = V\}$ *be such that*

$$
\mathcal{D} := \big(J^-(p) \cap J^+(q)\big) \setminus \{p\} \subset \mathcal{Q} \tag{3.85}
$$

for some point $q \in J^-(p)$. If the radius function is bounded in $\mathcal{D}$,

$$\sup_{\mathcal{D}} r < \infty \;, \tag{3.86}$$

then $p \in \mathcal{Q}$. In particular (since $\mathcal{Q}$ is an open past set), there exists $\epsilon > 0$ such that

$$[0, u(p) + \epsilon] \times [0, v(p) + \epsilon] \subset \mathcal{Q} \;. \tag{3.87}$$

3.3 Unboundedness of the radius function

Quite remarkably, we are almost ready to establish that, for the initial value problem $\mathrm{IVP}_{\mathcal{C}^+}$, the radius function has to blow up along any null ray of the form $v = v_1 \in (0, V]$. To do so we need the following lower bounds.

Lemma 19. *For the initial value problem* $\mathrm{IVP}_{\mathcal{C}^+}$*, we have in $\mathcal{Q}$*

$$\lambda(u,v) \geq \underline{\lambda_0}\, e^{2ku} \;, \tag{3.88}$$

where

$$\underline{\lambda_0} := \inf_{v \in [0,V]} \lambda(0,v) \;>\; 0 \,. \tag{3.89}$$

Moreover, there exists $\alpha > 0$ such that, for any $v \in [0, V]$,

$$\nu(u,v) \geq \alpha v \;. \tag{3.90}$$

Proof. Note that in view of (3.58) and (3.63) we have in $\mathcal{Q}$

$$\underline{\kappa}\, \partial_r (1 - \mu) \geq 2k \;. \tag{3.91}$$

It then follows that

$$\begin{aligned}\lambda(u,v) &= \lambda(0,v)\,\exp\left[\int_0^u \underline{\kappa}\,\partial_r(1-\mu)\,(u',v)\,du'\right]\\ &\geq \underline{\lambda_0}\,\exp\left[\int_0^u 2k\,du'\right]\\ &= \underline{\lambda_0}\,e^{2ku}\,.\end{aligned}\tag{3.92}$$

Consequently,

$$\begin{aligned}\nu(u,v) &= \nu(u,0) + \int_0^v \lambda\,\underline{\kappa}\,\partial_r(1-\mu)\,(u,v')dv'\\ &\geq \nu(u,0) + \underline{\lambda_0}\,2k\,e^{2ku}\,v\,,\end{aligned}\tag{3.93}$$

from which (3.90) follows.

□

We are now ready to obtain, in one go, two important characterizations of the behavior of the radius function: we will show that the radius function blows up along any null line $v = v_1 \neq 0$, and that all the (spacelike curves) $r = r_1 > r_c$ accumulate (in the topology of the plane) at $i^+ = (\infty, 0)$. More precisely:

Theorem 20. *Consider the initial value problem* $\mathrm{IVP}_{\mathcal{C}^+}$ *and let* $v_1 \in\,]0, V]$. *Then,*

$$\sup\{r(u,v_1) : (u,v_1) \in \mathcal{Q}\} = +\infty\,. \tag{3.94}$$

Moreover, for all $r_1 > r_c$ *there exists* $0 < V_{r_1} \leq V$ *such that the function* u_{r_1} *(recall* (3.70)*) is defined for all* $v \in\,]0, V_{r_1}[$ *and is such that*

$$u_{r_1}(v) \to \infty\,,\ \ \textit{as}\ v \to 0\,. \tag{3.95}$$

Proof. From (3.90) we have

$$
\begin{aligned}
r(u,v_1) &= r(0,v_1) + \int_0^u \nu(u',v_1)\,du' \\
&\geq r(0,v_1) + \alpha v_1 u \ ,
\end{aligned}
\tag{3.96}
$$

which clearly diverges as $u \to \infty$, provided that $v_1 \neq 0$. The first result (3.94) is now a consequence of the extension criterion (Proposition 18), and then the remaining conclusions follow. □

We will now show that, in fact, r blows up in finite u-"time".

Theorem 21. *Let $\mathcal{Q}$ be the domain of the maximal solution of the characteristic initial value problem with initial data satisfying* $\mathrm{IVP}_{\mathcal{C}^+}$. *Then:*

1. *For every $v_1 \in (0,V]$ there exists $u^*(v_1) > 0$ such that*

$$
\lim_{u \to u^*(v_1)} r(u,v_1) = +\infty \ . \tag{3.97}
$$

2. *For every $r_1 > r(0,0)$ there exists $0 < V_{r_1} \leq V$ such that the function u_{r_1} is defined for all $v \in (0,V_{r_1}]$. Moreover, $V_{r_1} = V$ and $u_{r_1}(V) < u^*(V)$ or $u_{r_1}(V_{r_1}) = 0$, i.e., the curve $r = r_1$ leaves $\mathcal{Q}$ to the right through $(\{u=0\} \cup \{v=V\}) \cap \mathcal{Q}$.*

3. *For any $r_1 > r(0,V)$, $V_{r_1} = V$ and*

$$
\mathcal{Q} \cap \{r \geq r_1\} \setminus \{v=0\} = \{(u,v) \,:\, 0 < v \leq V \text{ and } u_{r_1}(v) \leq u < u^*(v)\} \ . \tag{3.98}
$$

4.

$$
\lim_{(u,v) \to (u^*(v_1),v_1)} r(u,v) = \infty \ . \tag{3.99}
$$

Proof. From (3.63) and (3.73) we can guarantee the existence of $\bar{A} > 0$ such that

$$-\partial_r(1-\mu) \geq 2\bar{A}\,r\,. \tag{3.100}$$

Therefore, recalling (3.58) and that $\nu(u,0) \geq 0$, the equation for ν yields

$$\begin{aligned}
\nu(u,v) &= \nu(u,0) + \int_0^v \lambda\,\underline{\kappa}\,\partial_r(1-\mu)\,(u,v')\,dv' \\
&\geq 2\bar{A}\int_0^v \lambda\,r\,(u,v')\,dv' \\
&= \bar{A}\left[r^2(u,v')\right]_{v'=0}^{v'=v} \\
&= \bar{A}\,r^2(u,v)\left[1-\frac{r^2(u,0)}{r^2(u,v)}\right]\,.
\end{aligned} \tag{3.101}$$

So, for $r_c < r_1 \leq r(u,v)$, we have

$$\nu(u,v) \geq A\,r^2(u,v)\,, \tag{3.102}$$

with

$$A := \bar{A}\left[1-\frac{r_c^2}{{r_1}^2}\right]\,. \tag{3.103}$$

In view of Theorem 20, if r_1 is sufficiently large then for each $v \in \,]0,V]$ there exists $u_{r_1}(v) \geq 0$ such that $r(u_{r_1}(v),v) = r_1$. Consequently,

$$\begin{aligned}
&\int_{u_{r_1}(v)}^{u} \frac{\partial_u r(u',v)}{r^2(u',v)}du' \geq A\,(u-u_{r_1}(v)) \\
&\Leftrightarrow \frac{1}{r_1}-\frac{1}{r(u,v)} \geq A\,(u-u_{r_1}(v)) \\
&\Leftrightarrow r(u,v) \geq \frac{1}{\frac{1}{r_1}-A\,(u-u_{r_1}(v))}\,,
\end{aligned} \tag{3.104}$$

which tends to $+\infty$ when $u \to \left(u_{r_1}(v) + \frac{1}{Ar_1}\right)^-$. This establishes point 1.

Let V_{r_1} be the supremum of the set of points in $]0, V]$ for which u_{r_1} is well-defined in $]0, V_{r_1}[$. If point 2 were not true then, in view of the spacelike character of $r = r_1$, we would have

$$\lim_{v \to V_{r_1}} u_{r_1}(v) = u^*(V_{r_1}) \ . \tag{3.105}$$

Since r is unbounded along $v = V_{r_1}$, there must exist $r_2 > r_1$ and $0 < u_2 < u^*(V_{r_1})$ such that $r(u_2, V_{r_1}) = r_2$. But since $(u_2, V_{r_1}) \in \mathcal{Q}$, there exists $\epsilon > 0$ such that $r(u_{r_2}(V_{r_1} - \epsilon), V_{r_1} - \epsilon) = r_2$ and $u_{r_2}(V_{r_1} - \epsilon) < u_{r_1}(V_{r_1} - \epsilon)$, which contradicts the fact that $\nu > 0$ in $\mathcal{Q} \setminus \{v = 0\}$. This establishes point 2. Points 3 and 4 are then an immediate consequence. □

3.4 Behavior of the scalar field for large r

It is possible to control the geometry of the curves of constant r even further. To do this, we need precise knowledge about the behavior of the scalar field for large r, which is, of course, interesting in itself.

Note that, for $r_1 > r_c$, the function

$$(u, v) \mapsto (r(u, v), v) \ , \tag{3.106}$$

defines a diffeomorphism, mapping $\mathcal{Q} \cap \{r \geq r_1\}$ into $[r_1, \infty) \times (0, V]$, with inverse

$$(r, v) \mapsto (u_r(v), v) \ . \tag{3.107}$$

For any function $h = h(u,v)$ we define

$$\tilde{h}(r,v) := h(u_r(v),v)\ . \tag{3.108}$$

Now define the function $f : [r_1,\infty) \to \mathbb{R}$ by

$$f(r) := \sup_{v\in(0,V]} \max\left\{\left|\frac{\tilde{\theta}}{\tilde{\lambda}}(r,v)\right| , \left|\frac{\tilde{\zeta}}{\tilde{\nu}}(r,v)\right|\right\}\ ; \tag{3.109}$$

that this is a well-defined function follows from (3.77), (3.102) and Theorem 21.

Theorem 22. *There exists $r_1 > r_c$ such that, for all r_2, r with $r_1 \le r_2 \le r$ we have, for the function defined by* (3.109),

$$f(r) \le f(r_2)\frac{r_2}{r}\ . \tag{3.110}$$

Proof. We will start by establishing the following

Lemma 23. *There exists $r_1 > r_c$ such that, given $r_2 > r_1$, we have for any $r \ge r_2$*

$$f(r) \le C_2 \Rightarrow f(r) \le \frac{C_2}{2}\left(1 + \frac{{r_2}^2}{r^2}\right)\ . \tag{3.111}$$

Proof. Integrating (3.79) from $r = r_2$ towards the future gives

$$\begin{aligned}\frac{\theta}{\lambda}(u,v) =& \frac{\theta}{\lambda}(u_{r_2}(v),v)e^{-\int_{u_{r_2}(v)}^{u}\frac{\partial_r(1-\mu)}{1-\mu}\nu(u',v)du'}\\ &- \int_{u_{r_2}(v)}^{u}\frac{\zeta}{r}(u',v)e^{-\int_{u'}^{u}\frac{\partial_r(1-\mu)}{1-\mu}\nu(u'',v)\,du''}\,du'\ .\end{aligned} \tag{3.112}$$

To estimate the exponential we note that from Lemma 16 we have

$$
\begin{aligned}
\frac{\partial_r(1-\mu)}{1-\mu} &= \frac{-\partial_r(1-\mu)}{-(1-\mu)} \\
&\geq \frac{\frac{2\Lambda}{3}r - \frac{2\varpi(0,0)}{r^2} + \frac{2e^2}{r^3}}{\frac{\Lambda}{3}r^2 - 1 + \frac{2\varpi(0,0)}{r} - \frac{e^2}{r^2}} \\
&= \frac{2}{r}\,\frac{1 - \frac{3\varpi(0,0)}{\Lambda r^3} + \frac{3e^2}{\Lambda r^4}}{1 - \frac{3}{\Lambda r^2} + \frac{6\varpi(0,0)}{\Lambda r^3} - \frac{3e^2}{\Lambda r^4}} \\
&= \frac{2}{r}\left(1 + \frac{\frac{3}{\Lambda r^2} - \frac{9\varpi(0,0)}{\Lambda r^3} + \frac{6e^2}{\Lambda r^4}}{1 - \frac{3}{\Lambda r^2} + \frac{6\varpi(0,0)}{\Lambda r^3} - \frac{3e^2}{\Lambda r^4}}\right) \\
&\geq \frac{2}{r}\,,
\end{aligned}
\tag{3.113}
$$

for any $r \geq r_1$ with r_1 sufficiently large. As an immediate consequence, for $r_1 \leq r_a \leq r_b$, we have

$$
\exp\left[-\int_{r_a}^{r_b} \frac{\partial_r(1-\mu)}{1-\mu}dr\right] \leq \left(\frac{r_a}{r_b}\right)^2 . \tag{3.114}
$$

Using this estimate we find that

$$
\begin{aligned}
\left|\frac{\theta}{\lambda}\right|(u,v) \leq \left|\frac{\theta}{\lambda}\right|(u_{r_2}(v),v)\left(\frac{r_2}{r}\right)^2 + \\
+\frac{1}{r^2}\int_{u_{r_2}(v)}^{u}\left(\left|\frac{\zeta}{\nu}\right|\nu r\right)(u',v)du' .
\end{aligned}
\tag{3.115}
$$

Using the hypothesis of the lemma we get

$$
\begin{aligned}
\left|\frac{\tilde{\theta}}{\tilde{\lambda}}(r,v)\right| &\leq C_2\left(\frac{r_2}{r}\right)^2 + \frac{C_2}{r^2}\int_{r_2}^{r} r'dr' \\
&= C_2\left(\frac{r_2}{r}\right)^2 + \frac{C_2}{r^2}\frac{r^2 - {r_2}^2}{2} \\
&= \frac{C_2}{2}\left(1 + \frac{{r_2}^2}{r^2}\right) .
\end{aligned}
\tag{3.116}
$$

Analogously, by integrating the equation

$$\partial_v \frac{\zeta}{\nu} = -\frac{\theta}{r} - \frac{\zeta}{\nu}\frac{\partial_v \nu}{\nu} \,, \tag{3.117}$$

we obtain

$$\left|\frac{\tilde{\zeta}}{\tilde{\nu}}(r,v)\right| \leq \frac{C_2}{2}\left(1 + \frac{{r_2}^2}{r^2}\right) \,, \tag{3.118}$$

and the lemma follows. $\square$

We proceed by showing that f is non-increasing in $r \geq r_1$. For that purpose, assume there exist $r_1 < r_2 < r_3$ with $f(r_2) < f(r_3)$. Then, there exists $r_* \in (r_2, r_3]$ such that

$$C_* := \max_{r\in[r_2,r_3]} f(r) = f(r_*) \,. \tag{3.119}$$

Using Lemma 23 with $C_2 = C_*$ we obtain the contradiction

$$C_* = f(r_*) \leq \frac{C_*}{2}\left(1 + \frac{{r_2}^2}{{r_*}^2}\right) < C_* \,, \tag{3.120}$$

and the claimed monotonicity follows.

As a consequence, we can in fact write, for all r and r_2 such that $r_1 \leq r_2 \leq r$,

$$f(r) \leq \frac{f(r_2)}{2}\left(1 + \frac{{r_2}^2}{r^2}\right) \,. \tag{3.121}$$

This can in turn be rearranged to yield

$$f(r) - f(r_2) \leq \frac{f(r_2)}{2}\left(\frac{{r_2}^2}{r^2} - 1\right) \,. \tag{3.122}$$

Dividing by $r - r_2$ gives

$$\frac{f(r) - f(r_2)}{r - r_2} \leq \frac{f(r_2)}{2} \frac{{r_2}^2 - r^2}{r^2(r - r_2)} \ . \tag{3.123}$$

Since f is a Lipschitz-continuous function, for almost every $r_2 \geq r_1$,

$$\begin{aligned} f'(r_2) &\leq \frac{f(r_2)}{2} \lim_{r \to r_2} \frac{{r_2}^2 - r^2}{r^2(r - r_2)} \\ &= -\frac{f(r_2)}{2} \lim_{r \to r_2} \frac{r + r_2}{r^2} \\ &= -\frac{f\,(r_2)}{r_2} \ . \end{aligned} \tag{3.124}$$

Finally, integrating the last inequality we obtain, for all $r_1 \leq r_2 \leq r$,

$$f(r) \leq f(r_2) \exp\left(- \int_{r_2}^{r} \frac{dr'}{r'} \right) = f(r_2) \frac{r_2}{r} \ . \tag{3.125}$$

□

As an immediate consequence we have

Corollary 24. *There exists $r_1 > r_c$ and $C_1 \geq 0$, depending on r_1, such that, in $\mathcal{Q} \cap \{r \geq r_1\}$,*

$$\max \left\{ \left| \frac{\theta}{\lambda} \right| , \left| \frac{\zeta}{\nu} \right| \right\} \leq \frac{C_1}{r} \ . \tag{3.126}$$

3.5 The causal character of $r = \infty$

The estimates of the previous section will now allow us to obtain estimates for all geometric quantities in the region $r \geq r_1$, for $r_1 > r_c$ sufficiently large. Those in turn will give the necessary information to show that $r = \infty$ is, in an appropriate sense, a spacelike surface.

We start by fixing an r_1 as in Corollary 24. Then, by integrating (3.31) from $r = r_1$ while using (3.81) and (3.126) we get

$$\begin{aligned} |\underline{\kappa}(u,v)| &= |\underline{\kappa}(u, v_{r_1}(u))| \exp\left[\int_{v_{r_1}(u)}^{v} \left|\frac{\theta}{\lambda}\right|^2 \frac{\lambda}{r}(u,v')\,dv'\right] \\ &\leq C_{d,r_1} \exp\left[{C_1}^2 \int_{r_1}^{r} (r')^{-3}\,dr'\right] \\ &\leq C_{d,r_1}\,. \end{aligned} \tag{3.127}$$

From Lemma 16 there exists $C > 0$ such that

$$|1-\mu| \leq C\,r^2\,, \tag{3.128}$$

which, together with the previous estimate, immediately gives

$$\nu \leq C_{d,r_1} r^2\,. \tag{3.129}$$

We also have, in view of (3.88) and (3.114),

$$\begin{aligned} \lambda(u,v) &= \lambda(u_{r_1}(v), v) \exp\left[\int_{u_{r_1}(v)}^{u} \frac{\nu\,\partial_r(1-\mu)}{1-\mu}(u',v)du'\right] \\ &\geq \underline{\lambda_0}\left(\frac{r}{r_1}\right)^2\,. \end{aligned} \tag{3.130}$$

From (3.28) we have

$$\varpi(u,v) = \varpi(u, v_{r_1}(u)) + \frac{1}{2}\int_{v_{r_1}(u)}^{v} \left(\frac{\theta}{\lambda}\right)^2 \lambda\,(1-\mu)\,(u,v')\,dv'\,, \tag{3.131}$$

which we can estimate using (3.83), (3.126) and (3.128):

$$\begin{aligned}|\varpi(u,v)| &\leq |\varpi(u,v_{r_1}(u))| + \frac{1}{2}\int_{v_{r_1}(u)}^{v}\left|\frac{\theta}{\lambda}\right|^2 \lambda\,|1-\mu|\,(u,v')dv' \\ &\leq C_{d,r_1} + C_{d,r_1}\int_{v_{r_1}(u)}^{v}\frac{1}{r^2}\lambda r^2(u,v')dv' \\ &\leq C_{d,r_1} r\ .\end{aligned} \tag{3.132}$$

In view of this estimate, we have

$$\frac{\partial_r(1-\mu)}{1-\mu} = \frac{2}{r}\left(1+\frac{\frac{3}{\Lambda r^2}-\frac{9\varpi}{\Lambda r^3}+\frac{6e^2}{\Lambda r^4}}{1-\frac{3}{\Lambda r^2}+\frac{6\varpi}{\Lambda r^3}-\frac{3e^2}{\Lambda r^4}}\right) \leq \frac{2}{r}+\frac{C_{d,r_1}}{r^2}\,, \tag{3.133}$$

for $r \geq r_1$ with r_1 sufficiently large. Using this inequality together with (3.84) yields

$$\begin{aligned}\lambda(u,v) &= \lambda(u_{r_1}(v),v)\ \exp\left[\int_{u_{r_1}(v)}^{u}\frac{\nu\,\partial_r(1-\mu)}{1-\mu}(u',v)du'\right] \\ &= \lambda(u_{r_1}(v),v)\ \exp\left[\int_{r_1}^{r}\frac{\partial_r(1-\mu)}{1-\mu}(r',v)dr'\right] \\ &\leq C_{d,r_1}\exp\left(C_{d,r_1}u_{r_1}(v)\right)\ \exp\left[\log\left(\frac{r^2}{{r_1}^2}\right)+\frac{C_{d,r_1}}{r_1}\right] \\ &\leq C_{d,r_1}\exp\left(C_{d,r_1}u_{r_1}(v)\right)\ r^2 \\ &\leq C_{d,r_1}\exp\left(C_{d,r_1}u\right)\ r^2\,.\end{aligned} \tag{3.134}$$

Next, we consider the level sets of r. Differentiating the identity

$$r = r(u_r(v),v)\ , \tag{3.135}$$

leads to

$$\begin{aligned}0 &= \nu(u_r(v),v)\,u_r'(v) + \lambda(u_r(v),v) \\ \Rightarrow u_r'(v) &= -\frac{\lambda(u_r(v),v)}{\nu(u_r(v),v)}\ .\end{aligned} \tag{3.136}$$

This means that, in view of (3.102), (3.129), (3.130) and (3.134), we have the following result:

Proposition 25. *The curve* $r = \infty$ *is spacelike, in the sense that there exists* $r_1 > r_c$ *and a constant* $C_{d,r_1} > 0$*, depending only on* r_1 *and on the size of the initial data for* $\mathrm{IVP}_{\mathcal{C}^+}$*, such that, for any* $r \geq r_1$*,*

$$- C_{d,r_1} \exp\left(C_{d,r_1} u_{r_1}(v)\right) < u_r'(v) < -C_{d,r_1}^{-1} \ , \tag{3.137}$$

for all $v \in (0, V]$*.*

Remark 26. *Recall from* (3.95) *that* $u_{r_1}(v)$ *blows up as* $v \to 0$*.*

For future reference, we collect some more estimates that can be easily derived from the information gathered until now. From (3.126) and (3.129) we have

$$|\zeta| = \left|\frac{\zeta}{\nu}\right| |\nu| \leq C_{d,r_1} r \ . \tag{3.138}$$

Analogously, (3.126) and (3.134) give

$$|\theta| \leq C_{d,r_1} \exp\left(C_{d,r_1} u\right) r \ . \tag{3.139}$$

Note that, as an immediate consequence of the definitions of ζ and θ, there exists $C > 0$, such that in $\mathcal{Q}$

$$|\partial_u \phi| \leq C \ , \tag{3.140}$$

and

$$|\partial_v \phi| \leq C e^{Cu} \ . \tag{3.141}$$

Remark 27. *This last estimate is somewhat unpleasant and cruder than expected. Presumably one should be able to improve it by using the techniques of [15], but* *this might require imposing some extra decaying conditions on* $\zeta_0(u)$. *We will not pursue this here since* (3.141) *turns out to be sufficient for our goals.*

3.6 Cosmic No-Hair

Let $i = (u_\infty, v_\infty)$ be a point in $\{r = \infty\} \subset \overline{\mathcal{Q}}$. Define a new coordinate $\tilde{u}$ by demanding that

$$\underline{\tilde{\kappa}}(\tilde{u}, v_\infty) \equiv -1 \ , \tag{3.142}$$

where $\underline{\tilde{\kappa}} := \frac{\tilde{\nu}}{1-\mu}$ and $\tilde{\nu} = \partial_{\tilde{u}} r$. Then, $u = u(\tilde{u})$ is such that

$$\frac{du}{d\tilde{u}} = \frac{\tilde{\nu}(\tilde{u}, v_\infty)}{\nu(u(\tilde{u}), v_\infty)} = -\frac{(1-\mu)(\tilde{u}, v_\infty)}{\nu(u(\tilde{u})), v_\infty)} \sim 1 \ . \tag{3.143}$$

The last estimate follows from the results of the previous section and in particular shows that the function $\tilde{u} = \tilde{u}(u)$ has a finite limit when $u \to u_\infty$.

In the region $r \geq r_1$, for r_1 as before, we can define κ by

$$\kappa = \frac{\lambda}{1-\mu} \ . \tag{3.144}$$

Analogously to the definition of $\tilde{u}$, we define $\tilde{v} = \tilde{v}(v)$ by demanding that

$$\tilde{\kappa}(u_\infty, \tilde{v}) \equiv -1 \ , \tag{3.145}$$

which is equivalent to

$$\frac{dv}{d\tilde{v}} = \frac{\tilde{\lambda}(u_\infty, \tilde{v})}{\lambda(u_\infty, v(\tilde{v}))} = -\frac{(1-\mu)(u_\infty, \tilde{v})}{\lambda(u_\infty, v(\tilde{v}))} \sim 1 \ . \tag{3.146}$$

We can now shift our coordinates so that $i = (\tilde{u} = 0, \tilde{v} = 0)$.

In these new coordinates the spacetime metric,

$$g = -\tilde{\Omega}^2(\tilde{u},\tilde{v})d\tilde{u}d\tilde{v} + r^2(\tilde{u},\tilde{v})\mathring{g} \ , \tag{3.147}$$

satisfies

$$\tilde{\Omega}^2(\tilde{u},\tilde{v}) = -4\tilde{\kappa}\underline{\tilde{\kappa}}(1-\mu)(\tilde{u},\tilde{v}) \ . \tag{3.148}$$

Now consider a point i_{dS} in the future null infinity of the pure de Sitter spacetime with cosmological constant Λ. We can cover a neighborhood of the past of this point by null coordinates for which $i_{dS} = (\tilde{u} = 0, \tilde{v} = 0)$ and the metric takes the form

$$^{dS}g = -\Omega_{dS}^2(\tilde{u},\tilde{v})d\tilde{u}d\tilde{v} + r_{dS}^2(\tilde{u},\tilde{v})\mathring{g} \ , \tag{3.149}$$

with

$$\tilde{\Omega}_{dS}^2(\tilde{u},\tilde{v}) = -4\left(1 - \frac{\Lambda}{3}r_{dS}^2(\tilde{u},\tilde{v})\right) \ . \tag{3.150}$$

We identify $J^-(i)\cap\{r \geq r_1\} \subset \{(\tilde{u},\tilde{v}) \, : \, \tilde{u} \leq 0 \, , \tilde{v} \leq 0)\}\setminus\{(0,0)\}$, in our dynamic solution $(\mathcal{Q}, g)$, with the region of de Sitter space with the same range of the $(\tilde{u},\tilde{v})$ coordinates.

The first goal of this section is to show that the components of (the dynamic spacetime metric) g approach those of the de Sitter metric ^{dS}g in $J^-(i)\cap\{r \geq r_2\}$ as $r_2 \to \infty$. With that in mind, we start by establishing the following

Lemma 28. *In $J^-(i)\cap\{r \geq r_2\}$, for sufficiently large r_2,*

$$-1 \leq \underline{\tilde{\kappa}} \, , \tilde{\kappa} \leq -1 + \frac{C}{{r_2}^2} \ . \tag{3.151}$$

Proof. Concerning $\underline{\tilde{\kappa}}$, the estimate from below is an immediate consequence of the fact that $\partial_{\tilde{v}}\underline{\tilde{\kappa}} \leq 0$ (see (3.31)). To establish the other inequality, recall Theorem 22, and note that

$$\left|\frac{\tilde{\theta}}{\tilde{\lambda}}\right|(\tilde{u},\tilde{v}) \leq f(r(\tilde{u},\tilde{v})) \leq f(r_2)\frac{r_2}{r(\tilde{u},\tilde{v})}\ , \tag{3.152}$$

with

$$f(r_2) \leq f(r_1)r_1\frac{1}{r_2} =: C_1\frac{1}{r_2}\ . \tag{3.153}$$

Then, for $\tilde{v} \leq 0$ we get

$$\begin{aligned}
\underline{\tilde{\kappa}}(\tilde{u},\tilde{v}) &= \underline{\tilde{\kappa}}(\tilde{u},0)\exp\left(-\int_{\tilde{v}}^{0}\left|\frac{\tilde{\theta}}{\tilde{\lambda}}\right|^2\frac{\tilde{\lambda}}{r}(\tilde{u},v')\,dv'\right)\\
&\leq -\exp\left(-f^2(r_2){r_2}^2\int_{\tilde{v}}^{0}\frac{\tilde{\lambda}}{r^3}(\tilde{u},v')\,dv'\right)\\
&\leq -\exp\left(-f^2(r_2)\frac{{r_2}^2}{2}\left(\frac{1}{r^2(\tilde{u},\tilde{v})}-\frac{1}{r^2(\tilde{u},0)}\right)\right)\\
&\leq -\exp\left(-f^2(r_2)\right)\\
&\leq -\exp\left(-\frac{C}{{r_2}^2}\right)\\
&\leq -1+\frac{C}{{r_2}^2}\ .
\end{aligned} \tag{3.154}$$

The results concerning $\tilde{\kappa}$ follow in a dual manner by using the equation

$$\partial_{\tilde{u}}\tilde{\kappa} = \tilde{\kappa}\left(\frac{\tilde{\zeta}}{\tilde{\nu}}\right)^2\frac{\tilde{\nu}}{r}\ . \tag{3.155}$$

□

As an immediate consequence, in $J^-(i)\cap\{r\geq r_2\}$,

$$-\left(1-\frac{C}{{r_2}^2}\right)(1-\mu)\leq\tilde\lambda,\tilde\nu\leq-(1-\mu)\ . \tag{3.156}$$

Since, in view of (3.132), we have

$$-(1-\mu)\leq C+\frac{\Lambda}{3}r^2=\left(1+\frac{3C}{\Lambda r^2}\right)\frac{\Lambda}{3}r^2\leq\left(1+\frac{C}{{r_2}^2}\right)\frac{\Lambda}{3}r^2 \tag{3.157}$$

and, similarly,

$$-(1-\mu)\geq\left(1-\frac{C}{{r_2}^2}\right)\frac{\Lambda}{3}r^2\ , \tag{3.158}$$

estimates (3.156) give rise to

$$\left(1-\frac{C}{{r_2}^2}\right)\frac{\Lambda}{3}r^2(\tilde u,\tilde v)\leq\tilde\lambda,\tilde\nu\leq\left(1+\frac{C}{{r_2}^2}\right)\frac{\Lambda}{3}r^2(\tilde u,\tilde v)\ , \tag{3.159}$$

for sufficiently large r_2 and $r>r_2$.

For the de Sitter spacetime we have

$$\left(1-\frac{C}{{r_2}^2}\right)\frac{\Lambda}{3}r^2_{dS}(\tilde u,\tilde v)\leq\tilde\lambda_{dS},\tilde\nu_{dS}\leq\frac{\Lambda}{3}r^2_{dS}(\tilde u,\tilde v)\ . \tag{3.160}$$

Lemma 29. *Let A,B be positive constants and $r=r(u,v)$ a function in $J^-(0,0)=\{(u,v)\,:\,u\leq 0\ ,v\leq 0)\}\setminus\{(0,0)\}$ satisfying*

$$Br^2(u,v)\leq\partial_v r,\partial_u r\leq Ar^2(u,v)\ , \tag{3.161}$$

and

$$r(x,0)\to\infty\ \ ,\ \ r(0,x)\to\infty,\quad \textit{as } x\to 0\ . \tag{3.162}$$

Then, in $J^-(0,0)$,

$$-\frac{1}{A(u+v)} \leq r(u,v) \leq -\frac{1}{B(u+v)} \ . \tag{3.163}$$

Proof. We have

$$\begin{aligned}\int_u^0 \frac{1}{r^2(u',v)}\frac{\partial r}{\partial u}(u',v)du' \leq \int_u^0 A du' \Rightarrow \int_{r(u,v)}^{r(0,v)} \frac{1}{r^2}dr \leq -Au \\ \Rightarrow \frac{1}{r(u,v)} - \frac{1}{r(0,v)} \leq -Au\end{aligned} \tag{3.164}$$

and

$$\begin{aligned}\int_v^{v_1} \frac{1}{r^2(0,v')}\frac{\partial r}{\partial v}(0,v')dv' \leq \int_v^{v_1} A dv' \Rightarrow \int_{r(0,v)}^{r(0,v_1)} \frac{1}{r^2}dr \leq A(v_1-v) \\ \Rightarrow \frac{1}{r(0,v)} - \frac{1}{r(0,v_1)} \leq A(v_1-v) \\ \Rightarrow \frac{1}{r(0,v)} \leq -Av \ ,\end{aligned} \tag{3.165}$$

where the last implication follows by taking the limit $v_1 \to 0$. Then,

$$\frac{1}{r(u,v)} \leq -Au + \frac{1}{r(0,v)} \leq -A(u+v) \ , \tag{3.166}$$

and the result follows. The other bound is similar. □

In view of (3.159), (3.160) and the previous lemma, we conclude that, in $J^-(i) \cap \{r \geq r_2\}$,

$$|r(\tilde{u},\tilde{v}) - r_{dS}(\tilde{u},\tilde{v})| \leq -\frac{C}{{r_2}^2}\frac{1}{\tilde{u}+\tilde{v}} \leq \frac{C}{{r_2}^2}\, r_{dS}(\tilde{u},\tilde{v}) \ . \tag{3.167}$$

Then it is clear that

$$\begin{aligned}\frac{|r^2(\tilde{u},\tilde{v}) - r_{dS}^2(\tilde{u},\tilde{v})|}{r_{dS}^2(\tilde{u},\tilde{v})} &\leq \frac{C}{{r_2}^2}\frac{1}{r_{dS}(\tilde{u},\tilde{v})}(r(\tilde{u},\tilde{v}) + r_{dS}(\tilde{u},\tilde{v})) \\ &\leq \frac{C}{{r_2}^2}\left(\frac{r}{r_{dS}}(\tilde{u},\tilde{v}) + 1\right) \\ &\leq \frac{C}{{r_2}^2} \; .\end{aligned} \tag{3.168}$$

Recalling (3.148) and (3.150), we obtain, arguing as before,

$$\left(1 - \frac{C}{{r_2}^2}\right)\frac{4\Lambda}{3}r^2(\tilde{u},\tilde{v}) \leq \tilde{\Omega}^2(\tilde{u},\tilde{v}) \leq \left(1 + \frac{C}{{r_2}^2}\right)\frac{4\Lambda}{3}r^2(\tilde{u},\tilde{v}) \; , \tag{3.169}$$

from which we see that

$$\frac{|\tilde{\Omega}^2(\tilde{u},\tilde{v}) - \tilde{\Omega}_{dS}^2(\tilde{u},\tilde{v})|}{\tilde{\Omega}_{dS}^2(\tilde{u},\tilde{v})} \leq \frac{C}{{r_2}^2} \; . \tag{3.170}$$

Let us now rewrite these last estimates in terms of an orthonormal frame $\{e_a, e_A\}_{a\in\{0,1\},A\in\{3,4\}}$ of de Sitter, where $\{e_A\}$ is an orthonormal frame of $r_{dS}^2\mathring{g}$ and

$$e_0 = \tilde{\Omega}_{dS}^{-1}(\partial_{\tilde{u}} + \partial_{\tilde{v}}) \; , \tag{3.171}$$

$$e_1 = -\tilde{\Omega}_{dS}^{-1}(\partial_{\tilde{u}} - \partial_{\tilde{v}}) \; . \tag{3.172}$$

In the frame above the dynamic metric reads

$$g_{ab} = \frac{\tilde{\Omega}^2}{\tilde{\Omega}_{dS}^2}\eta_{ab} \tag{3.173}$$

and

$$g_{AB} = \frac{r^2}{r_{dS}^2}\delta_{AB} \; . \tag{3.174}$$

So we see that, in this frame, (3.168) and (3.170) imply that

$$\sup_{J^-(i)\cap\{r\geq r_2\}} |g_{\mu\nu} - {}^{dS}g_{\mu\nu}| \lesssim r_2^{-2} \ . \tag{3.175}$$

Let us now consider the Riemann curvature, whose components satisfy [18][Appendix A]

$$\begin{aligned} R^a_{bcd} &= K\left(\delta^a_c g_{bd} - \delta^a_d g_{bc}\right) \ , &(3.176)\\ R^a_{BcD} &= -r^{-1}\nabla^a\nabla_c r g_{BD} \ , &(3.177)\\ R^A_{BCD} &= r^{-2}\left(1-\partial_a r\partial^a r\right)\left(\delta^A_C g_{BD} - \delta^A_D g_{BC}\right) \ . &(3.178)\end{aligned}$$

The remaining components are zero. Also, the Gaussian curvature of the surfaces of fixed angular coordinates is given by [18][Appendix A]

$$K = 4\tilde{\Omega}^{-2}\partial_{\tilde{u}}\partial_{\tilde{v}}\log\tilde{\Omega} \ . \tag{3.179}$$

Recalling (3.16) and using (3.140), (3.141) and (3.169), we first obtain

$$K = 4\tilde{\Omega}^{-2}\left(O(1) + \frac{\tilde{\lambda}\tilde{\nu}}{r^2}\right) \ , \tag{3.180}$$

and then, using (3.159) and (3.169) again, we get, for $r > r_2$,

$$\left(1 - \frac{C}{{r_2}^2}\right)\frac{\Lambda}{3} \leq K \leq \left(1 + \frac{C}{{r_2}^2}\right)\frac{\Lambda}{3} \ . \tag{3.181}$$

The previous expression is valid in both spacetimes, consequently,

$$|K - K_{dS}| \lesssim r_2^{-2} \ . \tag{3.182}$$

In the $\{e_\mu\}$ frame we then have

$$\begin{aligned}\left|R^a_{bcd} - {}^{dS}R^a_{bcd}\right| &= \left|\frac{\tilde\Omega^2}{\tilde\Omega^2_{dS}}K - K_{dS}\right| |\delta^a_c\eta_{bd} - \delta^a_d\eta_{bc}| \\ &\lesssim \left|\frac{(\tilde\Omega^2 - \tilde\Omega^2_{dS})}{\tilde\Omega^2_{dS}}K\right| + |K - K_{dS}| \\ &\lesssim r_2^{-2} \,.\end{aligned} \tag{3.183}$$

The Hessian of the radius function is [18][Appendix A][8]

$$\nabla_a\nabla_b r = \frac{1}{2r}\left(1 - \partial_c r\partial^c r\right) g_{ab} - r(T_{ab} - \operatorname{tr} T g_{ab}) - \frac{1}{2}r\Lambda g_{ab} \,, \tag{3.184}$$

where $\operatorname{tr} T = g^{ab}T_{ab}$. We have

$$\partial_c r\partial^c r = 1 - \mu \,, \tag{3.185}$$

and, in view of (3.140) and (3.141),

$$|e_a\phi| \lesssim r_{dS}^{-1} \,. \tag{3.186}$$

Using this inequality, we can estimate the scalar field part of the energy momentum tensor by

$$|(e_a\phi)(e_b\phi) + 2(\partial_{\tilde u}\phi)(\partial_{\tilde v}\phi)\Omega^{-2}_{dS}\eta_{ab}| \lesssim r_{dS}^{-2} \,, \tag{3.187}$$

and the electromagnetic part is, in view of (3.9) and (3.13), bounded by Cr^{-4}. Consequently,

$$|T_{ab}| \lesssim r^{-4} + r_{dS}^{-2} \,. \tag{3.188}$$

[8]There is a difference in a factor of 4π due to our different convention for the value of Newton's gravitational constant.

Since $\operatorname{tr} T = 0$, we get

$$|\nabla_a \nabla_b r| \lesssim r \ , \tag{3.189}$$

and

$$\begin{aligned}\left|\nabla_a \nabla_b r - {}^{dS}\nabla^{dS}_a \nabla_b r_{dS}\right| &\leq \frac{\Lambda}{6}\left[\left|\frac{\tilde{\Omega}^2}{\tilde{\Omega}^2_{dS}} r - r_{dS}\right| + \frac{C}{{r_2}^2}\left(\frac{\tilde{\Omega}^2}{\tilde{\Omega}^2_{dS}} r + r_{dS}\right)\right] \\ &\quad + Cr\left|r^{-4} + r_{dS}^{-2}\right| + \frac{\Lambda}{2}\left|\frac{\tilde{\Omega}^2}{\tilde{\Omega}^2_{dS}} r - r_{dS}\right| \\ &\lesssim \frac{r + r_{dS}}{{r_2}^2} \ ,\end{aligned} \tag{3.190}$$

where we used the fact that $r, r_{r_{dS}} \gtrsim r_2$, together with (3.167) and (3.170).

We are now ready to estimate

$$\begin{aligned}\left|R^a_{BcD} - {}^{dS}R^a_{BcD}\right| &= \left|\left(r_{dS}^{-1} \ {}^{dS}\nabla^{dS}_a \nabla_c r_{dS} - r^{-1}\frac{\tilde{\Omega}^2_{dS}}{\tilde{\Omega}^2}\frac{r^2}{r^2_{dS}} \nabla_a \nabla_c r\right)\delta_{BD}\right| \\ &\leq \frac{1}{r_{dS}}\left|{}^{dS}\nabla^{dS}_a \nabla_c r_{dS} - \nabla_a \nabla_c r + \left(1 - \frac{\tilde{\Omega}^2_{dS}}{\tilde{\Omega}^2}\frac{r}{r_{dS}}\right)\nabla_a \nabla_c r\right| \\ &\lesssim \frac{1}{r_{dS}}\left[\frac{r + r_{r_{dS}}}{{r_2}^2} + \frac{r}{{r_2}^2}\right] \\ &\lesssim r_2^{-2} \ .\end{aligned} \tag{3.191}$$

The previous estimates and a similar reasoning also allow us to conclude that

$$|R^A_{BCD} - {}^{dS}R^A_{BCD}| \lesssim r_2^{-2}. \tag{3.192}$$

3.7 Proofs of Theorems 9 and 10

We can now easily prove Theorem 9:

Proof. We can use the existing gauge freedom $u \mapsto f(u)$ to impose the condition $\underline{\kappa}_0 := -\frac{1}{4}\Omega^2(\partial_v r)^{-1}|_{v=0} \equiv -1$. Then there exist $U, V > 0$ such that the conditions of IVP$_{\mathcal{C}^+}$of Subsection 3.1.3 are satisfied on $[U, +\infty[\, \times \{0\} \cup \{0\} \times [0, V]$ (in Subsection 3.1.3 we translate u so that $U = 0$): the regularity conditions are a consequence of the smoothness of the Cauchy initial data; the sign conditions follow by continuity; the constraints from the fact that $(\mathcal{M}, g, F, \phi)$ is a solution of the Einstein-Maxwell-scalar field system; and the subextremality conditions from the fact that the data is converging to the reference subextremal RNdS.

Then the results *1-4* are just a compilation of the results in Theorem 21, Corollary 24, Proposition 25 and Section 3.6. □

We end with the proof of Theorem 10:

Proof. Conditions *i'-v'* together with assumption I were tailored to ensure that IVP$_{\mathcal{C}^+}$ holds in $[0, +\infty[\times\{0\}\cup\{0\}\times[0, V]$, for all $V \geq 0$, and that the dual initial value problem, obtained by interchanging u and v, holds in $[0, U] \times \{0\} \cup \{0\} \times [0, \infty)$, for all $U > 0$. Then, as in the proof of Theorem 9, the results are valid in $\mathcal{Q} = \mathcal{Q}_1 \cup \mathcal{Q}_2$, where $\mathcal{Q}_1 = \tilde{\mathcal{Q}} \cap [0, \infty) \times [0, V)$ and $\mathcal{Q}_2 = \tilde{\mathcal{Q}} \cap [0, U) \times [0, \infty)$. Choosing U and V sufficiently large we can ensure that $\mathcal{Q} = \tilde{\mathcal{Q}} \cap [0, \infty)^2$, as desired.

Conditions *i'-v'* allow us to invoke conclusion *3* of Theorem 9 to conclude that, under assumption *II* we have that $\mathcal{N}_1$ is in fact empty; by the interchange symmetry between u and v, we also conclude that $\mathcal{N}_2 = \varnothing$. Then, the future boundary of $\mathcal{Q} = \tilde{\mathcal{Q}} \cap [0, \infty)^2$ is only composed of $\mathcal{B}_\infty$, so that conclusion *1* holds throughout and the remaining ones are valid near the points (∞, v_∞) and (u_∞, ∞). Now, Raychaudhuri's equations imply that in $D^-(\mathcal{B}_\infty) = \mathcal{Q}$ we have $\lambda > 0$, $\nu > 0$ and, consequently $1 - \mu < 0$. From the previous signs we see that the conclusions of Lemma 16 hold and hence (3.113) is valid in $\mathcal{Q} \cap \{r \geq r_1\}$, for a sufficiently large r_1. To see that the conclusions of Section 3.4 then hold in $\mathcal{Q}$ we just need to note that the function f defined in that section remains well-defined in the new ("larger") $\mathcal{Q}$; but this follows from compactness, after noting that the curve of constant r where one takes the supremum is the union of a subset near (∞, v_∞), a subset near (u_∞, ∞), and a compact subset connecting these two. This shows that *2* holds in $\mathcal{Q}$. The remaining conclusions now follow, since they are rooted in *2* and in the fact that all relevant quantities have appropriate bounds along $r = r_1$, which follows by a compactness argument as before. □

4

Conclusions and Future Work

Contents

Finally, we can summarize the progress achieved during the course of the research work made towards this thesis and point a few possibilities for how to proceed next.

4.1 Overview of achieved results

By modeling the accelerated expansion of the Universe with the addition of a positive cosmological constant to the Einstein field equations, our ultimate goal was the establishing of a cosmic no-hair theorem in spherical symmetry and in the presence of a cosmological charged black hole.

For this purpose, we started by studying the decay of solutions of the wave equation in some expanding cosmological spacetimes, namely flat Friedmann-Lemaître-Robertson-Walker (FLRW) models and the cosmological region of the Reissner-Nordström-de Sitter (RNdS) solution, as a proxy to the full system including the Einstein equations. By introducing a partial energy and using an iteration scheme, we found that, for initial data with finite higher order energies, the decay rate of the time derivative is faster than previously existing estimates, and that, for models undergoing accelerated expansion, our decay rate appears to be (almost) sharp (see Appendix A for FLRW and Appendix B and [51] for RNdS).

Next, we analyzed in detail the geometry and dynamics of the cosmological region arising in spherically symmetric black hole solutions of the Einstein-Maxwell-scalar field system with a positive cosmological constant. More precisely, we solved, for such a system, a characteristic initial value problem with data emulating a dynamic cosmological horizon. Our assumptions were fairly weak, in the sense that we only required the data to approach that of a subextremal Reissner-Nordström-de Sitter black hole, without imposing any rate of decay. We then showed that the radius (of symmetry) blows up along any null ray parallel to the cosmological horizon ("near" i^+), in such a way that $r = +\infty$ is, in an appropriate sense, a spacelike hypersurface. In the end, we succeeded in proving that a cosmic no-hair scenario does actually occur in this setting, by showing that in the past of any causal curve reaching infinity both the metric and the Riemann curvature tensor asymptote those of a de Sitter spacetime.

4.2 Ideas for future work

The new results presented above are naturally associated with several avenues for future research, some of which will be sketched in what follows.

Regarding the decay of the wave equation in expanding cosmological spacetimes, and in the spirit of what was carried out in Chapter 2 and presented in article [17], one can, for example, again study the decay of solutions but for different cosmological models instead, as well as adapt the developed techniques to other types of equations, such as Maxwell's equations or the linearized Einstein's equations, or even generalize them to the Klein-Gordon equation and to $f(R)$ theories with the methods of Philippe G. LeFloch and Yue Ma [37–43]. Moreover, one might also try to combine the aforementioned techniques with the vector field methods employed by Sergiu Klainerman, Mihalis Dafermos, and Igor Rodnianski, which can be found for example in [19], in order to improve the decay rates when the spatial sections are not compact (to illustrate this, consider how in the case of Minkowski our techniques do not allow us to obtain decay, even though it is known that solutions to the wave equation decay with $t^{-\frac{n-1}{2}}$).

As for the cosmic no-hair conjecture in the case described in Chapter 3 and set out in article [16], the immediately obvious possibility for future research is to try to find conditions under which mass inflation occurs generically; it is to be noted that thus far we managed to establish that the absolute value of the Hawking mass grows at most linearly with the radius, but not yet that it does grow to infinity (presumably at a linear rate). The ideas enumerated in the previous paragraph could also provide inspiration to explore cosmic no-hair-related phenomena for the other scenarios, such as, for instance, the Einstein-Klein-Gordon problem. Lastly, the natural final goal of this type of venture, and also a problem currently being tackled by Volker Schlue [57] concerning the stability of the Schwarzschild-de Sitter spacetime's cosmological region, should be to drop the requirement of spherical symmetry from the hypotheses entirely and obtain the respective cosmic no-hair theorems.

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A

Fourier modes in Friedmann-Lemaître-Robertson-Walker spacetimes

It should be clear from the proof of Theorem 1 that this result also holds for expanding flat Friedmann-Lemaître-Robertson-Walker models with toroidal spatial sections; this shows, in particular, that the underlying decay mechanism must be the cosmological expansion, as opposed to dispersion. In the toroidal case, the wave equation can be studied by performing a Fourier mode analysis, which gives valuable information about how sharp our estimates are.

Taking, for simplicity, $\mathbb{T}^n = \mathbb{R}^n/(2\pi\mathbb{Z})^n$, we can expand any smooth function $\phi : \mathbb{R} \times \mathbb{T}^n \to \mathbb{R}$ as

$$\phi(t,x) = \sum_{k\in\mathbb{Z}^n} c_k(t) e^{i\langle k,x\rangle} \,. \tag{A.1}$$

Substituting in (2.14) we obtain

$$\ddot{c}_k + \frac{n\dot{a}}{a}\dot{c}_k + \frac{k^2}{a^2}c_k = 0 \,, \tag{A.2}$$

or, equivalently,

$$\frac{d}{dt}\left(a^n \dot{c}_k\right) + k^2 a^{n-2} c_k = 0 \,. \tag{A.3}$$

If we change the independent variable to

$$\tau = \int \frac{dt}{a(t)} \,, \tag{A.4}$$

so that

$$\frac{d}{dt} = \frac{1}{a}\frac{d}{d\tau} \,, \tag{A.5}$$

equation (A.2) becomes

$$\left(a^{n-1} c_k'\right)' + k^2 a^{n-1} c_k = 0 \,, \tag{A.6}$$

where the prime denotes differentiation with respect to τ. Setting

$$c_k = a^{-\frac{n-1}{2}} d_k \,, \tag{A.7}$$

so that

$$c_k' = a^{-\frac{n-1}{2}} d_k' - \frac{(n-1)}{2} a^{-\frac{n-1}{2}-1} a' d_k \,, \tag{A.8}$$

we obtain

$$\left(a^{\frac{n-1}{2}} d_k' - \frac{(n-1)}{2} a^{\frac{n-1}{2}-1} a' d_k \right)' + k^2 a^{\frac{n-1}{2}} d_k = 0 \,, \tag{A.9}$$

or, equivalently,

$$d_k'' + \left[k^2 - \frac{(n-1)}{2} \frac{a''}{a} - \frac{(n-1)(n-3)}{4} \left(\frac{a'}{a} \right)^2 \right] d_k = 0 \,. \tag{A.10}$$

If $a(t) = t^p$, then

$$\tau = \int \frac{dt}{t^p} = \frac{t^{1-p}}{1-p} \Leftrightarrow t = [(1-p)\tau]^{\frac{1}{1-p}} \,, \tag{A.11}$$

whence

$$a = [(1-p)\tau]^{\frac{p}{1-p}} \,, \tag{A.12}$$

thus implying

$$a' = p \, [(1-p)\tau]^{\frac{p}{1-p}-1} \tag{A.13}$$

and

$$a'' = p(2p-1) \, [(1-p)\tau]^{\frac{p}{1-p}-2} \,. \tag{A.14}$$

We conclude that equation (A.10) can be written as

$$d_k'' + \left(k^2 - \frac{\mu}{\tau^2} \right) d_k = 0 \,, \tag{A.15}$$

where

$$\mu = \frac{(n-1)p(2p-1)}{2(1-p)^2} + \frac{(n-1)(n-3)p^2}{4(1-p)^2}\,. \tag{A.16}$$

The general solution of equation (A.15) is

$$d_k(\tau) = C_1\sqrt{\tau}J_\nu(k\tau) + C_2\sqrt{\tau}Y_\nu(k\tau)\,, \tag{A.17}$$

where $k = |k|$,

$$J_\alpha(z) = \sum_{m=1}^{+\infty} \frac{(-1)^m}{m!\,\Gamma(m+\alpha+1)}\left(\frac{z}{2}\right)^{2m+\alpha} \tag{A.18}$$

is the Bessel function of the first kind,

$$Y_\alpha(z) = \frac{J_\alpha(z)\cos(\alpha\pi) - J_{-\alpha}(z)}{\sin(\alpha\pi)} \tag{A.19}$$

is the Bessel function of the second kind, and

$$\nu^2 = \frac{1}{4} + \mu = \frac{(np-1)^2}{4(1-p)^2} \geq 0\,. \tag{A.20}$$

For $p > 1$ we have

$$d_k \sim C_1\tau^{\frac{1}{2}-\nu} + C_2\tau^{\frac{5}{2}-\nu} \tag{A.21}$$

as $t \to +\infty \Leftrightarrow \tau \to 0^-$ (note that $\nu > \frac{n}{2}$ for $p > 1$), whence

$$d_k \sim C_1 t^{(1-p)(\frac{1}{2}-\nu)} + C_2 t^{(1-p)(\frac{5}{2}-\nu)}\,. \tag{A.22}$$

This leads to

$$c_k \sim C_1 t^{-\frac{n-1}{2}p+(1-p)(\frac{1}{2}-\nu)} + C_2 t^{-\frac{n-1}{2}p+(1-p)(\frac{5}{2}-\nu)}\,, \tag{A.23}$$

that is,

$$c_k \sim C_1 + C_2 t^{-2p+2}\,, \tag{A.24}$$

implying in particular that

$$|\dot{c}_k| \lesssim t^{-2p+1}\,. \tag{A.25}$$

Comparing with (2.7), we see that (2.4) is (almost) sharp in the case $a(t) = t^p$ with $p > 1$.

For $p < 1$ we have[1]

$$d_k \sim C_1 \cos\left(k\tau - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) + C_2 \cos\left(k\tau + \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \tag{A.26}$$

as $t \to +\infty \Leftrightarrow \tau \to +\infty$, whence

$$d_k \sim C_1 \cos\left(\frac{kt^{1-p}}{1-p} - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) + C_2 \cos\left(\frac{kt^{1-p}}{1-p} + \frac{\nu\pi}{2} - \frac{\pi}{4}\right)\,. \tag{A.27}$$

This leads to

$$c_k \sim C_1 t^{-\frac{n-1}{2}p} \cos\left(\frac{kt^{1-p}}{1-p} - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) + C_2 t^{-\frac{n-1}{2}p} \cos\left(\frac{kt^{1-p}}{1-p} + \frac{\nu\pi}{2} - \frac{\pi}{4}\right)\,, \tag{A.28}$$

implying in particular that

$$|\dot{c}_k| \lesssim t^{-\frac{n+1}{2}p}\,. \tag{A.29}$$

Comparing with (2.7), we see that (2.4) is very far from sharp in the case $a(t) = t^p$

[1]More precisely, as $z \to \infty$ we have the asymptotic formulae

$$J_\nu(z) \sim \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right), \qquad Y_\nu(z) \sim \sqrt{\frac{2}{\pi z}} \sin\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right),$$

which hold up to the first derivative, as can be seen from the identities

$$\frac{dJ_\nu}{dz}(z) = \frac{1}{2}J_{\nu-1}(z) - \frac{1}{2}J_{\nu+1}(z)\,, \qquad \frac{dY_\nu}{dz}(z) = \frac{1}{2}Y_{\nu-1}(z) - \frac{1}{2}Y_{\nu+1}(z)\,.$$

with $p < 1$. Note that in this case we do obtain the exponent coming from the naïve physical argument governing the decay of the Fourier modes.

If $a(t) = e^t$, then[2]

$$\tau = \int \frac{dt}{e^t} = -e^{-t} \Leftrightarrow t = -\log(-\tau)\,, \tag{A.30}$$

whence

$$a = -\frac{1}{\tau}\,, \tag{A.31}$$

implying

$$a' = \frac{1}{\tau^2} \tag{A.32}$$

and

$$a'' = -\frac{2}{\tau^3}\,. \tag{A.33}$$

We conclude that equation (A.10) can be written as

$$d_k'' + \left(k^2 - \frac{\mu}{\tau^2}\right) d_k = 0\,, \tag{A.34}$$

where

$$\mu = n - 1 + \frac{(n-1)(n-3)}{4} \tag{A.35}$$

(the limit of the value in the $a(t) = t^p$ case as $p \to +\infty$). The general solution of equation (A.34) is

$$d_k(\tau) = C_1\sqrt{\tau}J_\nu(k\tau) + C_2\sqrt{\tau}Y_\nu(k\tau)\,, \tag{A.36}$$

where

$$\nu^2 = \frac{1}{4} + \mu = \frac{n^2}{4}\,. \tag{A.37}$$

[2]Note that we can always set $H = 1$ by choosing units such that $\Lambda = \frac{1}{2}n(n-1)$.

We have

$$d_k \sim C_1 \tau^{\frac{1}{2}-\nu} + C_2 \tau^{\frac{5}{2}-\nu} \tag{A.38}$$

as $t \to +\infty \Leftrightarrow \tau \to 0^-$, whence

$$d_k \sim C_1 e^{(\nu-\frac{1}{2})t} + C_2 e^{(\nu-\frac{5}{2})t} \,. \tag{A.39}$$

This leads to

$$c_k \sim C_1 e^{-\frac{n-1}{2}t+(\nu-\frac{1}{2})t} + C_2 e^{-\frac{n-1}{2}t+(\nu-\frac{5}{2})t} \,, \tag{A.40}$$

that is,

$$c_k \sim C_1 + C_2 e^{-2t} \,, \tag{A.41}$$

implying in particular that

$$|\dot{c}_k| \lesssim e^{-2t} \,. \tag{A.42}$$

Comparing with (2.6), we see that (2.4) is (almost) sharp in the case $a(t) = e^t$.

B

Fourier modes in the Reissner-Nordström-de Sitter spacetime

A Fourier component analysis along the lines of the one performed in Appendix A can also be made for the Reissner-Nordström-de Sitter case, which in particular is useful to infer what kind of decay estimates one should expect in Theorem 5 for $\partial_r \phi$ when $r \to +\infty$. In other words, we are interested in understanding the behavior of ϕ when r is arbitrarily large.

For these purposes, we start by writing the metric – we will assume three spatial dimensions, seeing as how the end result regarding the observed decay rate is pretty much the same –, which takes the form

$$
\begin{aligned}
g &= -(1-\mu)\, dt^2 + \frac{dr^2}{1-\mu} + r^2 \left(d\theta^2 + \sin^2\theta\, d\varphi^2\right) \\
&= -(1-\mu) \left(dt^2 - \frac{dr^2}{(1-\mu)^2}\right) + r^2 \left(d\theta^2 + \sin^2\theta\, d\varphi^2\right) .
\end{aligned}
\tag{B.1}
$$

It is useful to work with the "tortoise" coordinate,

$$
r^*(r) := \int_r^{+\infty} \frac{dr'}{1-\mu(r')} . \tag{B.2}
$$

Using the coordinate change (B.2), the metric (B.1) can then be rewritten as

$$
g = -(1-\mu)\left(dt^2 - (dr^*)^2\right) + r^2 \left(d\theta^2 + \sin^2\theta\, d\varphi^2\right) . \tag{B.3}
$$

In this frame, and in the cosmological region (where $1 - \mu < 0$),

$$\sqrt{|g|} = -(1-\mu)\, r^2 \sin\theta\,. \tag{B.4}$$

Let us now write the wave equation in the cosmological region and in spherical symmetry:

$$\begin{aligned}
\Box_g \phi = 0 \;\Leftrightarrow\; & \frac{1}{\sqrt{|g|}}\,\partial_\alpha\left(g^{\alpha\beta}\sqrt{|g|}\,\partial_\beta\phi\right) = 0 \\
\Leftrightarrow\; & \partial_\alpha\left(g^{\alpha\beta}\,(-(1-\mu))\, r^2 \sin\theta\,\partial_\beta\phi\right) = 0 \\
\Leftrightarrow\; & \partial_t\left(g^{tt}\,(-(1-\mu))\, r^2 \sin\theta\,\partial_t\phi\right) + \partial_{r^*}\left(g^{r^*r^*}\,(-(1-\mu))\, r^2 \sin\theta\,\partial_{r^*}\phi\right) = 0 \\
\Leftrightarrow\; & \partial_t\left(-\frac{1}{1-\mu}\,(-(1-\mu))\, r^2\,\partial_t\phi\right) + \partial_{r^*}\left(\frac{1}{1-\mu}\,(-(1-\mu))\, r^2\,\partial_{r^*}\phi\right) = 0 \\
\Leftrightarrow\; & r^2\,\partial_t^2\phi - \partial_{r^*}\left(r^2\,\partial_{r^*}\phi\right) = 0 \\
\Leftrightarrow\; & \partial_t^2\phi - \partial_{r^*}^2\phi - 2\,\frac{\partial_{r^*} r}{r}\,\partial_{r^*}\phi = 0\,.
\end{aligned} \tag{B.5}$$

Using the notation

$$\dot{} \equiv \frac{\partial}{\partial t}\,, \tag{B.6}$$

$$' \equiv \frac{\partial}{\partial r^*}\,, \tag{B.7}$$

equation (B.5) becomes

$$\ddot{\phi} - \phi'' - 2\frac{r'}{r}\phi' = 0\,. \tag{B.8}$$

Now recall that $r \to +\infty$. Under these circumstances, (B.2) gives

$$r^*(r) \approx \int_r^{+\infty} \frac{dr}{-r^2} = -\frac{1}{r}\,. \tag{B.9}$$

Also,

$$\frac{dr^*}{dr}(r) = -\frac{1}{1-\mu(r)}\,, \tag{B.10}$$

and

$$r'(r^*) = \frac{dr}{dr^*}(r^*) = -\left(1-\mu\left(r\left(r^*\right)\right)\right) \approx r^2(r^*) \approx \frac{1}{(r^*)^2}\,. \tag{B.11}$$

So, (B.8) can be approximated as

$$\ddot{\phi} - \phi'' + \frac{2}{r^*}\phi' = 0\,. \tag{B.12}$$

One can then take the Ansatz

$$\phi\,(t, r^*) \;=\; \int_{-\infty}^{+\infty} R\,(\omega, r^*)\; e^{i\omega t}\, \frac{d\omega}{2\pi}\,, \tag{B.13}$$

$$R\,(\omega, r^*) \;=\; \sum_{n=0}^{+\infty} a_n\,(\omega)\; (r^*)^n\,. \tag{B.14}$$

In this situation, solving (B.12) becomes equivalent to solving

$$\omega^2\, R \,+\, R'' \,-\, \frac{2}{r^*}\, R' \;=\; 0\,, \tag{B.15}$$

and, as such,

$$a_1 \;=\; 0\,, \tag{B.16}$$

and a_0 and a_3 are free functions of ω, while the other coefficients are determined by the iteration

$$\begin{aligned} &\omega^2\, a_{n-2} \,-\, 2na_n \,+\, n\,(n-1)\, a_n \;=\; 0 \;\Leftrightarrow \\ &\Leftrightarrow a_n \;=\; -\frac{\omega^2}{n(n-3)}\, a_{n-2}\,. \end{aligned} \tag{B.17}$$

One can therefore prove by induction that

$$a_{2n} = \frac{(-1)^{n+1}\,(2n-1)\,\omega^{2n}}{(2n)!}\,a_0\,, \tag{B.18}$$

$$a_{2n+1} = \frac{3\,(-1)^{n+1}\,(2n)\,\omega^{2n-2}}{(2n+1)!}\,a_3\,. \tag{B.19}$$

The radius of convergence is

$$\frac{1}{\limsup_{n\to+\infty}\sqrt[n]{|a_n|}} = +\infty\,, \tag{B.20}$$

meaning that $R(\omega, r^*)$ and $\phi(t, r^*)$ are defined for all r^*. Notice that the series expansion contains terms of order 0, 2, and 3 (but not 1), which indicates decay rates of $\dot{\phi} \sim 1$ and $\phi' \sim r^*$ when $r^* \to 0^-$, and is thus in line with what we obtained in Chapter 2, namely, $\left|\dot{\phi}\right| \lesssim 1$ and $|\phi'| \lesssim |r^*|^{1-\epsilon}$ (with the latter estimate being, as we can see now, almost sharp).

In addition, it is also interesting to check what happens in the homogeneous case (corresponding to $\ddot{\phi} \equiv 0$), where (B.8) is rendered as

$$\phi'' = -2\,\frac{r'}{r}\,\phi'\,. \tag{B.21}$$

In this situation,

$$\phi'\left(r^*\right) \;=\; \phi'\left(r_0^*\right)\,\left(\frac{r_0}{r}\right)^2\;, \tag{B.22}$$

that is,

$$\partial_r\phi\left(r^*\left(r\right)\right) \;=\; \frac{r_0^2\,\phi'\left(r^*\left(r_0\right)\right)}{r^2\left|1-\mu\left(r\right)\right|}\;\sim\;\frac{1}{r^4}\,, \tag{B.23}$$

which on its turn implies

$$\left|\phi\left(r^*\left(r\right)\right)-\phi\left(r^*\left(+\infty\right)\right)\right|\;\sim\;\frac{1}{r^3}\,. \tag{B.24}$$

C

The conformally invariant wave equation

The conformally invariant wave equation in $n+1$ dimensions is (see, for instance, [61])

$$\left(\Box_g + \frac{n-1}{4n} R_g\right) \phi = 0 \,, \tag{C.1}$$

where R_g is the scalar curvature of the metric g. If g is a FLRW metric with flat $n-$dimensional spatial sections, then it is conformally related with the Minkowski metric,

$$g = a^2(t) \left[-d\tau^2 + \left(\left(dx^1\right)^2 + \ldots + \left(dx^n\right)^2 \right) \right] \,, \tag{C.2}$$

with

$$\tau = \int \frac{dt}{a(t)} \,, \tag{C.3}$$

and so any solution of the conformally invariant wave equation is of the form

$$\phi = a^{1-\frac{n+1}{2}} \psi \,, \tag{C.4}$$

where ψ is a solution of the wave equation in Minkowski spacetime. Thus, we have

$$\dot{\phi} \sim a^{-\frac{n+1}{2}} \dot{a} \psi + a^{1-\frac{n+1}{2}} \frac{\partial \psi}{\partial \tau} \frac{d\tau}{dt} = a^{-\frac{n+1}{2}} \left(\dot{a}\psi + \frac{\partial \psi}{\partial \tau} \right) \,. \tag{C.5}$$

Both ψ and its time derivative are bounded for any topology of the flat spatial sections, and so

$$|\dot{\phi}| \lesssim a^{-\frac{n+1}{2}} \left(\dot{a} + 1 \right) \,. \tag{C.6}$$

If $a(t) = t^p$, then, for $p \leq 1$, we have

$$|\dot{\phi}| \lesssim a^{-\frac{n+1}{2}} \,, \tag{C.7}$$

replicating what was seen for the Fourier modes of the wave equation. For $p > 1$, however, we have

$$|\dot{\phi}| \lesssim a^{-\frac{n+1}{2}} t^{p-1} = a^{-\frac{n-1}{2} - \frac{1}{p}} \,, \tag{C.8}$$

quite different from the behavior of the wave equation. If $a(t) = e^t$, we have

$$|\dot{\phi}| \lesssim a^{-\frac{n-1}{2}}\,, \tag{C.9}$$

again quite different from the behavior of the wave equation. Note that for the metric (C.2) we have

$$R_g = \frac{2n\ddot{a}}{a} + \frac{n(n-1)\dot{a}^2}{a^2}\,, \tag{C.10}$$

and so

$$R_g = \frac{np\,[(n+1)p - 2]}{t^2} \tag{C.11}$$

for $a(t) = t^p$, and

$$R_g = n(n+1) \tag{C.12}$$

for $a(t) = e^t$.

D

Birkhoff's theorem

Theorem 30. *Every spherically symmetric solution (M, g) of the 4-dimensional Einstein's equations in electrovacuum is locally isommetric to a member of the Reissner-Nordström-(anti-)de Sitter family of solutions with parameters $\varpi, e, \Lambda \in \mathbb{R}$ or to one of the following constant radius solutions:*

(a) Nariai (if $\varpi > 0$, $\Lambda > 0$, $e = 0$), with $r = \frac{1}{\sqrt{\Lambda}}$ and $\varpi = \frac{1}{3\sqrt{\Lambda}}$;

(b) Bertotti-Robinson (if $\varpi > 0$, $\Lambda = 0$, $e \neq 0$), with $r = \varpi = |e|$;

(c) Charged Nariai (if $\varpi > 0$, $\Lambda > 0$, $e \neq 0$, $r > \sqrt{2}|e|$), with $r = \sqrt{\frac{1}{2\Lambda}\left(1 \pm \sqrt{1 - 4\Lambda e^2}\right)}$ and $\varpi = \frac{r}{3} + \frac{2e^2}{3r}$;

(d) Cosmological Bertotti-Robinson (if $\varpi > 0$, $\Lambda > 0$, $e \neq 0$, $r < \sqrt{2}|e|$), with $r = \sqrt{\frac{1}{2\Lambda}\left(1 - \sqrt{1 - 4\Lambda e^2}\right)}$ and $\varpi = \frac{r}{3} + \frac{2e^2}{3r}$;

(e) Plebánski-Hacyan (if $\varpi > 0$, $\Lambda > 0$, $4\Lambda e^2 = 1$), with $r = \frac{1}{\sqrt{2\Lambda}}$ and $\varpi = \frac{\sqrt{2}}{3\sqrt{\Lambda}}$;

(f) $AdS_2 \times S^2$ (if $\varpi < 0$, $\Lambda < 0$, $e \neq 0$), with $r = \sqrt{\frac{1}{2|\Lambda|}\left(-1 + \sqrt{1 + 4\Lambda e^2}\right)}$ and $\varpi = \frac{r}{3} + \frac{2e^2}{3r}$.

Proof. Recall the first order formulation of the Einstein-Maxwell-scalar field system from Subsection 3.1.1. Electrovacuum solutions can be studied by eliminating the scalar field components (that is, by setting ϕ to be identically equal to zero); as such, one has $\theta \equiv \zeta \equiv 0$, and so (3.27) and (3.28) imply that ϖ is constant (with μ thence depending only on r). The evolution equation (3.26) can thus be rewritten as

$$\partial_u \lambda = \frac{\lambda \, \partial_u r}{1 - \mu} \, \partial_r (1 - \mu) = \frac{\lambda \, \partial_u (1 - \mu)}{1 - \mu} , \tag{D.1}$$

whence

$$\lambda\,(u,v) \;=\; a_1\,(v)\cdot(1-\mu)\ . \tag{D.2}$$

In analogous fashion, it can also be seen that

$$\nu\,(u,v) \;=\; a_2\,(u)\cdot(1-\mu)\ . \tag{D.3}$$

Note that, due to this, if $\lambda \equiv 0$ one must then have $1-\mu \equiv 0$, and hence $\nu \equiv 0$ as well. In the same way, if $\nu \equiv 0$ then $\lambda \equiv 0$. These situations correspond to constant radius solutions, which satisfy the conditions $1-\mu \;=\; \partial_r\,(1-\mu) \;=\; 0$; that is:

$$3\varpi r \;=\; r^2 \,+\, 2e^2\,, \tag{D.4}$$

$$\Lambda r^4 \;=\; r^2 \,-\, e^2\,. \tag{D.5}$$

For $\Lambda > 0$ and $e = 0$, corresponding to the Nariai solution, one gets

$$
\begin{aligned}
r &= \frac{1}{\sqrt{\Lambda}}\,, && \text{(D.6)}\\
\varpi &= \frac{1}{3\sqrt{\Lambda}}\,. && \text{(D.7)}
\end{aligned}
$$

For $\Lambda = 0$ and $e \neq 0$, one has the Bertotti-Robinson solution, with

$$
r = \varpi = |e|\,. \qquad \text{(D.8)}
$$

For $\Lambda > 0$ and $e \neq 0$, one obtains the charged Nariai solution if $r > \sqrt{2}|e|$, the cosmological Bertotti-Robinson solution if $r > \sqrt{2}|e|$, or the Plebánski-Hacyan solution if $4\Lambda e^2 = 1$, each of these with

$$
\begin{aligned}
r &= \sqrt{\frac{1}{2\Lambda}\left(1 \pm \sqrt{1 - 4\Lambda e^2}\right)}\,, && \text{(D.9)}\\
\varpi &= \frac{r}{3} + \frac{2e^2}{3r}\,. && \text{(D.10)}
\end{aligned}
$$

For $\Lambda < 0$ and $e \neq 0$, one gets the anti-de Sitter metric, and in this particular context $AdS_2 \times S^2$, where

$$r = \sqrt{\frac{1}{2\,|\Lambda|}\left(-1 + \sqrt{1+4\Lambda e^2}\right)}\,, \tag{D.11}$$

$$\varpi = \frac{r}{3} + \frac{2e^2}{3r}\,. \tag{D.12}$$

On the other hand, for a non-constant radius, functions a_1 and a_2 can always be rescaled to become identically equal to 1 or -1 (according to the spacetime region one is in), thus yielding the Reissner-Nordström-(anti-)de Sitter metric.

□

E

The Nariai spacetime

The Nariai spacetime has the metric

$$\bar{g} \,=\, -ds^2 \,+\, \cosh^2 s\, dx^2 \,. \tag{E.1}$$

One can make a change of coordinates such that $d\sigma = \frac{ds}{\cosh s}$: for example, let us choose $\sigma = 2 \arctan \tanh \frac{s}{2}$. The metric (E.1) thus assumes the form

$$\begin{aligned} \bar{g} \,&:=\, -ds^2 \,+\, \cosh^2 s\, dx^2 \\ &=\, -4\cosh^2 s\, \frac{1}{2}\,(d\sigma \,-\, dx)\, \frac{1}{2}\,(d\sigma \,+\, dx) \\ &=:\, -\Omega^2\, du'\, dv' \,, \end{aligned} \tag{E.2}$$

where we picked double-null coordinates

$$du' \,=\, \frac{1}{2}\,(d\sigma \,-\, dx)\;, \tag{E.3}$$

$$dv' \,=\, \frac{1}{2}\,(d\sigma \,+\, dx)\;. \tag{E.4}$$

Therefore,

$$\Omega^2 \,=\, 4\cosh^2 s \,=\, \frac{4}{\cos\sigma} \,=\, \frac{4}{\cos\left(u' + v'\right)}\,, \tag{E.5}$$

and so (E.1) can be rewritten as

$$g = \frac{4\,du'dv'}{\cos\left(u'+v'\right)}\,. \tag{E.6}$$