TÉCNICO LISBOA

Construction of Penrose Diagrams for Static Spherically Symmetric Space-times

1st Cycle Integrated Scientific Project in Engineering Physics

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1 | Abstract

Penrose diagrams are one of the major tools to study the global causal structure of a wide variety of spacetimes, but often it is not clear how to compute them. With this in mind, we show how to construct the Penrose diagram for almost arbitrary static spherically symmetric spacetimes. We deduce how the metric determines the shape of the regions of the diagram separated by the horizons. We then show in what conditions we can extend this regions past the horizons such that the metric is continuous and differentiable and the radius is a differentiable function of the proper time along geodesics crossing the horizons. In some cases, we explicitly give coordinates to match two regions along the hypersurface corresponding to the horizon separating them.

Keywords: General Relativity, Penrose diagram, static spacetime, spacetime extendibility

2 | Introduction

One of the main tasks in General Relativity is understanding the causal structure of a given spacetime. Even though sometimes our spacetime might be highly nontrivial, making the analysis of its causality properties difficult, in some cases, namely for spherically symmetric spacetimes, there is an easy way to visualize the structure of our manifold. And that is by constructing its Penrose-Carter diagram - also known simply as Penrose diagram - whose name acknowledges both Brandon Carter and Roger Penrose, who were the first researchers to employ it. These are two-dimensional diagrams and each point corresponds to a two-sphere.

The advantage of the Penrose diagram comes from the fact that it is a finite diagram depicting the whole spacetime, including the points at infinity, such that the light rays move at 45 degrees, making it clear how the light cone is at each point of the spacetime. Due to this last property, it is much easier to determine if, for instance, an event might be able to influence another in the sense that there exists a causal future-pointing curve connecting the first to the latter. Despite the usefulness of Penrose diagrams in General Relativity, they also pose some problems, such as the fact that one might be difficult to construct, or the nonuniqueness of the maximal extensions of the spacetimes, introducing some additional questions when studying a specific spacetime. On that account, in this report we will analyse how to treat static spherically symmetric spacetimes and how to find the Penrose diagram for its maximal analytical extension, introducing an algorithm for its construction.

3 | Construction of the algorithm

3.1. Description of the problem

In this work, we will consider static spherically symmetric spacetimes whose metric can be written in the following form:

$$ds^2 = -f(r)dt^2 + \frac{1}{g(r)}dr^2 + r^2\, d\Omega^2 \tag{3.1}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$ is the metric for the unit 2-sphere. Additionally, we will consider that the functions f and g are continuous and differentiable, have finitely many roots and are analytic at their roots. Note that a spacetime is a Lorentzian manifold, so we will only be interested in the points where $f(r)$ and $g(r)$ have the same sign, such that the signature of our spacetime is $(-+++)$.

The main objective will be to construct the Penrose diagram for such spacetimes, by discussing its shape and extendibility at the roots of f and g, which will be the points that might lead to some kind of singularities.

3.2. Simple case - $f = g$

Firstly, we will start by analysing the case where the functions f and g are the same. Therefore our metric can be written as:

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\Omega^2 \tag{3.2}$$

Let us start by constructing the Penrose diagram for the Minkowski spacetime [5]. In this case we have that $f(r) = 1$ and let us consider the following coordinate transformation:

$$\begin{cases} u = t - r \\ v = t + r \end{cases} \tag{3.3}$$

In these coordinates, the metric can be written as

$$ds^2 = -du\, dv + r^2(u,v)d\Omega^2 \tag{3.4}$$

where

$$r(u,v) = \frac{v-u}{2} \tag{3.5}$$

The coordinates (u, v) are null coordinates, so their level sets are null cones corresponding to outgoing and ingoing null geodesics emanating from the center, respectively.

We have the constraint that $r \geqslant 0$, so we have $v \geqslant u$. Now performing the coordinate change

$$\begin{cases} \tilde{u} = \tanh(u) \\ \tilde{v} = \tanh(v) \end{cases} \Leftrightarrow \begin{cases} u = \operatorname{arctanh}(\tilde{u}) \\ v = \operatorname{arctanh}(\tilde{v}) \end{cases} \tag{3.6}$$

we obtain that

$$ds^2 = -\frac{1}{(1-\tilde{u}^2)(1-\tilde{v}^2)}d\tilde{u}\, d\tilde{v} + r^2(\tilde{u},\tilde{v})d\Omega^2 \tag{3.7}$$

This coordinate tranformation allowed us to make our spacetime finite, as the image of tanh is the interval $(-1, 1)$. Using the constraint $v \geqslant u \Leftrightarrow \tilde{v} \geqslant \tilde{u}$, we have that

$$-1 \leqslant \tilde{u} \leqslant \tilde{v} \leqslant 1 \tag{3.8}$$

and so we conclude that the Penrose diagram for the Minkowski spacetime is the following.

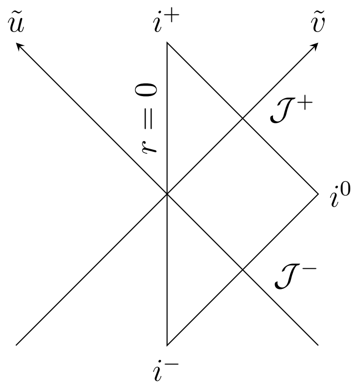

Figure 3.1: Penrose diagram for the Minkowski spacetime

Now, for a general function f the algorithm will be similar, with the additional difficulty that f might have roots, which will cause the diagram's shape to change. As it was assumed before, f has a finite number of roots. Therefore, let $0 < r_1 < ... < r_N$ be the nonzero roots of f. The origin could also possibly be a root of f, but we exclude it above for notation purposes. Now let us define the open intervals $I_i = (r_i, r_{i+1})$ for $0 < i < N$, $I_0 = (0, r_1)$ and $I_N = (r_N, +\infty)$. We see that the metric, in the coordinates (t, r) is not well defined at the roots of f, so we will need to construct the Penrose diagram for each interval separately, which we will call blocks, and then match them adequately, if possible.

Firstly, let us start by constructing the blocks and suppose that $N > 1$. For a given I_i $(0 < i < N)$, suppose that $f(r) > 0$ in this interval. Then the metric for $r \in I_i$ can be written as:

$$\begin{aligned} ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\Omega^2 = -f(r)\left(dt^2 - \frac{1}{f^2(r)}dr^2\right) + r^2 d\Omega^2 = \\ = -f(r)\left(dt - \frac{1}{f(r)}dr\right)\left(dt + \frac{1}{f(r)}dr\right) + r^2 d\Omega^2 = -f(r)du\,dv + r^2 d\Omega^2 \end{aligned} \tag{3.9}$$

where we performed the following coordinate transformation

$$\begin{cases} u = t - \int \frac{1}{f(r)}\,dr \\ v = t + \int \frac{1}{f(r)}\,dr \end{cases} \tag{3.10}$$

and r is a function of the null coordinates (u, v). Now we have that

$$\frac{v - u}{2} = \int \frac{1}{f(r)}\,dr \tag{3.11}$$

so to get a constraint for v and u we need to analyse the integral in 3.11. Given $R \in I_i$, we know that

$$\int_R^{r_{i+1}} \frac{1}{f(r)}\,dr = +\infty \quad \text{and} \quad \int_R^{r_i} \frac{1}{f(r)}\,dr = -\infty \tag{3.12}$$

because $f(r) > 0$ in I_i and f is analytic at its roots. We conclude what there is no constraint to v and u like there was in the Minkowski spacetime. So, by performing the coordinate transformation in 3.6 we get the full square and not just a triangle. To identify what values of r each side of the square corresponds to, we see that at the points where the integral is $+\infty$ we must have $\tilde{v} = 1$ or $\tilde{u} = -1$, so these sides of the square correspond to r_{i+1}. We do a similar analysis for the other sides of the square and we get the Penrose diagram for this block (Figure 3.2(a)). An example of this type of block is the region between the black hole and the cosmological horizon in the Schwarzschild-de Sitter spacetime [3].

If $f(r) < 0$, we need to start by performing the coordinate transformation $u' = -u$. That way, the metric takes the form:

$$ds^2 = f(r)du'\,dv + r^2 d\Omega^2 \tag{3.13}$$

Now we do can the transformation 3.6 and we will also get a square, but now its orientation changes and the Penrose diagram is the one presented in Figure 3.2(b), and one classical example of this kind of block is the region between the event horizon and the Cauchy horizon in the non-extremal Reissner-Nordström spacetime.

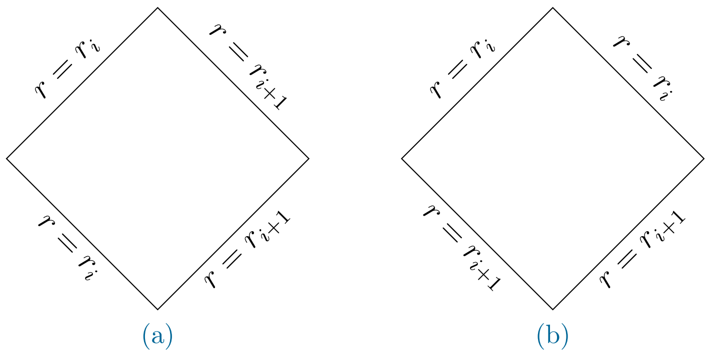

Figure 3.2: Penrose diagrams for the interval I_i and (a) $f(r) > 0$ (b) $f(r) < 0$

To better understand why the orientation of the block changes, we see that now we have

$$\frac{1}{2}(\operatorname{arctanh}(\tilde{u}) + \operatorname{arctanh}(\tilde{v})) = \int \frac{1}{f(r)}\,dr \tag{3.14}$$

and, due to the fact that $f(r) < 0$,

$$\int_R^{r_{i+1}} \frac{1}{f(r)}\,dr = -\infty \quad \text{and} \quad \int_R^{r_i} \frac{1}{f(r)}\,dr = +\infty \tag{3.15}$$

for a given $R \in I_i$. By a direct analysis, we conclude that at the points where the integral is $-\infty$ we must have $\tilde{u} = -1$ or $\tilde{v} = -1$, so these sides of the square must correspond to $r = r_{i+1}$. Once again, we can perfom a similar analysis to conclude that the other sides correspond to $r = r_i$.

Let us now proceed to analyse what happens in the interval I_0, assuming that $f(r) > 0$. We do the same coordinate transformation as in 3.3 and once again we need to analyse the same integral. Given a fixed $R \in I_0$, we know that

$$\int_R^{r_1} \frac{1}{f(r)}\,dr = +\infty \tag{3.16}$$

for the same reasons as before. If $f(0) \neq 0$ or even if f is not defined at 0 but $\lim_{x\to 0} f(x) = +\infty$, we know that the integral from R to 0 converges and is equal to a constant $k \in \mathbb{R}$. Now our constraint is

$$\frac{v-u}{2} \geqslant k \Leftrightarrow v \geqslant u + 2k \tag{3.17}$$

By doing the coordinate transformation $u' = u + 2k$ followed by the one in 3.6, we get the same constraints as in the Minkowski spacetime, so the Penrose diagram is a triangle oriented like the one in Minkowski spacetime (Figure 3.3(a)), which can be seen for example in the interior of the Reissner-Nordström black hole. If $f(r) < 0$ in I_0, we get a triangle oriented horizontally (Figure 3.3(b)), just like the black hole or the white hole in the Schwarzschild spacetime. If $f(0) = 0$, we need to calculate the integral to see if it converges, and we will also get a triangle, or if it diverges, and in that case we get a square as before.

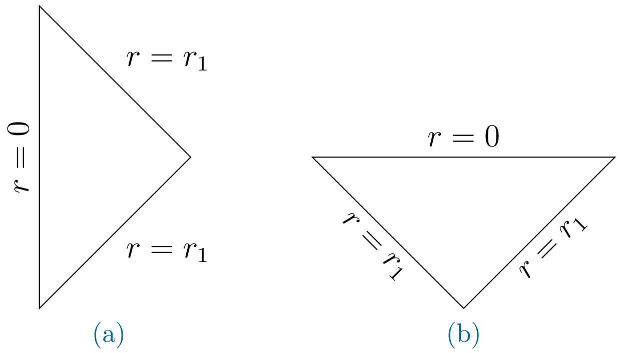

Figure 3.3: Penrose diagrams for the interval I_0 if $\lim_{x\to\infty} f(x) \neq 0$ and (a) $f(r) > 0$ (b) $f(r) < 0$

An analogous procedure can be made for the interval I_N, but in this case we calculate the integral from R to $+\infty$. If $\lim_{x\to\infty} f(x) \neq \infty$, the integral diverges and we get a square. If $\lim_{x\to\infty} f(x) = \infty$, we need to calculate the integral to see if it converges, and we will also get a triangle (like the region outside the cosmological horizon in the Schwarzschild-de Sitter spacetime), or if it diverges, and in that case we get a square (for example the region outside the Schwarzschild black hole).

An important remark is that for a given block with coordinates $(\tilde{u}, \tilde{v})$, one can perform the transformation $(\tilde{u}, \tilde{v}) \to (-\tilde{u}, -\tilde{v})$ and we get a similar but mirrored block, so as a matter of fact for each interval I_i there are two blocks which are a mirrored version of each other - for example the black hole and the white hole in the Schwarzschild spacetime.

Now, we need to see if we can match two blocks corresponding to successive intervals together. To do so let us consider the coordinate change $(t, r) \to (u, r)$, such that the metric is:

$$ds^2 = -f(r)du\left(du + \frac{2}{f(r)}dr\right) + r^2 d\Omega^2 = -f(r)du^2 - 2du\,dr + r^2 d\Omega^2 \tag{3.18}$$

We see that, in these coordinates, the metric doesn't have any singularity at the roots and that it also is nondegenerate, so we check that we can match two blocks along a line of constant v, i.e. along an ingoing null line. In order to match along an outgoing null line, we consider the coordinate transformation $(t, r) \to (v, r)$, such that the metric is:

$$ds^2 = -f(r)dv\left(dv - \frac{2}{f(r)}dr\right) + r^2 d\Omega^2 = -f(r)dv^2 + 2dv\,dr + r^2 d\Omega^2 \tag{3.19}$$

Once again, we see that in these coordinates the metric doesn't have any problem at the roots and we are allowed to match two blocks along a line of constant u, i.e. along an outgoing null line.

This procedure shows that we don't have any real singularity at the roots of f, so we just match all the blocks together without any problem. Note that we must only match an outgoing (ingoing) null line to another outgoing (ingoing) null line, both corresponding to the same radius. Additionally, when we match two blocks together, we must do it so that the radius varies monotonically along a null line going through both the blocks. Taking this into account, we can construct the Penrose diagram, corresponding to the maximal analytic extension solution by allowing repetitions of the blocks.

It remains to consider the case where $N = 0$, i.e., f doesn't have any nonzero roots. Once again, one needs to change to null coordinates as in 3.3 and then, for a given $R \neq 0$ analyse the following integrals:

$$\int_0^R \frac{1}{f(r)}\,dr \quad \text{and} \quad \int_R^{+\infty} \frac{1}{f(r)}\,dr \tag{3.20}$$

If the first integral converges, there will be a timelike surface (if $f(r) > 0$) or a spacelike surface (if $f(r) < 0$) corresponding to $r = 0$. However, if the integral diverges, there will be two null surfaces corresponding to $r = 0$. It is analogous for the second integral, so one concludes that if both integrals diverge the Penrose diagram will be a square, if only one of them converges we get a triangle and if both of them converge we get a strip.

3.3. General case - $f \neq g$

Now we will proceed to construct the Penrose diagram for the general case where f and g are arbitrary functions which might be different and therefore have different roots. First of all, let us denote the nonzero roots of $f(r)g(r)$ by $0 < r_1 < ... < r_N$ and the intervals I_i are defined as in the section 3.2. Note that now the two functions might have opposite signs, but in this case the manifold will not be Lorentzian, as the metric 3.1 will not have signature $(-+++)$. Therefore, we will not be interested in the intervals I_i where $f(r)g(r) < 0$ and we will assume from now on that both functions have the same sign.

The analysis of the shape of the blocks corresponding to each interval will be similar to the previous one. With that in mind we will start by constructing the block for an interval I_i with $0 < i < N$. Assuming f and g are positive in I_i, we note that the metric can be written as:

$$\begin{aligned} ds^2 &= -f(r)dt^2 + \frac{1}{g(r)}dr^2 + r^2 d\Omega^2 = -f(r)\left(dt^2 - \frac{1}{f(r)g(r)}dr^2\right) + r^2 d\Omega^2 = \\ &= -f(r)\left(dt - \frac{1}{\sqrt{f(r)g(r)}}dr\right)\left(dt + \frac{1}{\sqrt{f(r)g(r)}}dr\right) + r^2 d\Omega^2 = \\ &= -f(r)du\,dv + r^2 d\Omega^2 \end{aligned} \tag{3.21}$$

where u and v are given by

$$\begin{cases} u = t - \int \frac{1}{\sqrt{f(r)g(r)}}\,dr \\ v = t + \int \frac{1}{\sqrt{f(r)g(r)}}\,dr \end{cases} \tag{3.22}$$

Now we see that

$$\frac{v-u}{2} = \int \frac{1}{\sqrt{f(r)g(r)}}\,dr \tag{3.23}$$

and to infer what is the shape of the block corresponding to I_i we need to analyse the convergence of the integrals

$$\int_{r_i}^{R} \frac{1}{\sqrt{f(r)g(r)}}\,dr \quad \text{and} \quad \int_{R}^{r_{i+1}} \frac{1}{\sqrt{f(r)g(r)}}\,dr \tag{3.24}$$

The first (second) integral will diverge if r_i (r_{i+1}) is a root of both functions and will converge if it is a root of order 1 of only one of the functions. Now just like in section 3.2 for the case $N = 0$, if both integrals diverge we have a square, if only one diverges we get a triangle and if both converge we get a strip. For the case where $f(r) < 0$ in I_i we note that the only difference is the orientation of the block just like before.

Furthermore, we now need to see if we can extend a given block I_i across a surface of constant $r = r_i$ and match it with the block corresponding to I_{i-1}.

In the first place, if $f(r_i) = 0$ and $g(r_i) \neq 0$ we can conclude there is a curvature singularity by analysing the behaviour of the Kretschamann scalar [4]:

$$\begin{aligned} K = {} & \frac{g^2 f''^2}{f^2} + \frac{1}{4}\frac{f'^2 g^2}{f^2} + \frac{1}{4}\frac{f'^4 g^2}{f^4} - \frac{1}{2}\frac{f'^3 g g'}{f^3} + \frac{f'' f' g' g}{f^2} - \\ & - \frac{f'' f'^2 g^2}{f^3} + \frac{2}{r^2}\frac{g^2 f'^2}{f^2} + \frac{2}{r^2} g'^2 + \frac{4}{r^4}(g-1)^2 \end{aligned} \tag{3.25}$$

Secondly, if $f(r_i) \neq 0$ and $g(r_i) = 0$, we see immeditately that the Kretschmann scalar doesn't diverge, so we cannot conclude if the spacetime is extendible. Instead, let us examine the behaviour of the affine parameter of null geodesics crossing the surface with $r = r_i$. Thus, let us consider the metric in coordinates r and u:

$$ds^2 = -f(r)du^2 - 2\sqrt{\frac{f(r)}{g(r)}}du\,dr + r^2 d\Omega^2 \tag{3.26}$$

As we are analysing a spherically symmetric spacetime we want a geodesic with zero angular velocity. A null geodesic with for instance u constant will identically satisfy one of the geodesic equations and the other one is the following:

$$\frac{d}{ds}\left(-2\sqrt{\frac{f(r)}{g(r)}}\dot{r}\right) = 0 \Leftrightarrow \sqrt{\frac{f(r)}{g(r)}}\dot{r} = E \Rightarrow \Delta s = \frac{1}{E}\int_{R}^{r_i}\sqrt{\frac{f(r)}{g(r)}}\,dr \tag{3.27}$$

where E is an integration constant. One can see that the affine parameter goes to infinity if the order of the root of g is greater than one, so we conclude that we cannot extend the geodesic past the surface $r = r_i$ in that case.

Following these calculations, one can conclude that when r_i is a root of f and g and the order of g minus the order of f is greater than 1, the affine parameter also goes to infinity and we also cannot extend the spacetime in these conditions.

Furthermore, we analyse the case where r_i is a root of both functions and the order of f is greater than the order of g. We know that if the spacetime is extendible across the surface $r = r_i$, a geodesic crossing it should be well behaved. One of the geodesics equations gives us

$$f(r)\dot{t} = E \tag{3.28}$$

where E is a constant. We also know that a geodesic must satisfy

$$1 = f(r)\dot{t}^2 - \frac{1}{g(r)}\dot{r}^2 \Leftrightarrow 1 = \frac{E^2}{f(r)} - \frac{1}{g(r)}\dot{r}^2 \Leftrightarrow \dot{r}^2 = E^2\frac{g(r)}{f(r)} - g(r) \tag{3.29}$$

As we know that r is a quantity with a geometrical and physical meaning, it must be a C^∞ function along the geodesic. However, we notice that if the order of f is greater than the order of g, $\dot{r}$ diverges at the surface $r = r_i$, indicating we cannot extend the spacetime.

At this point, we still didn't analyse the case where r_i is a root of f and g, with the same order. In this case, the integral in 3.24 diverges and we get two null surfaces corresponding to $r = r_i$. To match this block to the neighbour one we just need to change to coordinates (u, r) or (v, r) just like in section 3.2 and the metric takes the form in 3.26 for the first case and for the latter one it takes the following form:

$$ds^2 = -f(r)dv^2 + 2\sqrt{\frac{f(r)}{g(r)}}dv\,dr + r^2 d\Omega^2 \tag{3.30}$$

We see that although the functions are zero at $r = r_i$, the determinant of the metric is different from zero due to the fact that r_i is a root of the same order of f and g.

Finally, we will discuss the case where $f(r_i) \neq 0$ and r_i is a root of order 1 of g. Firstly, one can see that the previous analysis doesn't allow us to conclude whether the spacetime can be extended, as the affine parameter of null geodesics crossing this surface does not diverge and the derivative of r with respect to the proper time does not diverge either. So we need a different approach to deal with this case. By calculating the first integral in 3.24, we see that it converges, so assuming $f(r) > 0$ in I_i we get a timelike surface corresponding to the points such that $r = r_i$. Moreover, the function $f(r)g(r)$ is negative in I_{i-1} because f does not change its sign in r_i, but g does, so the manifold in this interval is not Lorentzian. For this reason, we cannot match the block corresponding to I_i to the one associated to the neighbour interval. Instead, we will see if we can match it with a reflection of itself. To do so, we can calculate the second fundamental form at the timelike surface $r = r_i$ and we see that it is zero, allowing us to match the two blocks along this surface [2]. After we match the two blocks we get a wormhole, whose radius is equal to r_i (e.g., the ultra-static wormhole of Ellis and Bronnikov [6] or the Sultana spacetime [1] - Figure 3.4). If $f(r) < 0$, the analysis is similar but now we match the block with a reflection of itself along a spacelike surface corresponding to $r = r_i$.

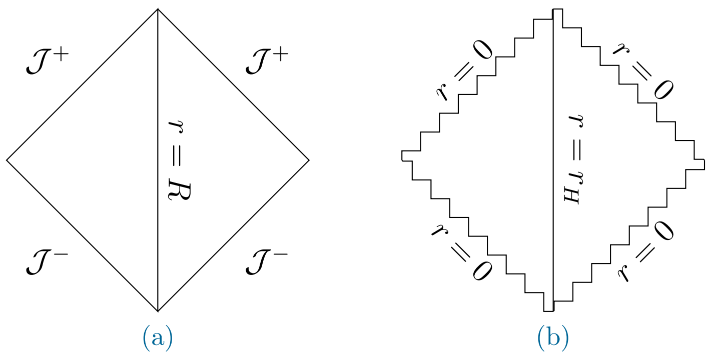

Figure 3.4: Penrose diagrams for the (a) ultra-static wormhole of Ellis and Bronnikov with radius R (b) maximal analytic extension of Sultana spacetime with cosmological horizon at r_H

4 | Conclusion

In this work, we were able to construct the Penrose diagram for almost arbitrary static and spherically symmetric spacetimes. We started by exploring the simpler case where f and g are the same function and we found out that we didn't have any problems concerning the matching of different blocks as the spacetime was always extendible at the surfaces corresponding to the roots of f. Moreover, we deduced that the blocks associated with intervals whose both extremes were roots of f would be squares and its orientation would depend on the sign of f. Furthermore, we managed to do a similar analysis for the general case where f and g are different functions. That said, we deduced the shape of the blocks where the manifold is Lorentzian but, in this case, we showed that the spacetime might be extendible or not. In addition, we came to the conclusion that the extendibility of the blocks was completely determined by the order of the roots of f and g, so we deduced that we can only extend the spacetime if f and g have the same order or if the order of g is the order of f plus 1. Actually, in this last case we only proved that the block is extendible if f is not zero and g has a root of order 1. For higher orders, we didn't prove neither disprove if the block is extendible, hence this case could be considered in future research. Further work could also drop the assumption that f and g are analytic at their roots, thus covering a wider variety of spacetimes.

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