Master's Thesis

The Wave Equation in Cosmological Spacetimes

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Introduction

The objective of this work is to study the decay of solutions of the wave equation

$$\Box_g \phi = 0,$$

with ϕ scalar field which does not affect the spacetime curvature and g a metric describing a homogeneous and isotropic universe. Such cosmological models are the Friedmann–Lemaître–Robertson–Walker (FLRW) models: those are considered, for large scales, an accurate version of the universe in which we live.

This decay problem can be regarded with interest due to several reasons. First, it consists of investigating the behaviour of classical waves in a dynamical background, which is mathematically interesting per se. The decay of the wave is affected by the expansion of the space sections and by dispersion (i.e. the non-compactness of space). Moreover, the asymptotic behaviour of solutions is strictly related to the causal structure of the spacetime. The respective analysis is therefore more involved than the one for the wave equation in Minkowski.

On the other hand, our classical problem is relevant to develop a full-fledged quantum field theory (QFT) in curved spacetime. If we consider an analogy with the flat case, indeed, the role of $\Box_g \phi = 0$ is comparable to the one of the wave equation for classical fields when studying QFTs in Minkowski. Understanding the behaviour of classical fields is a prerequisite to quantization. Such a quantum theory in curved backgrounds can be understood as a first approximation, in which the curvature of spacetime is still a classical phenomenon, of a full theory of quantum gravity. In this sense, several types of backgrounds are significant, the de Sitter solution being one of the most prominent ones in the analysis of the early universe. For instance, the cosmic inflation theory [19] describes an early universe which underwent an expansion of exponential acceleration between $t \sim 10^{-36}$ and $t \sim 10^{-33}$ seconds after the Big Bang. The theory solves cosmological problems as

- The *horizon problem*: why do parts of the universe apparently belonging to causally disconnected regions seem to have come in contact with each other in the past?
- The *flatness problem*: why are the space sections of our universe approximately flat today?

A first attempt to describe such a period of exponential expansion is through a de Sitter spacetime containing a scalar field, the *inflaton*, whose equations of motion in the most basic cases are given by the Klein-Gordon equation, see e.g. [23]. Other fields such as the electromagnetic (vector) field can be studied, the common denominator in all these contexts being the wave equation.

Furthermore, solving the wave equation is one of the first steps to a rigorous approach for the Einstein's equations:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \,,$$

where $R_{\mu\nu}$ and R are, respectively, the Ricci tensor and the Ricci scalar, G is the gravitational constant, $T_{\mu\nu}$ is the energy-momentum tensor of the matter present in the spacetime and Λ is the cosmological constant. In dimension 3+1, they are nothing else than a system of ten, independent, non-linear hyperbolic partial differential equations (PDEs). Their linearization leads, in some sense, to the wave equation. In perturbation theory, this can be seen after considering a small perturbation around a cosmological metric $\tilde{g}_{\mu\nu}$:

$$g_{\mu\nu} = \tilde{g}_{\mu\nu} + h_{\mu\nu}$$

for $|h_{\mu\nu}| \ll 1$ for every μ, ν. If we ignore higher-order terms and make a gauge choice, Einstein's equations imply that, in vacuum, the perturbation h satisfies [20]:

$$\Box_{\tilde{g}} h_{\mu\nu} - 2R_{\alpha\mu\nu\beta}\, h^{\alpha\beta} = 0.$$

The latter is more involved than the wave equation for a scalar field, since the covariant derivatives are now applied to tensors of rank greater or equal than two, and therefore more terms containing Christoffel symbols are present. Moreover, a curvature term appears in the general case that we are considering. Those described by last equation are the gravitational waves that were (indirectly) detected through observations of systems of binary pulsars in the '70s and '80s and (directly) detected through the LIGO experiments of 2015.

Additional applications of the wave equation in the realm of General Relativity can be found, for instance, when studying black holes and the stability of spacetimes (see e.g. [8], [7] and [9]). A related problem is the Klein-Gordon equation in cosmological spacetimes, analysed e.g. in [21].

A basic analysis of our problem follows from physical intuition. First, we know that the decay of the wave is related to decay of its energy, since the energy density is proportional to $(\partial_t \phi)^2$. What is more, we know [5] that redshift causes the wave energy to decay as $a^{-1}(t)$ in 3+1 dimensions, where a is the scale factor which denotes the size of the space sections of the universe. Intuitively, $\partial_t \phi$ should then decay as a^{-2}, since

$$E \sim \int (\partial_t \phi)^2 \sqrt{-g} d^3 x$$

and $\sqrt{-g} = a^3$. Thus, we expect a dependence on a in our results. Indeed, we got a decay rate of $t^{-\frac{2}{3}}$ for ϕ for the dust case with toroidal space sections (see chapter 1), when the scale factor is $a(t) = t^{\frac{2}{3}}$. However, when the space sections of our universe are non-compact, dispersion plays an important role as well and the above physical description turns out to be an oversimplification. For instance, we got a decay rate of t^{-1} for non-compact flat cases as the Minkowski, dust and radiation solutions analysed in chapter 2 in dimension 3+1, even though their respective scale factors are not the same: $a(t) = t^p$, where $p = 0$ for Minkowski, $p = \frac{2}{3}$ for the dust case, $p = \frac{1}{2}$ for the radiation case. In Minkowski, in particular, there is no redshift and thus the entire decay rate of t^{-1} is due to dispersion. Similar physical arguments were taken into consideration during this work.

The three chapters of the thesis contain the three different approaches that we followed. We worked mostly in the physically relevant case of dimension 3+1, but some results for general dimension n were also obtained in chapters 1 and 3. In chapter 1, the decay rates for the wave equation were obtained through a Fourier analysis. Although this is an efficient way to gain insights about universes with toroidal and spherical space sections, the method is suited for linear equations and, thus, cannot be extended to the case of Einstein's equations.

Chapter 2 deals with a different approach. The wave equation in cosmological spacetimes is studied through some operators defined on the space of infinitely differentiable functions. The problem of the decay rate of a wave in some particular spacetime is, in this way, reduced to a similar problem but in a much simpler spacetime. This argument was first used by Klainerman and Sarnak in [16] for the dust models in the flat and hyperbolic case. We generalized this method for the spherical case and found suitable operators for a range of spacetimes (radiation-filled, de Sitter, anti-de Sitter and Milne universe). The bright side of this "operator trick" is that it lets us obtain a spherical means formula for the solution of the wave equation (analogously to Kirchhoff's formula for Minkowski), which explicitly represents the behaviour of the solution. On the other hand, the needed operator is very specific to the spacetime and to the curvature of its space sections. Although more general solutions cannot be excluded, such operators could be found only for the above-mentioned universes.

Lastly, chapter 3 regards the most general approach of this thesis: the energy method. The decay rate of the solutions is found through the divergence theorem applied to the *domain of dependence* of the hypersurface of initial data, i.e. the region of spacetime which is influenced by the initial data at some instant of time. The integrand term is given by a current J, defined as the contraction between the energy-momentum tensor associated to the wave equation and a proper *multiplier vector field* X. Different results could be achieved using different multipliers. The approach has its origins in the energy methods used for PDE theory in $\mathbb{R}^n$. In our case, however, the method requires more care since it is applied in the context of Lorentzian geometry on spacelike, timelike and null domains.

Notations

In the thesis, notations from both General Relativity and Differential Geometry are used. Indeed, in Mathematical Relativity a spacetime is a couple (M, g), where M is a differentiable Lorentzian manifold and g a metric which satisfies Einstein's equations. In our conventions, the Minkowski metric in dimension 3+1 has signature $(-, +, +, +)$. We will use Einstein's summation convention throughout the three chapters and we will make no distinction between a tensor and its components - e.g. $T_{\mu\nu}$ will be referred to both as the "energy-momentum tensor" and as "the components of the energy-momentum tensor", depending on the context. As a matter of convenience, we will often work with a modified energy for the wave, defined in dimension n as

$$E(t) := a(t)^{2-n} E_{\text{phys.}}(t)$$

with respect to the physical energy $E_{\text{phys.}}$. From now on, we will work in Planck's units, i.e. $G = c = 1$.

Notations from Mathematical Analysis are also used in the thesis. In particular, we use the symbols

$$f(x) \lesssim g(x) \text{ as } x \to x_0 \quad \Longleftrightarrow \quad \limsup_{x \to x_0} \frac{f(x)}{g(x)} \leq C,$$

$$f(x) \gtrsim g(x) \text{ as } x \to x_0 \quad \Longleftrightarrow \quad \liminf_{x \to x_0} \frac{f(x)}{g(x)} \geq C,$$

$$f(x) \sim g(x) \text{ as } x \to x_0 \quad \Longleftrightarrow \quad C_1 \leq \liminf_{x \to x_0} \frac{f(x)}{g(x)} \leq \limsup_{x \to x_0} \frac{f(x)}{g(x)} \leq C_2,$$

for some constants C, C_1, C_2 greater than zero. As done here, we will often use the same letter to denote constants that are different in general, if such constants are not relevant in the computations.

Main Results

We have obtained the following decay rates for solutions ϕ of the wave equation.
Friedmann universe with flat space sections, scale factor $a(t) = t^p, 0 < p \leq 1$:

- For toroidal space sections of dimension n (see chapter 1), we considered the Fourier coefficients $c_k(t)$ of ϕ and obtained:

$$|c_k(t)| \lesssim t^{-\frac{n-1}{2}p} \text{ as } t \to +\infty, \text{ for } p \neq 1,$$

$$|c_k(t)| \sim C, \text{ as } t \to +\infty, \text{ for } p = 1,$$

and:

$$|c_k(t)| \sim \begin{cases} C, & 0 < p < \frac{1}{n}, \\ \left|\log\left(\frac{k}{1-p}t^{1-p}\right)\right|, & p = \frac{1}{n}, \\ t^{1-np}, & \frac{1}{n} < p < 1, \\ t^{-\frac{n-1}{2}}, & p = 1, \end{cases}$$

as $t \to 0^+$.

- For non-compact space sections of dimension n, using energy methods (see chapter 3):

$$\|\phi(t,\cdot)\|_{L^\infty(\mathbb{R}^n)} \lesssim \begin{cases} t^{-\frac{n-1}{2}p}, & 0 < p \leq \frac{2}{n+1} \text{ and } p = 1, \\ t^{-\frac{n-1}{2}p(1-\delta)}, & \frac{2}{n+1} < p < 1, \end{cases}$$

for any $\delta > 0$. The above estimates include the dust case ($p = \frac{2}{n}$) and the radiation case ($p = \frac{2}{n+1}$).

- For non-compact space sections of dimension 3, using the operator trick (see chapter 2):

$$|\phi(t,x)| \lesssim \begin{cases} t^{-1}, & \text{dust case}, \\ t^{-1}, & \text{radiation case}, \end{cases}$$

as $t \to +\infty$.

Friedmann universe with flat space sections, scale factor $a(t) = t^p$, $p > 1$:

- For toroidal space sections of dimension n (see chapter 1), using Fourier analysis we got:

$$|\dot{c}_k(t)| \sim t^{1-2p}$$

as $t \to +\infty$ and:

$$|c_k(t)| \lesssim t^{-\frac{n-1}{2}p}$$

as $t \to 0^+$.

- For non-compact space sections of dimension 3, using energy methods (see chapter 3):

$$\|\partial_t\phi(t,\cdot)\|_{L^\infty(\mathbb{R}^n)} \lesssim t^{1-2p}$$

as $t \to +\infty$.

Dust and radiation case for hyperbolic space sections:

For non-compact space sections of dimension 3, using the operator trick (see chapter 2):

$$|\phi| \lesssim t^{-\frac{3}{2}} \text{ as } t \to \infty \text{ for the dust case}$$

and

$$|\partial_t\phi| \lesssim t^{-2} \text{ as } t \to \infty \text{ for the radiation case.}$$

Dust and radiation case for spherical space sections:

For spherical space sections of dimension 3, using the operator trick (see chapter 2):

$$\phi \sim \frac{1}{\sin\left(\frac{\tau}{2}\right)} \text{ as } \tau \to 0^+ \text{ and as } \tau \to 2\pi^-, \text{for the dust case}$$

and

$$\phi \sim \frac{1}{\sin(\tau)} \text{ as } \tau \to 0^+ \text{ and as } \tau \to \pi^-, \text{for the radiation case.}$$

Here, $\tau = \int \frac{dt}{a}$ is the conformal time. The solutions diverge at the physical singularities: the Big Bang ($\tau = 0$) and the Big Crunch ($\tau = 2\pi$ for the dust, $\tau = \pi$ for radiation).

De Sitter universe, for flat, spherical and hyperbolic space sections:

Using energy methods (see chapter 3):

$$\|\partial_t \phi(t,\cdot)\|_{L^\infty(\Sigma)} \lesssim e^{-2t},$$

where $\Sigma = \mathbb{R}^3, S^3$ or $\mathbb{H}^3$. Same result obtained for general dimension n using Fourier analysis (see chapter 1) and the operator trick (see chapter 2).

Anti de-Sitter universe:

Using the operator trick, for $n = 3$:

$$\phi(t,x) \to C \text{ as } t \to +\infty,$$

and C does not depend on x.

Milne universe:

Using the operator trick, for $n = 3$:

$$|\phi| \lesssim t^{-2} \text{ as } t \to +\infty.$$

Energy decay for bounded domains:

In chapter 3 we have obtained an estimate for the local energy of the waves restricted to flat domains which are bounded in the radial variable. The total energy in this case is defined as

$$E(t) = \frac{1}{2}\int_{\mathbb{R}^3} \left(a^2(t)\left(\partial_t\phi(t,x)\right)^2 + \left(\partial_r\phi(t,x)\right)^2 + \left|\nabla\!\!\!/\,\phi(t,x)\right|^2\right) d^3x.$$

and the result can be expressed as follows.

Theorem (Morawetz estimate in the Friedmann universe)**.** *Let ϕ be a solution of the wave equation $\Box_g \phi = 0$, where*

$$g = -dt^2 + t^{2p}\left(dr^2 + r^2 d\theta^2 + r^2 \sin^2(\theta) d\varphi^2\right)$$

and $0 < p \leq 1$. Let us denote the spatial hypersurfaces of the Friedmann universe by $\Sigma_t = \{t\} \times \mathbb{R}^3$ for $t \geq t_0 > 0$ and let $\mathcal{R}(\tau) = \cup_{t\in[t_0,\tau]} \Sigma_t$. Then, for every $R > 0$ and for every $\tau > t_0$:

$$\int_{\mathcal{R}(\tau)\cap\{r\leq R\}} a(t)\left(a^2(t)\,(\partial_t\phi)^2 + (\partial_r\phi)^2 + |\nabla\!\!\!/\,\phi|^2\right) r^2 dt\,dr\,d\Omega \leq C_R\, a(t_0) E(t_0),$$

for some constant C_R depending on R, where $|\nabla\!\!\!/\,\phi|^2 = \frac{1}{r^2}(\partial_\theta\phi)^2 + \frac{1}{r^2\sin^2(\theta)}(\partial_\varphi\phi)^2$ and $d\Omega = \sin(\theta) d\theta d\varphi$.

1

Mode Expansion for FLRW Spacetimes

1 Introduction

Experimental observations of the Cosmic Microwave Background suggest that the universe in which we live is homogeneous and isotropic. If we describe the spatial slices of spacetime through a Riemannian manifold, as it is done in General Relativity with remarkable experimental confirmations, the isotropy and homogeneity assumptions imply that such a manifold has constant sectional curvature K. Therefore, the Killing-Hopf theorem holds (see also [17], theorem 12.4): every spatial manifold that satisfies these assumptions falls in one of the following three classes:

- Positive curvature manifolds ($K = 1$, spherical case),
- Zero curvature manifolds ($K = 0$, flat case),
- Negative curvature manifolds ($K = -1$, hyperbolic case).

These classes include quotient spaces: for instance, the torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ is a zero curvature manifold. A prominent role in this sense is played by FLRW (Friedmann–Lemaître–Robertson–Walker) models. Given a function $a(t)$ (also called "scale factor"), such models are given by the metric

$$g = -dt^2 + a^2(t)d\Sigma_n^2, \tag{1.1}$$

where $d\Sigma_n^2$ is the metric for the n-dimensional space sections of constant curvature K. In the following, the space dimension n will be greater than one. In the summation convention, we will use Latin characters when indices vary in $\{1, \dots, n\}$ and Greek characters when indices vary in $\{0, 1, \dots, n\}$. The metric (1.1) has a physical singularity for $a(t) = 0$*, corresponding either to the *Big Bang* or the *Big Crunch* (the latter being described by FLRW solutions with spherical space sections).

*The Ricci scalar is $R = 6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2}\right]$, see e.g. [5].

In this setting, we will study the wave equation

$$\Box_g \phi = 0. \tag{1.2}$$

Using the so-called *conformal time variable*

$$\tau = \int \frac{dt}{a(t)} \tag{1.3}$$

we can easily show that the d'Alembertian can be written as

$$\Box_g \phi = -\frac{1}{a^2}\partial_\tau^2 \phi - (n-1)\frac{a'}{a^3}\partial_\tau \phi + \frac{1}{a^2}\Delta_{\Sigma_n}\phi, \tag{1.4}$$

where Δ_{Σ_n} is the Laplace-Beltrami operator for the Riemannian manifold Σ_n and differentiation by τ is represented by primed characters. Indeed, this can be shown in the following way. The metric (1.1) can be written using the conformal time as

$$g = a^2(\tau)\left(-d\tau^2 + d\Sigma_n^2\right), \tag{1.5}$$

where $a(\tau) = a(t(\tau))$ with a small abuse of notation. Now,

$$\begin{aligned}
\Box_g \phi &= \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi\right) && \text{by definition}\\
&= \frac{1}{\sqrt{-g}}\partial_\tau\left(\sqrt{-g}g^{\tau\nu}\partial_\nu\phi\right) + \frac{1}{\sqrt{-g}}\partial_i\left(\sqrt{-g}g^{i\nu}\partial_\nu\phi\right) && \text{with } \tau \text{ not summed}\\
&= \frac{1}{\sqrt{-g}}\partial_\tau\left(\sqrt{-g}g^{\tau\tau}\partial_\tau\phi\right) + \frac{1}{\sqrt{-g}}\partial_i\left(\sqrt{-g}g^{ij}\partial_j\phi\right) && \text{since } g^{\tau i} = 0, g^{i\tau} = 0\\
&= -\frac{1}{\sqrt{-g}}\partial_\tau\left(\sqrt{-g}a^{-2}\partial_\tau\phi\right) + \frac{1}{\sqrt{-g}}\partial_i\left(\sqrt{-g}g^{ij}\partial_j\phi\right) && \text{using } g^{\tau\tau} = -a^{-2}\\
&= -\frac{1}{a^{n+1}\cancel{\sqrt{\tilde{g}}}}\partial_\tau\left(a^{n+1}\cancel{\sqrt{\tilde{g}}}a^{-2}\partial_\tau\phi\right) + a^{-2}\frac{1}{\sqrt{\tilde{g}}}\partial_i\left(\sqrt{\tilde{g}}\tilde{g}^{ij}\partial_j\phi\right), && (1.6)
\end{aligned}$$

where in the last step we used that $\sqrt{-g} = a^{n+1}\sqrt{\tilde{g}}$, with $\tilde{g}$ metric of Σ_n (for instance, $\tilde{g} = \left(dx^1\right)^2 + \ldots + \left(dx^n\right)^2$ for the flat n-dimensional case). Simplifications in the last line follow from the fact that a does not depend on spatial variables and that $\tilde{g}$ does not depend on conformal time. We notice that the elements of the inverse metric were computed using the fact that the inverse of a block-diagonal matrix is the matrix of the inverted blocks. Using the Leibniz rule for the first term of (1.6), and noticing that

$$\Delta_{\Sigma_n}\phi = \frac{1}{\sqrt{\tilde{g}}}\partial_i\left(\sqrt{\tilde{g}}\tilde{g}^{ij}\partial_j\phi\right), \tag{1.7}$$

we finally have equation (1.4). From (1.4), we can rewrite the d'Alembertian in the time variable t:

$$\Box_g \phi = -\ddot{\phi} - n\frac{\dot{a}}{a}\dot{\phi} + \frac{1}{a^2}\Delta_{\Sigma_n}\phi, \tag{1.8}$$

where we used the definition (1.3) of the conformal time variable: $\partial_\tau = a\partial_t$ and $\partial_\tau^2 = a\dot{a}\partial_t + a^2\partial_t^2$.

2 Toroidal Space Sections

We will now consider the flat case, so:

$$d\Sigma_n^2 = \left(dx^1\right)^2 + \ldots + \left(dx^n\right)^2 \tag{2.1}$$

and thus

$$g = -dt^2 + a^2(t)\left(\left(dx^1\right)^2 + \ldots + \left(dx^n\right)^2\right). \tag{2.2}$$

In this context, the dot above functions will denote differentiation with respect to t. Having this background and knowing that the Laplacian operator on $\mathbb{R}^n$ is $\Delta = \delta^{ij}\partial_i\partial_j$, we will study the wave equation (1.8):

$$-\ddot{\phi} - \frac{n\dot{a}}{a}\dot{\phi} + \frac{1}{a^2}\delta^{ij}\partial_i\partial_j\phi = 0. \tag{2.3}$$

Let us now analyse equation (2.3) through a Fourier mode analysis on the torus $\mathbb{T}_b^n = \mathbb{R}/(b_1\mathbb{Z}) \times \ldots \times \mathbb{R}/(b_n\mathbb{Z})$, with $b_i \in \mathbb{R}_{>0}$ for $i = 1, \ldots, n$. First of all, we define

$$\langle m, x\rangle_b := \frac{2\pi}{b_1}m_1x_1 + \ldots + \frac{2\pi}{b_n}m_nx_n \tag{2.4}$$

for $m \in \mathbb{Z}^n$, $x \in \mathbb{T}_b^n$. Now, any smooth function $\phi\colon \mathbb{R} \times \mathbb{T}_b^n \to \mathbb{R}$ can be expanded as

$$\phi(t, x) = \sum_{m\in\mathbb{Z}^n} c_m(t)e^{i\langle m,x\rangle_b}, \tag{2.5}$$

due to the fact that

$$\left\{\prod_{j=1}^{n}\frac{1}{\sqrt{b_j}}e^{i\langle m,x\rangle_b}\right\}_{m\in\mathbb{Z}^n} \tag{2.6}$$

is an orthonormal basis of $L^2(\mathbb{T}_b^n)$. The functions

$$c_m(t) = \frac{1}{\sqrt{b_1 \cdot \ldots \cdot b_n}}\int_{\mathbb{T}_b^n}\phi(t, x)e^{-i\langle m,x\rangle_b}d^nx \tag{2.7}$$

are the Fourier coefficients of $\phi(t, x)$. By replacing the Fourier expansion of $\phi(t, x)$ in (2.3), we get:

$$\ddot{c}_k + \frac{n\dot{a}}{a}\dot{c}_k + \frac{k^2}{a^2}c_k = 0, \tag{2.8}$$

where $k := 2\pi\left(\frac{m_1}{b_1}, \ldots, \frac{m_n}{b_n}\right)$, so $k^2 = \langle k, k\rangle = \sum_{j=1}^{n}\left(\frac{2\pi m_j}{b_j}\right)^2$, and we are denoting the Fourier coefficients by c_k with a small abuse of notation. We will study solutions of the differential equation (2.8) for different choices of the scale factor $a(t)$, generalizing the results in [8]. In the following we will have $n > 1$.

Let us consider the **case $a(t) = t$**. Equation (2.8) becomes:

$$\ddot{c}_k + \frac{n}{t}\dot{c}_k + \frac{k^2}{t^2}c_k = 0, \tag{2.9}$$

which is the Euler differential equation. Its solution is given by

$$c_k(t) = t^{-\frac{n-1}{2}}\left(C_1\, t^{-\frac{1}{2}\sqrt{(n-1)^2-4k^2}} + C_2\, t^{\frac{1}{2}\sqrt{(n-1)^2-4k^2}}\right). \tag{2.10}$$

It is interesting to analyse the behaviour of $c_k(t)$ for small and large times, corresponding to the behaviour of solutions at the metric singularity and infinitely far away in the future.

•) **Behaviour for $t \to \infty$:** If $(n-1)^2 - 4k^2 \geq 0$, we have that the dominant term has exponent

$$\begin{cases} \sqrt{(n-1)^2 - 4k^2} - (n-1) = 0, & \text{for } k = 0, \\ \sqrt{(n-1)^2 - 4k^2} - (n-1) < 0, & \text{for } k > 0. \end{cases} \tag{2.11}$$

So:

$$|c_k(t)| \sim \begin{cases} C, & k = 0, \\ C t^{\frac{1}{2}\left(\sqrt{(n-1)^2-4k^2}-(n-1)\right)}, & k > 0, \end{cases} \tag{2.12}$$

as $t \to +\infty$.

On the other hand, if $(n-1)^2 - 4k^2 < 0$, we can write the solution to the Euler equation as:

$$c_k(t) = t^{-\frac{n-1}{2}}\left(C_1\, t^{-\frac{i}{2}\sqrt{4k^2-(n-1)^2}} + C_2\, t^{\frac{i}{2}\sqrt{4k^2-(n-1)^2}}\right), \tag{2.13}$$

where the square roots contain positive quantities. Hence:

$$|c_k(t)| \sim C t^{-\frac{n-1}{2}} \text{ as } t \to \infty. \tag{2.14}$$

Since the first Fourier coefficient goes to a constant, while the remaining coefficients go to zero, we have that the solution of the wave equation converges to a constant.

•) **Behaviour at the Big Bang:** If $(n-1)^2 - 4k^2 \geq 0$, we saw that (2.11) holds. Thus:

$$|c_k(t)| \sim C t^{\frac{1}{2}\left(-\sqrt{(n-1)^2-4k^2}-(n-1)\right)} \text{ as } t \to 0^+, \tag{2.15}$$

i.e. the coefficients blow up at the Big Bang.

If $(n-1)^2 - 4k^2 < 0$, we can repeat the analysis that led to (2.13) and get the same expression. Thus, we have that

$$|c_k(t)| \sim C t^{-\frac{n-1}{2}} \text{ as } t \to 0^+ \tag{2.16}$$

and, again, the coefficients diverge at the Big Bang. The divergence of c_k, for every k, implies that the solution ϕ diverges as well at the Big Bang.

Let us now consider the **case $a(t) = t^p$**, $p \in \mathbb{R}_{>0} \setminus \{1\}$. We notice that the value $p = 0$, which we excluded, corresponds to the case in which the (2.8) gives a well-known wave equation in Minkowski spacetime. Whereas in the present case our differential equation (2.8) becomes:

$$\ddot{c}_k + \frac{np}{t}\dot{c}_k + \frac{k^2}{t^{2p}}c_k = 0. \tag{2.17}$$

For convenience, we define $\nu := \frac{1-np}{2(p-1)}$. The solutions are given by

$$c_k(t) = t^{\frac{1}{2}(1-np)}\left[C_1\, J_\nu\left(\frac{k t^{1-p}}{1-p}\right) + C_2\, Y_\nu\left(\frac{k t^{1-p}}{1-p}\right)\right], \tag{2.18}$$

where $k = |k| = \sqrt{k_1^2 + \ldots + k_n^2}$,

$$J_\alpha(z) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+\alpha+1)}\left(\frac{z}{2}\right)^{2m+\alpha} \tag{2.19}$$

is the Bessel function of the first kind and

$$Y_\alpha(z) = \frac{J_\alpha(z)\cos(\alpha\pi) - J_{-\alpha}(z)}{\sin(\alpha\pi)} \tag{2.20}$$

is the Bessel function of the second kind, for $\alpha \in \mathbb{R}$, $z \in \mathbb{C}$. When $\nu = m \in \mathbb{Z}$, $Y_m(z)$ is defined as

$$Y_m(z) = \lim_{\alpha \to m} Y_\alpha(z). \tag{2.21}$$

Now, we want to study the behaviour of the solution (2.18) in two cases: for $t \to \infty$ and at the Big Bang. Therefore, we will make use of the following estimates (see also [2]):

$$\begin{cases} J_\nu(z) = C z^\nu + O(z^{\nu+2}) \sim C z^\nu, & \text{for } \nu \neq -1, -2, \ldots, \text{ as } z \to 0 \\ J_\nu(z) \sim \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right), & \text{for every } \nu, \text{ as } z \to \pm\infty, \end{cases} \tag{2.22}$$

for some constant C, and

$$\begin{cases} Y_0(z) \sim C \log(z), & \text{as } z \to 0 \\ Y_\nu(z) = \left(A z^\nu + O(z^{\nu+2})\right) + \left(B z^{-\nu} + C z^{-\nu+2} + O(z^{-\nu+4})\right), & \text{for } \nu > 0, \text{ as } z \to 0 \\ Y_\nu(z) \sim \sqrt{\frac{2}{\pi z}} \sin\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right), & \text{for every } \nu, \text{ as } z \to \pm\infty, \end{cases} \tag{2.23}$$

for some constants A, B and C. We will often use the same letters as placeholders for different constants, since we are currently interested in the asymptotic analysis of the solutions. Estimates (2.22) and (2.23) show that our results for the asymptotic behaviour of the Fourier coefficients will not depend on the property of ν of being an integer, except for the case $\nu = 0$ (i.e. when $p = \frac{1}{n}$).

Now, let us suppose to have $\boxed{\boldsymbol{p > 1}}$.

•) **Behaviour for $t \to \infty$:** We have that the argument of the Bessel functions in (2.18) tends to 0: $\frac{t^{1-p}}{1-p} \underset{t\to\infty}{\longrightarrow} 0^-$. Here, the assumption $p > 1$ implies that $\nu > \frac{n}{2}$, so:

$$c_k(t) \sim t^{\frac{1}{2}(1-np)} \left(C_1 \left(t^{1-p}\right)^{\nu} + C_2 \left(t^{1-p}\right)^{\nu+2} \right) \text{ as } t \to \infty. \tag{2.24}$$

Using that $\nu = \frac{1-np}{2(p-1)}$:

$$c_k(t) \sim C_1 + C_2 t^{2-2p} \text{ as } t \to \infty. \tag{2.25}$$

Therefore, the wave ϕ converges to a constant as $t \to \infty$.

•) **Behaviour at the Big Bang:** The Big Bang is at $t = 0$. We have that $\frac{t^{1-p}}{1-p} \underset{t\to 0^+}{\longrightarrow} -\infty$. Therefore:

$$c_k(t) \sim t^{-\frac{n-1}{2}p} \left[C_1 \cos\left(\frac{k t^{1-p}}{1-p} - \frac{\nu\pi}{2} - \frac{\pi}{4} \right) + \right. \\ \left. + C_2 \cos\left(\frac{k t^{1-p}}{1-p} + \frac{\nu\pi}{2} - \frac{\pi}{4} \right) \right] \text{ as } t \to 0^+. \tag{2.26}$$

Therefore, the solution ϕ diverges at the Big Bang.

Now, let us consider the case $\boxed{\boldsymbol{0 < p < 1}}$. We defined

$$\nu = \frac{1-np}{2(p-1)} \tag{2.27}$$

and we noticed that the solution of equation (2.17), when $\nu \in \mathbb{Z}$, is different from the one that we have when $\nu \in \mathbb{R} \setminus \mathbb{Z}$. We will see that such a difference is irrelevant in the limit $t \to \infty$, whereas it becomes noticeable at the Big Bang.

•) **Behaviour for $t \to \infty$:** In this case, $\frac{t^{1-p}}{1-p} \underset{t\to\infty}{\longrightarrow} \infty$. First, we suppose that

$$\nu = \frac{1-np}{2(p-1)} \in \mathbb{R} \setminus \mathbb{Z}. \tag{2.28}$$

Therefore, the coefficients are:

$$c_k(t) \sim t^{-\frac{n-1}{2}p}\left[C_1 \cos\left(\frac{kt^{1-p}}{1-p}-\frac{\nu\pi}{2}-\frac{\pi}{4}\right)+ + C_2 \cos\left(\frac{kt^{1-p}}{1-p}+\frac{\nu\pi}{2}-\frac{\pi}{4}\right)\right] \text{ as } t \to \infty. \tag{2.29}$$

On the other hand, if

$$\nu = \frac{1-np}{2(p-1)} \in \mathbb{Z}, \tag{2.30}$$

we can check from the estimates (2.22) and (2.23) that the above result does not change. We can conclude that the solution of the wave equation decays with the above rate as $t \to \infty$.

•) **Behaviour at the Big Bang:** We have $\frac{t^{1-p}}{1-p} \underset{t\to 0^+}{\longrightarrow} 0^+$. We notice that now $\nu = \frac{1-np}{2(p-1)} \in (-\frac{1}{2}, +\infty)$, differently from what happened for $p \geq 1$. In particular, $\nu \in (-\frac{1}{2}, 0)$ if $0 < p < \frac{1}{n}$, whereas $\nu \in [0, +\infty)$ if $\frac{1}{n} \leq p < 1$. In order to simplify our analysis, we will only consider the first dominant term of the expansion of the Bessel functions.

If $0 < p < \frac{1}{n}$, for every value of ν we have:

$$c_k(t) \sim C\, t^{\frac{1}{2}(1-np)} t^{(1-p)\nu} = C \text{ as } t \to 0^+, \tag{2.31}$$

i.e. the solution ϕ goes to a constant.

If $\frac{1}{n} \leq p < 1$, let us assume that ν is not an integer. Then:

$$c_k(t) \sim C\, t^{\frac{1}{2}(1-np)} t^{(1-p)(-\nu)} = C\, t^{1-np} \text{ as } t \to 0^+. \tag{2.32}$$

So, since $np - 1 > 0$, $|c_k(t)|$ diverges polynomially as $t \to 0^+$ and the same can be said about the solution ϕ of the wave equation.

If ν is an integer, we can see from the estimates (2.22) and (2.23) that the result (2.32) will change only for $\nu = 0$. In that case, we have:

$$|c_k(t)| \sim \left| C \log\left(\frac{k}{1-p} t^{1-p}\right)\right| \text{ as } t \to 0^+, \tag{2.33}$$

i.e. the solution ϕ diverges logarithmically.

Let us now consider the **case $a(t) = e^t$** (flat de Sitter spacetime). The differential equation (2.8) becomes

$$\ddot{c}_k(t) + n\dot{c}_k(t) + k^2 e^{-2t} c_k(t) = 0. \tag{2.34}$$

For convenience, let us define $\nu = \frac{n}{2}$. The solution is given by:

$$c_k(t) = e^{-\nu t}\left[C_1\, J_\nu(ke^{-t}) + C_2\, Y_\nu(ke^{-t})\right] \tag{2.35}$$

where $J_\nu(z)$ and $Y_\nu(z)$ are the Bessel functions of the first and second kind defined, respectively, in (2.19) and (2.20). Since $\nu = \frac{n}{2} > 0$, we can suppose, without loss of generality, that ν is not an integer. Indeed, as we saw for the case $a(t) = t^p$, we have meaningful changes for the asymptotic behaviours of the Bessel functions only for $\nu = 0$.

We will now consider two possible limits.

•) **Behaviour for $t \to \infty$:** We have that the argument of the Bessel functions in (2.35) tends to 0: $e^{-t} \underset{t\to+\infty}{\longrightarrow} 0$. So, if we use the first estimate in (2.22):

$$c_k(t) \sim e^{-\nu t}\left(C_1(e^{-t})^{-\nu} + C_2(e^{-t})^{-\nu+2}\right) = C_1 + C_2 e^{-2t} \text{ as } t \to +\infty. \tag{2.36}$$

In particular, the decay of $\partial_t \phi$ as e^{-2t} follows from this result.

•) **Behaviour at the "Big Bang":** In this case, the Big Bang is infinitely far away in the past. We have that $e^{-t} \underset{t\to-\infty}{\longrightarrow} +\infty$. So, if we use the second estimate in (2.22):

$$c_k(t) \sim e^{-\nu t} e^{\frac{t}{2}}\left(C_1 \cos\left(ke^{-t} - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) + C_2 \cos\left(ke^{-t} + \frac{\nu\pi}{2} - \frac{\pi}{4}\right)\right) \tag{2.37}$$

as $t \to -\infty$. Since $\nu = \frac{n}{2}$:

$$c_k(t) \sim e^{-\frac{n-1}{2}t}\left(C_1 \cos\left(ke^{-t} - \frac{n\pi}{4} - \frac{\pi}{4}\right) + C_2 \cos\left(ke^{-t} + \frac{n\pi}{4} - \frac{\pi}{4}\right)\right). \tag{2.38}$$

So, $|c_k(t)|$ diverges at the Big Bang and the same is true for the solution ϕ of the wave equation.

3 Spherical Space Sections

We want to consider an FLRW model with n-dimensional spherical space sections. Its metric is given by

$$g = -dt^2 + a^2(t) d\Sigma_n^2, \tag{3.1}$$

where $d\Sigma_n^2$ is now the metric of the n-sphere. Using the expression (1.8) for the d'Alembertian, we now consider the wave equation:

$$-\ddot{\phi} - \frac{n\dot{a}}{a}\dot{\phi} + \frac{1}{a^2}\Delta_{S^n}\phi = 0. \tag{3.2}$$

For the spherical case, instead of expanding our function ϕ in terms of Fourier coefficients, we will use spherical harmonics. In fact, the latter form an orthonormal basis of $L^2(S^n)$. They are defined in the following way (see also [22]). Let $l \in \mathbb{N}$. We define

$$\begin{aligned} P_l &:= \{p\colon \mathbb{R}^{n+1} \to \mathbb{C} \mid p \text{ is a homogenous polynomial of degree } l\} = \\ &= \left\{p\colon \mathbb{R}^{n+1} \to \mathbb{C} \mid p \text{ is a polynomial, } p(\alpha x) = \alpha^l p(x)\; \forall\, \alpha \in \mathbb{R}, \forall\, x \in \mathbb{R}^{n+1}\right\}. \end{aligned} \tag{3.3}$$

Let us also consider

$$A_l := \{p \in P_l \mid \Delta p = 0\}. \tag{3.4}$$

Finally, the set

$$H_l := \{f : S^n \to \mathbb{C} \mid f(x) = p(x)\; \forall\, x \in S^n, \text{ for some } p \in A_l\} \tag{3.5}$$

is the set of spherical harmonics of degree l. We want to prove the following.

Claim 1.1 (Eigenfunctions of Δ_{S^n})**.** *For every $f \in H_l$, we have:*

$$\Delta_{S^n} f = -l(l+n-1)f. \tag{3.6}$$

Proof. Every $p \in P_l$ can be written in spherical coordinates as $p(r,\theta) = r^l g(\theta)$ for some function g depending on the angles. In fact, it follows from the fact that p is a homogenous polynomial of degree l and that the spherical coordinates are defined as

$$\begin{cases} x^1 = r\cos(\theta^1), \\ x^2 = r\sin(\theta^1)\cos(\theta^2), \\ \quad\vdots \\ x^{n-1} = r\sin(\theta^1)\cdot\ldots\cdot\sin(\theta^{n-1})\cos(\theta^n), \\ x^n = r\sin(\theta^1)\cdot\ldots\cdot\sin(\theta^{n-1})\sin(\theta^n). \end{cases} \tag{3.7}$$

Now, using the Euclidean metric in spherical coordinates

$$g = dr^2 + r^2 d\Sigma_{S^n} \tag{3.8}$$

and the formula (1.7), we can write the Laplacian on $\mathbb{R}^{n+1}$ in spherical coordinates:

$$\begin{aligned} \Delta_{\mathbb{R}^{n+1}}\phi &= \frac{1}{\sqrt{g}}\partial_\mu\left(\sqrt{g}g^{\mu\nu}\partial_\nu\phi\right) && \mu,\nu = r,1,2,\ldots \\ &= \frac{1}{\sqrt{g}}\partial_r\left(\sqrt{g}g^{r\nu}\partial_\nu\phi\right) + \frac{1}{\sqrt{g}}\partial_i\left(\sqrt{g}g^{i\nu}\partial_\nu\phi\right), && i = 1,2,\ldots \\ &= \frac{1}{r^n\sqrt{\tilde{g}}}\partial_r\left(r^n\sqrt{\tilde{g}}g^{rr}\partial_r\phi\right) + \frac{1}{\cancel{r^n}\sqrt{\tilde{g}}}\partial_i\left(\cancel{r^n}\sqrt{\tilde{g}}g^{ij}\partial_j\phi\right) && \text{where } r \text{ not summed} \\ &= \frac{n}{r}\partial_r\phi + \partial_r^2\phi + \frac{1}{r^2\sqrt{\tilde{g}}}\partial_i\left(\sqrt{\tilde{g}}\tilde{g}^{ij}\partial_j\phi\right) && \text{since } g^{rr} = 1, g^{ij} = \frac{1}{r^2}\tilde{g}^{ij} \\ &= \frac{n}{r}\partial_r\phi + \partial_r^2\phi + \frac{1}{r^2}\Delta_{S^n}\phi, \end{aligned} \tag{3.9}$$

where $\tilde{g}$ is the metric on S^n and we used that $\sqrt{g} = r^n \sqrt{\tilde{g}}$ and that $g^{ir} = 0$ since the metric g is block-diagonal. Therefore, we obtained:

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{n}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_{S^n}. \tag{3.10}$$

Using the above expression, we can evaluate the Laplacian of a homogeneous polynomial that is also a spherical polynomial (i.e. belongs to A_l):

$$\begin{aligned}\Delta(r^l g) &= \frac{\partial^2}{\partial r^2}\left(r^l g\right) + \frac{n}{r}\frac{\partial}{\partial r}\left(r^l g\right) + \frac{1}{r^2}\Delta_{S^n}(r^l g) = \\ &= l(l+n-1)r^{l-2}g(\theta) + \frac{1}{r^2}\Delta_{S^n}(r^l g(\theta)).\end{aligned} \tag{3.11}$$

If evaluated at $r = 1$, the (3.11) becomes

$$0 = l(l+n-1)g(\theta) + \Delta_{S^n}(g(\theta)). \tag{3.12}$$

By definition of the set H_l, there exists a function $f \in H_l$ that can be written as $f = g(\theta)$. Therefore:

$$\Delta_{S^n} f = -l(l+n-1) f. \tag{3.13}$$

Since the above reasoning is valid for every spherical polynomial, and since for every $f \in H_l$ we can find a spherical polynomial associated with it, the result is valid for every spherical harmonic. □

Now, every smooth function $\phi\colon \mathbb{R} \times S^n \to \mathbb{R}$ can be expanded as

$$\phi(t,x) = \sum_{l=0}^{\infty}\sum_{m=1}^{a_l} c_m^{(l)}(t) Y_m^{(l)}(x), \tag{3.14}$$

where $Y_m^{(l)} \in H_l$ for every $m = 1, \ldots, a_l$. Here, $a_l = \binom{n+l}{n} - \binom{n+l-2}{n}$. If we consider $n = 2$, we get $a_l = 2l + 1$ and our basis is the familiar $\{Y_0^{(0)}, Y_{-1}^{(1)}, Y_0^{(1)}, Y_1^{(1)}, \ldots\}$ by making a proper relabelling of the indices. For generic n, our equation (3.2) becomes

$$\ddot{c}_m^{(l)} + \frac{n\dot{a}}{a}\dot{c}_m^{(l)} + \frac{l(l+n-1)}{a^2}c_m^{(l)} = 0, \tag{3.15}$$

where we used the claim 1.1. Again, we can repeat the analysis done in section 2, with the only difference being the presence of $l(l+n-1)$ instead of k in the sinusoidal and logarithmic estimates. Therefore, the estimates for the coefficients $c_m^{(l)}$ only depend on the degree l.

So, let us analyse the **case $a(t) = \cosh(t)$** (corresponding to the de Sitter universe). We define $s := l(l+n-1)$ and $c_s := c_m^{(l)}$ as a shorthand notation. Now, let us replace $a(t)$ with $\cosh(t)$ in (3.15) :

$$\cosh^2(t)\,\ddot{c}_s + n\sinh(t)\cosh(t)\,\dot{c}_s + s\,c_s = 0. \tag{3.16}$$

This is solved by

$$c_s(t) = C_1 \left(\tanh^2(t) - 1\right)^{\frac{n}{4}} P_\beta^\alpha \left(\tanh(t)\right) + C_2 \left(\tanh^2(t) - 1\right)^{\frac{n}{4}} Q_\beta^\alpha \left(\tanh(t)\right), \tag{3.17}$$

where $P_\beta^\alpha(z)$ and $Q_\beta^\alpha(z)$ are, respectively, the Legendre functions of the first kind and of the second kind. We also defined

$$\alpha := \frac{n}{2}, \quad \beta := \frac{1}{2}\left(\sqrt{(n-1)^2 + 4s} - 1\right). \tag{3.18}$$

For $z \in \mathbb{C}, |z-1| < 2$ and for $\nu \in \mathbb{C}$, $\mu \in \mathbb{C} \setminus \{\pm 1, \pm 2, \ldots\}$, we have[†] (see also § 8, § 15 and eqn. 6.1.22 of [2]):

$$P_\nu^\mu(z) = \frac{1}{\Gamma(1-\mu)} \left(\frac{z+1}{z-1}\right)^{\frac{\mu}{2}} F\left(-\nu, \nu+1, 1-\mu, \frac{1-z}{2}\right) \tag{3.19}$$

and, when the arguments of the Gamma functions are not negative integers:

$$\begin{aligned} Q_\nu^\mu(z) = e^{i\mu\pi} \frac{\Gamma(1+\mu+\nu)\Gamma(-\mu)}{2\Gamma(1+\nu-\mu)} \left(\frac{z-1}{z+1}\right)^{\frac{\mu}{2}} F\left(-\nu, 1+\nu, 1+\mu, \frac{1-z}{2}\right) + \\ + \frac{1}{2} e^{i\mu\pi}\Gamma(\mu) \left(\frac{z+1}{z-1}\right)^{\frac{\mu}{2}} F\left(-\nu, 1+\nu, 1-\mu, \frac{1-z}{2}\right). \end{aligned} \tag{3.20}$$

Here, F is the Gauss hypergeometric series

$$F(a,b,c,z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}, \tag{3.21}$$

where $(a)_k$ represents the Pochhammer's symbol and it is defined by

$$\begin{cases} (a)_0 = 1, \\ (a)_k = a(a+1) \cdot \ldots \cdot (a+k-1), \quad \text{if } k > 0 \end{cases} \tag{3.22}$$

and the hypergeometric series converges for $|z| < 1$. The series in F might be ill-defined (for some values of a and b) when $c = m, m \in \{0, -1, -2, \ldots\}$, i.e. when $\mu \in \{1, 2, \ldots\}$. This is not a problem since we have already excluded such cases in the above definition. On the other hand, cases in which μ is a positive integer can be considered. In such contexts, we have different definitions of $P_\nu^\mu(z)$ and $Q_\nu^\mu(z)$ (e.g. see eqn. 14.3.4 in [11]). In the following, we will consider cases in which n is simultaneously an odd number and greater than one, so that $\alpha = \frac{n}{2}$ is not an integer. So, we can stick to the definitions (3.19) and (3.20). This includes the physically

[†]For $m, l \in \mathbb{N}$, $m < l$, the Legendre functions of the first kind define the spherical harmonics in S^2, i.e. we have either $Y_m^{(l)}(\theta, \phi) \propto \cos(m\phi) P_l^m(\cos\theta)$ or $Y_m^{(l)}(\theta, \phi) \propto \sin(m\phi) P_l^m(\cos\theta)$. See also [2], § 8.

relevant case $n = 3$. Now, we can rewrite P^α_β and Q^α_β in a different way. By using the above definitions, we get:

$$P^\alpha_\beta(z) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{z+1}{z-1}\right)^{\frac{\alpha}{2}} \sum_{k=0}^{\infty} \frac{(-\beta)_k(\beta+1)_k}{(1-\alpha)_k} \frac{1}{k!} \left(\frac{1-z}{2}\right)^k \tag{3.23}$$

and

$$Q^\alpha_\beta(z) = \frac{e^{i\alpha\pi}\Gamma(1+\beta+\alpha)\Gamma(-\alpha)\Gamma(1+\alpha)}{2\Gamma(1+\beta-\alpha)} P^{-\alpha}_\beta(z) + \frac{e^{i\alpha\pi}}{2}\Gamma(\alpha)\Gamma(1-\alpha)P^\alpha_\beta(z). \tag{3.24}$$

We stress that $Q^\alpha_\beta(z)$ is well defined since the arguments of the Gamma functions are never negative integers. In particular: $1+\beta-\alpha > 0$ and $1+\beta+\alpha > 0$. Indeed, using the assumption $n = 2k+1, k \in \{1, 2, \ldots\}$ and recalling that $s = l(l+n-1)$, we have:

$$2(\beta-\alpha) = \sqrt{(n-1)^2+4s} - 1 - n = \sqrt{(n-1)^2 + 4l(l+n-1)} - (n-1) \geq 0, \tag{3.25}$$

which implies: $\beta - \alpha \geq 0$. Then $\beta + \alpha \geq 0$ as well, since $\alpha = \frac{n}{2}$. Moreover, we notice that

$$P^\alpha_\beta(z) \sim C \left(\frac{z+1}{z-1}\right)^{\frac{\alpha}{2}} \quad \text{as } z \to 1^- \tag{3.26}$$

and

$$Q^\alpha_\beta(z) \sim D \left(\frac{z+1}{z-1}\right)^{\frac{\alpha}{2}} + E \left(\frac{z+1}{z-1}\right)^{\frac{\alpha}{2}} (z-1) \quad \text{as } z \to 1^- \tag{3.27}$$

for some constants C, D and E. In particular, the estimate for Q^α_β comes from equation (3.24) and it is determined by the first two terms of the expansion of P^α_β. Therefore, our solution $c_s(t)$ is such that:

$$c_s(t) \sim C_1 \left(\tanh^2(t) - 1\right)^{\frac{n}{4}} \left(\frac{\tanh(t)+1}{\tanh(t)-1}\right)^{\frac{n}{4}} + C_2 \left(\tanh^2(t)-1\right)^{\frac{n}{4}} \frac{(\tanh(t)+1)^{\frac{n}{4}}}{(\tanh(t)-1)^{\frac{n}{4}-1}} \tag{3.28}$$

as $t \to \infty$. By using that $\tanh(t) - 1 = -2\frac{e^{-t}}{e^t+e^{-t}}$, we get that

$$\begin{aligned} c_s(t) &\sim C_1 (\tanh(t)+1)^{\frac{n}{2}} + C_2 (\tanh(t)+1)^{\frac{n}{2}} (\tanh(t)-1) = \\ &= C_1 (\tanh(t)+1)^{\frac{n}{2}} + C_2 (\tanh(t)+1)^{\frac{n}{2}} \left(\frac{e^{-2t}}{1+e^{-2t}}\right), \end{aligned} \tag{3.29}$$

which is, the coefficients behave as a sum of a term that goes to a constant plus an exponentially decaying term. So, the asymptotic behaviour is the same that we got for $a(t) = e^t$ in the flat case (see (2.36)). In fact, for large times the two scale factors are asymptotically the same:

$$a(t) = \cosh(t) \sim e^t \text{ as } t \to \infty \tag{3.30}$$

and we found that the asymptotic behaviour of the waves is also the same independently of the space sections being spherical or flat.

We notice that there is no Big Bang for the scale factor $a(t) = \cosh(t)$, since $\cosh(t) \geq 1 \forall\, t \in \mathbb{R}$. Moreover, the behaviour as $t \to -\infty$ is analogous to the one described in (3.29) if we replace t with $-t$. To obtain such a result, we just need to repeat the above procedure using the estimates of $P_\alpha^\beta(z)$ and $Q_\alpha^\beta(z)$ as $z \to -1^+$. The relevant contributions come from the term $P_\beta^{-\alpha}$ in the expression (3.24) of Q_β^α.

2

An Operator Trick for FLRW Spacetimes

4 Introduction

When considering FLRW spacetimes to describe a homogeneous and isotropic universe, solutions of the wave equation can be investigated through operators that take them to solutions of the same equation in a simpler spacetime, the latter characterized by the same spatial curvature of the original universe. Klainerman and Sarnak used such a stratagem in [16] in order to study the solution of the wave equation in cosmological dust models for the flat and hyperbolic cases. We will follow this approach for the case $n = 3$ and the FLRW metric

$$g = -dt^2 + a^2(t)d\Sigma_3^2, \tag{4.1}$$

where $d\Sigma_3^2$ is the metric either for S^3 (spherical case, $K = 1$), $\mathbb{R}^3$ (flat case, $K = 0$) or for the hyperbolic space $\mathbb{H}^3$ (hyperbolic case, $K = -1$). The conformal time coordinate is defined as

$$\tau = \int \frac{dt}{a(t)}. \tag{4.2}$$

Differentiation with respect to τ will be represented by primed characters, whereas differentiation by t will be denoted by dotted characters. For the moment, we will consider smooth functions. Therefore, exchange of limits will be justified by Lebesgue's dominated convergence theorem. As we saw in section 1, equation (1.4), the wave equation $\Box_g \phi = 0$ becomes:

$$\partial_\tau^2 \phi + 2\frac{a'}{a}\partial_\tau \phi = \Delta_{\Sigma_3}\phi. \tag{4.3}$$

Now, let $\hat{O}$ be an operator on the space of smooth functions that commutes with Δ. If the assumption

$$\hat{O}\left(\partial_\tau^2 + 2\frac{a'}{a}\partial_\tau\right)\phi = \partial_\tau^2 \hat{O}\phi + K\hat{O}\phi \tag{4.4}$$

holds, where $K \in \{-1, 0, 1\}$ represents the curvature of the space we are considering, then equation (4.3) implies that

$$\partial_\tau^2 \hat{O}\phi = (\Delta_{\Sigma_3} - K)\hat{O}\phi. \tag{4.5}$$

So, if we suppose that a general function ϕ is a solution of the wave equation in the cosmological spacetime described by g and we choose the operator $\hat{O}$ such that (4.4) is true, then $\hat{O}\phi$ satisfies a generalized wave equation. In particular, $\hat{O}\phi$ satisfies the wave equation in Minkowski spacetime when $K = 0$, whereas it satisfies a wave equation containing an additional curvature term when we have either $K = -1$ (spatially-hyperbolic spacetime) or $K = 1$ (spherical space sections). We know the behaviour of $\hat{O}\phi$ since the Minkowski case and the static cases with hyperbolic or spherical space sections are easier to study. When $\hat{O}$ is simple enough, we can gain insight about the behaviour of ϕ from that of $\hat{O}\phi$. This is the modus operandi that we will follow.

5 The Friedmann Equations

Different choices of the scale factor $a(t)$ in the metric (4.1) will give rise to different spacetimes and will require different operators $\hat{O}$. Meaningful choices for $a(t)$ are given by inspecting the Friedmann equations, capable of describing an ideal cosmological fluid. The Friedmann equations in dimension 3 are given by (see also [6]):

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}\rho - \frac{K}{a^2} \tag{5.1}$$

and

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3}(\rho + 3p), \tag{5.2}$$

where K is the sectional curvature of the space, ρ is the effective energy density of the fluid, p is the effective pressure, and we used Planck's units. These equations are obtained by inserting our FLRW metric in the Einstein's equations with energy-momentum tensor

$$T_\mu{}^\nu = \mathrm{diag}(-\rho_m, p_m, p_m, p_m), \tag{5.3}$$

where

$$\rho = \rho_m + \frac{\Lambda}{8\pi}, \quad p = p_m - \frac{\Lambda}{8\pi}, \tag{5.4}$$

with ρ_m energy density, p_m pressure of the fluid and Λ is the cosmological constant. Now, the conservation of the energy-momentum tensor ($\nabla_\nu T_\mu{}^\nu = 0$) implies

$$\dot{\rho}_m + 3(\rho_m + p_m)\frac{\dot{a}}{a} = 0. \tag{5.5}$$

If we also assume the equation of state

$$p_m = w\rho_m \tag{5.6}$$

for a constant w, then equation (5.5) becomes

$$\dot{\rho}_m + 3(1 + w)\rho_m\frac{\dot{a}}{a} = 0. \tag{5.7}$$

The above can be solved for ρ_m by noticing that, if we multiply both sides by $a^{3(1+w)}$:

$$a^{3(1+w)}\dot{\rho}_m + 3(1+w)a^{3(1+w)-1}\dot{a}\rho_m = 0, \tag{5.8}$$

which is:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\rho_m a^{3(1+w)}\right) = 0. \tag{5.9}$$

Therefore: $\rho_m = \rho_0 \left(\frac{a_0}{a}\right)^{3(1+w)}$ or, in a shorter way:

$$\rho_m = \rho_0 a^{-3(1+w)} \tag{5.10}$$

for a constant ρ_0. Using this result and the definition of ρ in equation (5.4), the first Friedmann equation (5.1) can be written as

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}\rho_0 a^{-3(1+w)} + \frac{\Lambda}{3} - \frac{K}{a^2}, \tag{5.11}$$

or, using conformal time ($d\tau = a^{-1}dt \Rightarrow \partial_t = a^{-1}\partial_\tau$):

$$\left(\frac{a'}{a^2}\right)^2 = \frac{8\pi}{3}\rho_0 a^{-3(1+w)} + \frac{\Lambda}{3} - \frac{K}{a^2}. \tag{5.12}$$

Interesting values of w are the following (see also [13], Appendix A):

w	Fluid type
$w = 0$	dust
$w = \frac{1}{3}$	radiation
$w = 1$	stiff fluid
$w > 1$	unphysical (speed of sound > 1)

Figure 2.1: Fluid solutions to Friedmann's equations for different values of w

For $\Lambda = 0$, $K = 0$ (cosmological constant set to zero, flat case), solutions of (5.12) are given by

$$a(\tau) = \begin{cases} e^\tau, & \text{if } w = -\frac{1}{3}, \\ \tau^{\frac{2}{1+3w}}, & \text{otherwise,} \end{cases} \tag{5.13}$$

with an appropriate choice of ρ_0. We will analyse more in-depth the following scenarios:

$$a(\tau) = \begin{cases} \tau^{-1} & \text{(de Sitter)}, \\ \tau^2 & \text{(dust)}, \\ \tau & \text{(radiation)}. \end{cases} \tag{5.14}$$

More precisely, the de Sitter solution corresponds to the case $\Lambda > 0$, $\rho_m = 0$. However, part of it can be described as a Friedmann universe for the case $w = -1$.

For $\Lambda = 0$, $K = -1$ (cosmological constant set to zero, hyperbolic case), solutions of (5.12) are given by

$$a(\tau) = \begin{cases} e^{\tau}, & \text{if } w = -\frac{1}{3}, \\ \left[\sinh\left(\frac{1}{2}(3w+1)\tau\right)\right]^{\frac{2}{3w+1}}, & \text{otherwise.} \end{cases} \tag{5.15}$$

In particular, in the next sections we will analyse the cases

$$a(\tau) = \begin{cases} \cosh(\tau) - 1 & \text{(dust)}, \\ \sinh(\tau) & \text{(radiation)}, \\ \operatorname{sech}(\tau) & \text{(anti-de Sitter)}, \\ -\operatorname{csch}(\tau) & \text{(de Sitter)}, \\ e^{\tau} & \text{(Milne universe)}. \end{cases} \tag{5.16}$$

which are exact solutions of (5.15) for a suitable choice of ρ_0.

Finally, for $\Lambda = 0$, $K = 1$ (cosmological constant set to zero, spherical case), we will consider the following solutions of (5.12):

$$a(\tau) = \begin{cases} 1 - \cos(\tau), & \text{(dust)}, \\ \sin(\tau), & \text{(radiation)}, \end{cases} \tag{5.17}$$

which are obtained again after choosing a proper ρ_0.

6 First Order Operators

We will start our research for a suitable operator $\hat{O}$ by considering

$$\hat{O}\phi = f(\tau)\partial_\tau\phi + g(\tau)\phi, \tag{6.1}$$

where $f(\tau)$ and $g(\tau)$ are functions to be determined. We notice that $\hat{O}$ commutes with the Laplacian. In order to satisfy (4.4), we need that

$$\hat{O}\left(\partial_\tau^2 + 2\frac{a'}{a}\partial_\tau\right)\phi - \partial_\tau^2\hat{O}\phi - K\hat{O}\phi = \left(2\frac{a'}{a}f - 2f'\right)\partial_\tau^2\phi + \\ + \left[2\frac{a'}{a}g - 2g' - f'' + 2\left(\frac{a'}{a}\right)' f - Kf\right]\partial_\tau\phi - (g'' + Kg)\phi \tag{6.2}$$

is equal to 0. This holds if and only if the coefficients of $\partial_\tau^2\phi$, $\partial_\tau\phi$ and ϕ are equal to 0. We will now distinguish among the cases $K = 0$ (flat space), $K = -1$ (hyperbolic space) and $K = 1$ (spherical space).

7 First Order Operators for the Flat Case

Now, let $K = 0$. The right hand side of equation (6.2) is equal to 0 if and only if we assume that

$$\begin{cases} f(\tau) = \kappa a, \\ g(\tau) = \alpha\tau + \beta, \end{cases} \tag{7.1}$$

and

$$2\frac{a'}{a}g - 2g' - f'' + 2\left(\frac{a'}{a}\right)' f = 0, \tag{7.2}$$

with $\alpha, \beta, \kappa \in \mathbb{R}$. Let us consider the case

$$a(\tau) = \tau^j, \tag{7.3}$$

so that

$$f(\tau) = \kappa\tau^j. \tag{7.4}$$

Then, equation (7.2) becomes

$$2j(\alpha + \beta\tau^{-1}) - 2\alpha - \kappa j(j+1)\tau^{j-2} = 0. \tag{7.5}$$

We can analyse the coefficients for each power of τ. In particular, if $j \neq 1, 2$, the above equation is satisfied if

$$\begin{cases} \alpha(j-1) = 0, \\ j\beta = 0, \\ \kappa j(j+1) = 0. \end{cases} \tag{7.6}$$

On the other hand, if $j = 2$, the equation is satisfied when

$$\begin{cases} \alpha = 3\kappa, \\ \beta = 0. \end{cases} \tag{7.7}$$

Finally, if $j = 1$, we need:

$$\beta = \kappa. \tag{7.8}$$

So, there are non-trivial solutions f and g that satisfy the constraint (7.5) for $j \in \{0, -1, 1, 2\}$, choosing α, β, κ appropriately. We will analyse the four possibilities in the next sections.

7.1 Minkowski Space

If $j = 0$, the conditions (7.6) force us to choose $\alpha = 0$ so that the functions f and g become

$$\begin{cases} f(\tau) = \kappa, \\ g(\tau) = \beta. \end{cases} \tag{7.9}$$

In this case $a(\tau) = 1$, so this corresponds to Minkowski space. Indeed, if ϕ is a solution of the wave equation in Minkowski space, then

$$\hat{O}\phi = \kappa\partial_\tau\phi + \beta\phi \tag{7.10}$$

is a solution as well.

7.2 De Sitter Space

If $j = -1$, we choose $\alpha = \beta = 0$ so that the functions f and g become

$$\begin{cases} f(\tau) = \kappa\tau^{-1}, \\ g(\tau) = 0. \end{cases} \tag{7.11}$$

These lead to the operator $\hat{O}\phi = \tau^{-1}\partial_\tau\phi$, where we have put $\kappa = 1$ since such a factor cancels out in the wave equation (4.5). In this case:

$$a(\tau) = \tau^{-1} \tag{7.12}$$

and so

$$\frac{dt}{d\tau} = a(\tau) \Leftrightarrow t = \log\tau. \tag{7.13}$$

By changing the time coordinate from t to $-t$:

$$a(t) = e^t. \tag{7.14}$$

This case corresponds to the expanding flat de Sitter space. We want to study the behaviour for $\tau \to \infty$ and compare it with estimates obtained through different methods. If we consider the following initial data for the wave equation at $t = t_0$:

$$\begin{cases} \phi(t_0, x) = \phi_0(x), \\ \partial_t\phi(t_0, x) = \phi_1(x), \end{cases} \tag{7.15}$$

the initial data for

$$\hat{O}\phi = \tau^{-1}\partial_\tau\phi = \tau^{-1}\frac{dt}{d\tau}\partial_t\phi = \tau^{-2}\partial_t\phi \tag{7.16}$$

at the corresponding conformal time $\tau = \tau_0$ is given by

$$\begin{cases} \hat{O}\phi(\tau_0, x) = \tau_0^{-2}\phi_1(x), \\ \partial_\tau\hat{O}\phi(\tau_0, x) = \tau_0^{-1}\Delta\phi_0(x) + \tau_0^{-3}\phi_1(x), \end{cases} \tag{7.17}$$

where we used that $\partial_t = \tau\partial_\tau$, $\partial_t^2\phi = \Delta\phi + 2\tau^{-1}\partial_\tau\phi$ (see equation (4.3)) and therefore:

$$\partial_\tau\hat{O}\phi = \partial_\tau(\tau^{-1}\partial_\tau\phi) = \tau^{-1}\partial_\tau^2\phi - \tau^{-2}\partial_\tau\phi = \tau^{-1}\Delta\phi + \tau^{-2}\partial_\tau\phi = \tau^{-1}\Delta\phi + \tau^{-3}\partial_t\phi. \tag{7.18}$$

Now, if the appropriate Sobolev norms of the initial data and of the Laplacian $\Delta\phi_0$ are finite, then by the boundedness of the solutions to the wave equation in finite time intervals of Minkowski space, we have

$$|\hat{O}\phi| \lesssim 1 \Leftrightarrow |\partial_t\phi| \lesssim \tau^2 = e^{-2t}, \tag{7.19}$$

where we used that $\tau = e^{-t}$ (recall that we inverted the time flow to get the expanding space-time).

7.3 Einstein-de Sitter Universe (Dust-filled Flat FLRW Model)

If $j = 2$, the conditions (7.7) force the functions f and g to be

$$\begin{cases} f(\tau) = \kappa\tau^2, \\ g(\tau) = 3\kappa\tau, \end{cases} \tag{7.20}$$

and therefore $\hat{O}\phi = \tau^2\partial_\tau\phi + 3\tau\phi$, where again we have put $\kappa = 1$ without loss of generality. In this case:

$$a(\tau) = \tau^2 \tag{7.21}$$

and

$$\frac{dt}{d\tau} = a(\tau) \Leftrightarrow t \propto \tau^3. \tag{7.22}$$

It follows that

$$a(t) \propto t^{\frac{2}{3}}. \tag{7.23}$$

The scale factor given by equation (7.23) is a solution for the Friedmann equation (5.11) (case $K = 0, \Lambda = 0$) if $w = 0$. Therefore, it corresponds to a dust-filled universe.

Given initial data for the wave equation at $t = t_0$,

$$\begin{cases} \phi(t_0, x) = \phi_0(x), \\ \partial_t\phi(t_0, x) = \phi_1(x), \end{cases} \tag{7.24}$$

we have the following initial data for

$$\hat{O}\phi = \tau^2\partial_\tau\phi + 3\tau\phi = \tau^2\frac{dt}{d\tau}\partial_t\phi + 3\tau\phi = \tau^4\partial_t\phi + 3\tau\phi \tag{7.25}$$

at the corresponding conformal time $\tau = \tau_0$:

$$\begin{cases} \hat{O}\phi(\tau_0, x) = \tau_0^4\,\phi(x) + 3\tau_0\phi_0(x), \\ \partial_\tau\hat{O}\phi(\tau_0, x) = \tau_0^2\,\Delta\phi_0(x) + \tau_0^3\,\phi_1(x) + 3\phi_0(x). \end{cases} \tag{7.26}$$

Here, we used that $\partial_t = \tau^{-2}\partial_\tau$, $\partial_\tau^2\phi = \Delta\phi - 4\tau^{-1}\partial_\tau\phi$ (see equation (4.3)) and therefore:

$$\partial_\tau \hat{O}\phi = \partial_\tau(\tau^2\partial_\tau\phi + 3\tau\phi) = \\ = \tau^2\partial_\tau^2\phi + 5\tau\partial_\tau\phi + 3\phi = \tau^2\Delta\phi + \tau\partial_\tau\phi + 3\phi = \tau^2\Delta\phi + \tau^3\partial_t\phi + 3\phi. \quad (7.27)$$

If the appropriate Sobolev norms of the initial data and of the Laplacian $\Delta\phi_0$ are finite, then by the standard decay of the wave equation in Minkowski space we have, as $\tau \to +\infty$,

$$|\hat{O}\phi| \lesssim \tau^{-1} \Leftrightarrow |\tau^3\partial_\tau\phi + 3\tau^2\phi| \lesssim 1 \Leftrightarrow |\partial_\tau(\tau^3\phi)| \lesssim 1. \quad (7.28)$$

Integrating:

$$|\phi| \lesssim \tau^{-2} \sim t^{-\frac{2}{3}}, \quad (7.29)$$

which is the result that can be derived from the Fourier mode analysis as well (see equation (2.29), for $n = 3$, $p = \frac{2}{3}$). A decay rate of t^{-1}, the same as in Minkowski space, can be obtained from a more refined analysis using the Kirchhoff's (spherical means) formula. Renaming the initial data in (7.26) for the operator $\hat{O}$:

$$\begin{cases} \psi_0(x) = \tau_0^4\phi_1(x) + 3\tau_0\phi_0(x), \\ \psi_1(x) = \tau_0^2\Delta\phi_0(x) + \tau_0^3\phi_1(x) + 3\phi_0(x), \end{cases} \quad (7.30)$$

Kirchhoff's formula gives

$$\hat{O}\phi(\tau, x) = \frac{1}{4\pi(\tau-\tau_0)^2}\int_{\partial B_{\tau-\tau_0}(x)} \psi_0(y)dV_2(y) + \\ + \frac{1}{4\pi(\tau-\tau_0)}\int_{\partial B_{\tau-\tau_0}(x)} \nabla\psi_0(y)\cdot\frac{y-x}{|y-x|}dV_2(y) + \\ + \frac{1}{4\pi(\tau-\tau_0)}\int_{\partial B_{\tau-\tau_0}(x)} \psi_1(y)dV_2(y). \quad (7.31)$$

Now, using equation (7.25):

$$\hat{O}\phi = \tau^{-1}\partial_\tau(\tau^3\phi). \quad (7.32)$$

By replacing the above in equation (7.31), multiplying both sides by τ and integrating, we have

$$\tau^3\phi(\tau, x) - \tau_0^3\phi_0(x) = \int_{\tau_0}^{\tau}\frac{s}{4\pi(s-\tau_0)^2}\int_{\partial B_{s-\tau_0}(x)} \psi_0(y)dV_2(y)ds + \\ + \int_{\tau_0}^{\tau}\frac{s}{4\pi(s-\tau_0)}\int_{\partial B_{s-\tau_0}(x)} \nabla\psi_0(y)\cdot\frac{y-x}{|y-x|}dV_2(y)ds + \\ + \int_{\tau_0}^{\tau}\frac{s}{4\pi(s-\tau_0)}\int_{\partial B_{s-\tau_0}(x)} \psi_1(y)dV_2(y)ds \quad (7.33)$$

We notice that:

$$\left| \int_{\tau_0}^{2\tau_0} \frac{s}{4\pi(s-\tau_0)^2} \int_{\partial B_{s-\tau_0}(x)} \psi_0(y) dV_2(y) ds \right| =$$
$$= \left| \int_{\tau_0}^{2\tau_0} \frac{s}{4\pi} \int_{S^2} \psi_0(x+(s-\tau_0)z) dV_2(z) ds \right| \lesssim \|\psi_0\|_\infty, \tag{7.34}$$

where we used the change of variable $z = \frac{y-x}{s-\tau_0}$. Moreover, since $\frac{s}{4\pi(s-\tau_0)^2} \lesssim 1$ for $s > 2\tau_0$, we have that

$$\left| \int_{\tau_0}^{\tau} \frac{s}{4\pi(s-\tau_0)^2} \int_{\partial B_{s-\tau_0}(x)} \psi_0(y) dV_2(y) ds \right| \lesssim \|\psi_0\|_\infty + \|\psi_0\|_1, \tag{7.35}$$

(simply by splitting the integral $\int_{\tau_0}^{\tau} \dots = \int_{\tau_0}^{2\tau_0} \dots + \int_{2\tau_0}^{\tau} \dots$), and similarly for the other integrals in (7.33). Therefore:

$$|\phi| \lesssim \tau^{-3}(\|\psi_0\|_\infty + \|\psi_0\|_1 + \|\nabla\psi_0\|_\infty + \|\nabla\psi_0\|_1 + \|\psi_1\|_\infty + \|\psi_1\|_1 + \|\phi_0\|_\infty). \tag{7.36}$$

Here, $\|\cdot\|_\infty$ and $\|\cdot\|_1$ are the usual L^∞ and L^1 norms, i.e.

$$\begin{cases} \|\psi_0\|_\infty = \|\psi_0\|_{L^\infty(\mathbb{R}^n)} = \operatorname{ess\,sup}_{x\in\mathbb{R}^n} |\psi_0(x)|, \\ \|\psi_0\|_1 = \|\psi_0\|_{L^1(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |\psi_0(x)| dx. \end{cases} \tag{7.37}$$

So, provided that the relevant norms are finite:

$$|\phi| \lesssim t^{-1}, \tag{7.38}$$

since $\tau \propto t^{\frac{1}{3}}$. This decay rate can be divided into a part ($t^{-\frac{2}{3}}$) which is due to the expansion (it occurs even when the spatial sections are compact flat tori) and a part (the remaining $t^{-\frac{1}{3}}$) which is due to dispersion: since the space is expanding, the dispersion is not as effective as in Minkowski space, where it accounts for the whole decay rate t^{-1}.

Since the FLRW metric is singular at $\tau = 0$, it is interesting to study the solutions for $\tau_0 \to 0$, as addressed in [1]. In general, we expect the solutions to diverge due to the presence of the physical singularity. However, it is possible to construct specific solutions which are well-defined in the limit $\tau \to 0$. If we fix ϕ_0, ϕ_1, τ and take the limit $\tau_0 \to 0$ in equation (7.33), we obtain:

$$\tau^3\phi(\tau,x) = \int_0^\tau \frac{1}{4\pi s} \int_{\partial B_s(x)} \lim_{\tau_0\to 0} \psi_0(y) dV_2(y) ds +$$
$$+ \int_0^\tau \frac{1}{4\pi} \int_{\partial B_s(x)} \lim_{\tau_0\to 0} \nabla\psi_0(y) \cdot \frac{y-x}{|y-x|} dV_2(y) ds + \tag{7.39}$$
$$+ \int_0^\tau \frac{1}{4\pi} \int_{\partial B_s(x)} \lim_{\tau_0\to 0} \psi_1(y) dV_2(y) ds,$$

or, defining $z = \frac{y-x}{s}$:

$$\begin{aligned}\tau^3\phi(\tau, x) = &\int_0^\tau \frac{s}{4\pi}\int_{S^2}\lim_{\tau_0\to 0}\psi_0(x+sz)dV_2(z)ds + \\ &+\int_0^\tau \frac{s^2}{4\pi}\int_{S^2}\lim_{\tau_0\to 0}\nabla\psi_0(x+sz)\cdot z\, dV_2(z)ds + \\ &+\int_0^\tau \frac{s^2}{4\pi}\int_{S^2}\lim_{\tau_0\to 0}\psi_1(x+sz)dV_2(z)ds.\end{aligned} \tag{7.40}$$

Furthermore, from (7.30) we can see that $\psi_0 \underset{\tau_0\to 0}{\longrightarrow} 0$ and $\psi_1 \underset{\tau_0\to 0}{\longrightarrow} 3\phi_0$, whence the limiting solution is

$$\phi(\tau, x) = \frac{1}{\tau^3}\int_0^\tau \frac{3s^2}{4\pi}\int_{S^2}\phi_0(x+sz)dV_2(z)ds. \tag{7.41}$$

As expected, L'Hôpital's rule gives

$$\lim_{\tau\to 0}\phi(\tau, x) = \lim_{\tau\to 0}\frac{1}{3\tau^2}\frac{3\tau^2}{4\pi}\int_{S^2}\phi_0(x+\tau z)dV_2(z) = \phi_0(x). \tag{7.42}$$

A different way to construct a well-defined solution at the Big Bang is the following. If ψ_0 and ψ_1, instead of ϕ_0 and ϕ_1, are kept constant when taking the limit as $\tau_0 \to 0$, equation (7.40) and L'Hôpital's rule imply

$$\begin{aligned}\lim_{\tau\to 0}\phi(\tau, x) = \lim_{\tau\to 0}\frac{1}{3\tau^2}\bigg(&\frac{1}{4\pi\tau}\int_{S^2}\psi_0(x+\tau z)dV_2(z) + \\ &+\frac{\tau^2}{4\pi}\int_{S^2}\nabla\psi_0(x+\tau z)\cdot z dV_2(z) + \\ &+\frac{\tau^2}{4\pi}\int_{S^2}\psi_1(x+\tau z)dV_2(z)\bigg).\end{aligned} \tag{7.43}$$

Thus, ψ_0 must vanish so that $\phi(\tau, x)$ does not diverge near $\tau = 0$. Recalling the definitions of ψ_0 and ψ_1 in (7.30), it means that we must choose $\phi_1 = -3\tau_0^{-3}\phi_0$ and $\Delta\phi_0 = \tau_0^{-2}\psi_1$. In this case both ϕ_0 and ϕ_1 blow up as $\tau_0 \to 0$, but the limiting solution is well-defined and tends to ψ_1 as $\tau \to 0$.

7.4 Radiation-filled Flat FLRW Model

If $j = 1$, the conditions (7.8) tell us that the functions f and g must be

$$\begin{cases} f(\tau) = \kappa\tau, \\ g(\tau) = \alpha\tau + \kappa. \end{cases} \tag{7.44}$$

In this case:

$$a(\tau) = \tau \tag{7.45}$$

and

$$\frac{dt}{d\tau} = a(\tau) \Leftrightarrow t \propto \tau^2, \tag{7.46}$$

which implies

$$a(\tau) \propto t^{\frac{1}{2}}. \tag{7.47}$$

The scale factor $a(t) \propto t^{\frac{1}{2}}$ is a solution for the Friedmann equation (5.11) (case $K = 0, \Lambda = 0$) if $w = \frac{1}{3}$. Therefore, the current model corresponds to a radiation fluid. We notice that in this case $\hat{O}$ can be taken as the multiplicative operator

$$\hat{O}\phi = \tau\phi \tag{7.48}$$

(i.e. taking $g(\tau) = \tau$). The reason for this is that the energy-momentum tensor of a radiation fluid is traceless:

$$T = T_\mu{}^\mu \overset{(5.3)}{=} -\rho + 3p \overset{(5.6)}{=} 0 \tag{7.49}$$

and so the scalar curvature of the radiation-filled FLRW model vanishes:

$$R = -8\pi T = 0. \tag{7.50}$$

Consequently, the wave equation coincides with the conformally invariant wave equation and so, the scalar field $a\phi$ satisfies the wave equation in Minkowski space (with the conformal time coordinate τ). Notice that, in this case:

$$\hat{O}\phi = \tau\phi = a\phi \tag{7.51}$$

from (7.45). The other independent possibility for $\hat{O}$ that we can get from (7.44) is

$$\hat{O}\phi = \tau\partial_\tau\phi + \phi = \partial_\tau(\tau\phi), \tag{7.52}$$

which is the composition of a time derivative with the operator (7.51). In fact, any operator of the form

$$\hat{O}\phi = \partial_\tau^k(\tau\phi) = \tau\partial_\tau^k\phi + k\partial_\tau^{k-1}\phi \tag{7.53}$$

will satisfy the assumption (4.4).

Given initial data for the wave equation at $t = t_0$:

$$\begin{cases} \phi(t_0, x) = \phi_0(x), \\ \partial_t\phi(t_0, x) = \phi_1(x), \end{cases} \tag{7.54}$$

we have the following initial data for $\hat{O}\phi = \tau\phi$ at the corresponding conformal time $\tau = \tau_0$:

$$\begin{cases} \hat{O}\phi(\tau_0, x) = \tau_0\phi_0(x), \\ \partial_\tau\hat{O}\phi(\tau_0, x) = \tau_0^2\phi_1(x) + \phi_0(x), \end{cases} \tag{7.55}$$

where we used that $\partial_\tau = \tau\partial_t$ and thus:

$$\partial_\tau \hat{O}\phi = \partial_\tau(\tau\phi) = \tau\partial_\tau\phi + \phi = \tau^2\partial_t\phi + \phi. \tag{7.56}$$

Therefore, if the appropriate Sobolev norms of the initial data are finite, then by the standard decay of the wave equation in Minkowski space (see e.g. [15]) we have, as $\tau \to +\infty$:

$$|\hat{O}\phi| \lesssim \tau^{-1} \Leftrightarrow |\phi| \lesssim \tau^{-2} \sim t^{-1}, \tag{7.57}$$

the same decay rate as in Minkowski space. This is better than the result that can be derived from the Fourier mode analysis (see equation (2.29) with $n = 3$, $p = \frac{1}{2}$). Again, the extra decay rate can be attributed to dispersion.

The behaviour of ϕ near the Big Bang can be analysed using the Kirchhoff's (spherical means) formula. Setting

$$\begin{cases} \psi_0(x) = \tau_0\phi_0(x), \\ \psi_1(x) = \tau_0^2\phi_1(x) + \phi_0(x), \end{cases} \tag{7.58}$$

Kirchhoff's formula gives

$$\begin{aligned} \hat{O}\phi(\tau, x) = & \frac{1}{4\pi(\tau-\tau_0)^2}\int_{\partial B_{\tau-\tau_0}(x)} \psi_0(y)dV_2(y) + \\ & + \frac{1}{4\pi(\tau-\tau_0)}\int_{\partial B_{\tau-\tau_0}(x)} \nabla\psi_0(y)\cdot\frac{y-x}{|y-x|}dV_2(y) + \\ & + \frac{1}{4\pi(\tau-\tau_0)}\int_{\partial B_{\tau-\tau_0}(x)} \psi_1(y)dV_2(y). \end{aligned} \tag{7.59}$$

In order to study the solution at the Big Bang, we fix ϕ_0, ϕ_1, τ and take the limit as $\tau_0 \to 0$ to obtain:

$$\begin{aligned} \tau\phi(\tau, x) = & \frac{1}{4\pi\tau^2}\int_{\partial B_\tau(x)} \lim_{\tau_0\to 0} \psi_0(y)dV_2(y) + \\ & + \frac{1}{4\pi\tau}\int_{\partial B_\tau(x)} \lim_{\tau_0\to 0} \nabla\psi_0(y)\cdot\frac{y-x}{|y-x|}dV_2(y) + \\ & + \frac{1}{4\pi\tau}\int_{\partial B_\tau(x)} \lim_{\tau_0\to 0} \psi_1(y)dV_2(y), \end{aligned} \tag{7.60}$$

or, defining $z = \frac{y-x}{\tau}$:

$$\begin{aligned} \phi(\tau, x) = & \frac{1}{4\pi\tau}\int_{S^2} \lim_{\tau_0\to 0} \psi_0(x+\tau z)dV_2(z) + \\ & + \frac{1}{4\pi}\int_{S^2} \lim_{\tau_0\to 0} \nabla\psi_0(x+\tau z)\cdot z\, dV_2(z) + \\ & + \frac{1}{4\pi}\int_{S^2} \lim_{\tau_0\to 0} \psi_1(x+\tau z)dV_2(z). \end{aligned} \tag{7.61}$$

Moreover, we see from (7.58) that $\psi_0 \underset{\tau_0 \to 0}{\longrightarrow} 0$ and $\psi_1 \underset{\tau_0 \to 0}{\longrightarrow} \phi_0$, whence the limiting solution is

$$\phi(\tau, x) = \frac{1}{4\pi} \int_{S^2} \phi_0(x + \tau z) dV_2(z). \tag{7.62}$$

and so:

$$\lim_{\tau \to 0} \phi(\tau, x) = \phi_0(x). \tag{7.63}$$

Similarly to what we saw for the dust case, if ψ_0 and ψ_1 are kept constant instead of ϕ_0 and ϕ_1 when taking the limit $\tau_0 \to 0$, ψ_0 must vanish so that the solution does not diverge at the Big Bang. From the initial data given by (7.58), we must choose $\phi_0 = 0$ and $\phi_1(x) = \tau_0^{-2}\psi_1(x)$. In this case ϕ_1 blows up as $\tau_0 \to 0$, but the limiting solution is well defined and tends to ψ_1 as $\tau \to 0$.

8 First Order Operators for the Hyperbolic Case

Now, we want to find a suitable operator $\hat{O}$ for the hyperbolic case, $K = -1$. First, we notice that the right hand side of (6.2) becomes:

$$\left[2\frac{a'}{a} f - 2f'\right] \partial_\tau^2 \phi + \left[2\frac{a'}{a} g - 2g' - f'' + 2\left(\frac{a'}{a}\right)' f + f\right] \partial_\tau \phi - (g'' - g)\phi \tag{8.1}$$

and must be equal to 0. For this to happen, we need the following conditions:

$$\begin{cases} \frac{a'}{a} f - f' = 0, \\ g'' - g = 0, \\ 2\frac{a'}{a} g - 2g' - f'' + 2\left(\frac{a'}{a}\right)' f + f = 0. \end{cases} \tag{8.2}$$

The first two constraints define the functions for the operator $\hat{O} = f(\tau)\partial_\tau + g(\tau)$:

$$\begin{cases} f(\tau) = \kappa a(\tau), \\ g(\tau) = \alpha \cosh(\tau) + \beta \sinh(\tau). \end{cases} \tag{8.3}$$

If we replace f and g in the third constraint of (8.2), we get:

$$2\left(\alpha \cosh(\tau) + \beta \sinh(\tau)\right) a' - 2\left(\alpha \sinh(\tau) + \beta \cosh(\tau)\right) a + \kappa a a'' - 2\kappa (a')^2 + \kappa a^2 = 0. \tag{8.4}$$

Suitable values of α and β can be found if we choose scale factors $a(\tau)$ that satisfy the Friedmann equation, as seen in section 5. In these cases, it is possible to get an explicit expression for the solution $\hat{O}\phi$. Such an expression is given in spherical means, analogously to the Kirchhoff's spherical means formula in $\mathbb{R}^n$.

8.1 Spherical Means in a Static Spacetime with $K = -1$

In the following, we will find the solution of the generalized wave equation (4.5) in the spacetime given by the metric

$$g = -dt^2 + d\Sigma_3^2, \tag{8.5}$$

where $d\Sigma_3^2$ is the line element for the hyperbolic space $\mathbb{H}^3$. What we will find is an expression in terms of spherical means which is analogous to Kirchhoff's formula for the Minkowski case. Following [16], we will consider the Cauchy problem

$$\begin{cases} \partial_t^2\phi - L\phi = 0, \\ \phi(0,x) = g(x),\ \dot{\phi}(0,x) = h(x), \end{cases} \tag{8.6}$$

where $L\phi = \Delta\phi + \phi$ and $x = (x^1, x^2, x^3)$ are the spatial coordinates. Using geodesic polar coordinates about a point x, we know that

$$d\Sigma_3^2 = dr^2 + \sinh(r)^2 d\Omega^2, \tag{8.7}$$

where $d\Omega^2$ is the line element for S^2. So, we have

$$ds^2 = dr^2 + \sinh(r)^2 d\Omega^2 = dr^2 + \sinh(r)^2 d\theta^2 + \sinh(r)^2 \sin(\theta)^2 d\varphi^2. \tag{8.8}$$

Now, using that

$$\Delta = \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^i} \left(\sqrt{|g|} g^{ij} \frac{\partial}{\partial x^j} \right) \tag{8.9}$$

and that $\sqrt{|g|} = \sinh(r)^2 \sin(\theta)$, $g_{rr} = 1$, $g_{\theta\theta} = \sinh(r)^2$, $g_{\varphi\varphi} = \sinh(r)^2 \sin(\theta)^2$, we can write the Laplacian as:

$$\begin{aligned} \Delta = & \frac{1}{\sinh(r)^2 \sin(\theta)} \left[\frac{\partial}{\partial r} \left(\sinh(r)^2 \sin(\theta) \frac{\partial}{\partial r} \right) + \frac{\partial}{\partial \theta} \left(\sinh(r)^2 \sin(\theta) \frac{1}{\sinh(r)^2} \frac{\partial}{\partial \theta} \right) + \right. \\ & \left. + \frac{\partial}{\partial \varphi} \left(\sinh(r)^2 \sin(\theta) \frac{1}{\sinh(r)^2 \sin(\theta)^2} \frac{\partial}{\partial \varphi} \right) \right] = \\ = & \frac{\partial^2}{\partial r^2} + 2\coth(r) \frac{\partial}{\partial r} + \frac{1}{\sinh(r)^2} \Delta_S, \end{aligned} \tag{8.10}$$

where we used that

$$\Delta_S = \frac{1}{\sin\theta} \frac{\partial}{\partial \theta} \left(\sin(\theta) \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin(\theta)^2} \frac{\partial^2}{\partial \varphi^2} \tag{8.11}$$

is the Laplacian on S^2. The geodesic sphere $S(x,r)$ about the point x is defined as

$$S(x,r) := \exp_x \left(\{ y \in T_x\mathbb{H}^3 \text{ such that } \|y\| = r \} \right) = \exp_x(\partial B_r(0)), \tag{8.12}$$

where $B_r(0)$ is the ball of radius r about 0 in the tangent space. We define the spherical mean of a function $f(t, y)$ over the geodesic sphere of radius r about x as

$$M_f(t,r,x) := \frac{1}{4\pi \sinh(r)^2}\int_{S(x,r)} f(t,y)dA(y) = \frac{1}{4\pi}\int_{S^2} f(t,\exp_x(rz))dV_2(z) \tag{8.13}$$

where dA is the area element of the geodesic sphere. The last equality can be proved in the following way:

$$\begin{aligned}
M_f(t,r,x) &= \frac{1}{4\pi \sinh(r)^2}\int_{S(x,r)} f(t,y)dA(y) \\
&= \frac{1}{4\pi \sinh(r)^2}\int_{\exp_x(\partial B_r(0))} f(t,y)dA(y) && \text{by def. (8.12)} \\
&= \frac{1}{4\pi \sinh(r)^2}\int_{\partial B_r(0)} f(t,\exp_x(v))(\exp_x^* dA)(v) && \text{using the pullback of } \exp_x \\
&= \frac{1}{4\pi r^2}\int_{\partial B_r(0)} f(t,\exp_x(v))dV_2(v) && \text{since } r^2(\exp_x^* dA) = \sinh(r)^2 dV_2 \\
&= \frac{1}{4\pi}\int_{S^2} f(t,\exp_x(rz))dV_2(z). && \text{using } v = rz
\end{aligned} \tag{8.14}$$

Moreover, we define $M_f(t,-r,x) = M_f(t,r,x)$ in order to extend it to negative values of the second argument. It is natural to extend it as an even function, since we can think of any term $(-r)z$ in the last integral of (8.13) as $r(-z)$, i.e. as if we consider the vector antipodal to z, still belonging to $S^2 \subset T_x\mathbb{H}^3$. We also define the spherical mean of a function $g = g(y)$ in a similar way:

$$M_g(r;x) := \frac{1}{4\pi \sinh(r)^2}\int_{S(x,r)} g(y)dA(y) = \frac{1}{4\pi}\int_{S^2} g(\exp_x(rz))dV_2(z). \tag{8.15}$$

Now, given a solution ϕ for the problem (8.6), we have:

$$\begin{aligned}
\frac{\partial^2}{\partial t^2}M_\phi(t,r,x) &= \frac{1}{4\pi}\int_{S^2}\frac{\partial^2}{\partial t^2}\phi(t,\exp_x(rz))dV_2(z) && \text{(by (8.13))} \\
&= \frac{1}{4\pi}\int_{S^2} L\phi(t,\exp_x(rz))dV_2(z) && \text{(by (8.6))} \\
&= \frac{1}{4\pi}\int_{S^2}\left[\frac{\partial^2}{\partial r^2}\phi(t,\exp_x(rz)) + 2\coth(r)\frac{\partial}{\partial r}\phi(t,\exp_x(rz)) + \right. \\
&\quad \left. + \frac{1}{\sinh(r)^2}\Delta_S\phi(t,\exp_x(rz))\right]dV_2(z) + \\
&\quad + \frac{1}{4\pi}\int_{S^2}\phi(t,\exp_x(rz)dV_2(z) && \text{(by (8.10))} \\
&= \left(\frac{\partial^2}{\partial r^2} + 2\coth(r)\frac{\partial}{\partial r} + 1\right)M_\phi(t,r,x)
\end{aligned} \tag{8.16}$$

where in the last step we used the divergence theorem on the sphere:

$$\int_{S^2} \Delta_S \phi(t, \exp_x(rz)) dV_2(z) = \int_{S^2} \nabla_S \cdot \big(\nabla_S \phi(t, \exp_x(rz))\big) dV_2(z) = 0. \tag{8.17}$$

From (8.6), it follows that we have initial data

$$\begin{cases} M_\phi(0, r, x) = M_g(r; x), \\ \partial_t M_\phi(0, r, x) = M_h(r; x). \end{cases} \tag{8.18}$$

Ignoring for the moment the dependence of M_ϕ on x, if $M_\phi(t, r)$ satisfies (8.16), it can be checked that $\omega(t, r) := \sinh(r) M_\phi(t, r)$ satisfies

$$\partial_t^2 \omega - \partial_r^2 \omega = 0. \tag{8.19}$$

Given initial data $\omega(0, r) = \gamma(r) = \sinh(r) M_g(r; x)$ and $\dot{\omega}(0, r) = \psi(r) = \sinh(r) M_h(r; x)$, the solution of equation (8.19) is given by the d'Alembert formula:

$$\omega(t, r) = \frac{1}{2} \left[\gamma(r + t) + \gamma(r - t) + \int_{r-t}^{r+t} \psi(s) ds \right]. \tag{8.20}$$

Since $M_\phi(t, r) = \frac{1}{\sinh(r)} \omega(t, r)$ and, recovering the x-dependence:

$$M_\phi(t, r, x) = \frac{1}{2 \sinh(r)} \left[\gamma(r + t) + \gamma(r - t) + \int_{r-t}^{r+t} \psi(s) ds \right]. \tag{8.21}$$

Letting $r \to 0$, we have that $M_\phi(t, r, x) \to \phi(t, x)$ (as can be checked from (8.13)) and the term in (8.21) inside the square brackets tends to 0 since γ and ψ are odd functions. Thus, taking the limit $r \to 0$ and using L'Hôpital's rule:

$$\begin{aligned} \phi(t, x) &= \lim_{r \to 0} \frac{1}{2 \cosh(r)} \left[\gamma'(r + t) + \gamma'(r - t) + \psi(r + t) - \psi(r - t) \right] \\ &= \gamma'(t) + \psi(t) = \partial_t (\sinh(t) M_g(t; x)) + \sinh(t) M_h(t; x), \end{aligned} \tag{8.22}$$

where we used again the symmetries of γ and ψ. Therefore, the solution of the Cauchy problem

$$\begin{cases} \partial_t^2 \Phi - L\Phi = 0, \\ \Phi(t_0, x) = g(x),\ \dot{\Phi}(t_0, x) = h(x) \end{cases} \tag{8.23}$$

is given by

$$\Phi(t, x) = \partial_t (\sinh(t - t_0) M_g(t - t_0; x)) + \sinh(t - t_0) M_h(t - t_0; x). \tag{8.24}$$

Using definition (8.15) and the definition of the exponential map, the above can be written as

$$\begin{aligned}\Phi(t,x) = &\frac{\cosh(t-t_0)}{4\pi}\int_{S^2} g\big(\exp_x((t-t_0)z)\big)\,dV_2(z) + \\ &+ \frac{\sinh(t-t_0)}{4\pi}\int_{S^2} dg(\dot{c}_z(t-t_0))\,dV_2(z) + \\ &+ \frac{\sinh(t-t_0)}{4\pi}\int_{S^2} h\big(\exp_x((t-t_0)z)\big)\,dV_2(z),\end{aligned} \tag{8.25}$$

where we used that

$$\partial_t g(\exp_x((t-t_0)z)) = \frac{\partial}{\partial t}(g\circ c_z)(t-t_0) = dg(\dot{c}_z(t-t_0)), \tag{8.26}$$

with $c_z(t) = \exp_x(tz)$, i.e. $c_z(t)$ parametrizes the geodesic that fulfils $c_z(0) = x$, $\dot{c}_z(0) = z$. By applying the expression (8.15) for the spherical means, we can also rewrite the (8.25) via integrals over the geodesic spheres:

$$\begin{aligned}\Phi(t,x) = &\frac{\cosh(t-t_0)}{4\pi\sinh(t-t_0)^2}\int_{S(x,t-t_0)} g(y)dA(y) + \\ &+ \frac{1}{4\pi\sinh(t-t_0)}\int_{S(x,t-t_0)} [dg(X)]\,(y)dA(y) + \\ &+ \frac{1}{4\pi\sinh(t-t_0)}\int_{S(x,t-t_0)} h(y)dA(y),\end{aligned} \tag{8.27}$$

where X is the outward unit normal to $S(x,t-t_0)$

8.2 Dust-filled Hyperbolic FLRW Model

Let us consider the case

$$a(\tau) = \cosh(\tau) - 1. \tag{8.28}$$

This corresponds to the dust-filled model, since $a(\tau)$ solves Friedmann's equation (5.12) for $w = 0$. With this choice of the scale factor, our wave equation (4.3) becomes

$$\partial_\tau^2\phi + \frac{2\sinh(\tau)}{\cosh(\tau)-1}\partial_\tau\phi = \Delta\phi. \tag{8.29}$$

In order to find the operator $\hat{O} = f(\tau)\partial_\tau + g(\tau)$ such that f and g satisfy equation (8.4), i.e.

$$(2\beta - 3\kappa)(-1+\cosh(\tau)) + 2\alpha\sinh(t) = 0, \tag{8.30}$$

we need to consider $\kappa = 1, \alpha = 0, \beta = \frac{3}{2}$. Therefore, in order to get this solution we need to consider the operator

$$\hat{O} = \frac{3}{2}\sinh(\tau) + (\cosh(\tau)-1)\partial_\tau. \tag{8.31}$$

which indeed satisfies $\partial_\tau^2 \hat{O}\phi(\tau, x) = L\hat{O}\phi(\tau, x)$ by construction. We notice that, using $2\sinh(x)\cosh(x) = \sinh(2x)$ and $\sinh^2(x) = \frac{1}{2}(\cosh(2x) - 1)$, we can rewrite the above in the variable $\gamma = \frac{\tau}{2}$:

$$\hat{O}\phi = \frac{1}{\sinh(\gamma)}\partial_\gamma(\sinh^3(\gamma)\phi) = \frac{2}{\sinh\left(\frac{\tau}{2}\right)}\partial_\tau\left(\sinh^3\left(\frac{\tau}{2}\right)\phi\right), \tag{8.32}$$

which is the expression that can be found in [16]. Given initial data for the wave equation at $t = t_0$:

$$\begin{cases} \phi(t_0, x) = \phi_0(x), \\ \partial_t\phi(t_0, x) = \phi_1(x), \end{cases} \tag{8.33}$$

we have the following initial data for $\hat{O}\phi = \frac{3}{2}\sinh(\tau)\phi + (\cosh(\tau) - 1)\partial_\tau\phi$ at the corresponding conformal time $\tau = \tau_0$:

$$\begin{cases} \psi_0(x) = \hat{O}\phi(\tau_0, x) = \frac{3}{2}\sinh(\tau_0)\phi_0(x) + (\cosh(\tau_0) - 1)\phi_1(x), \\ \psi_1(x) = \partial_\tau\hat{O}\phi(\tau_0, x) = \frac{3}{2}\cosh(\tau_0)\phi_0(x) + \frac{1}{2}\sinh(\tau_0)(\cosh(\tau_0) - 1)\phi_1(x) + \\ \qquad\qquad + (\cosh(\tau_0) - 1)\Delta\phi_0(x), \end{cases} \tag{8.34}$$

where we used $\partial_\tau = (\cosh(\tau) - 1)\partial_t$, $\partial_\tau^2\phi = \Delta\phi - \frac{2\sinh(\tau)}{\cosh(\tau)-1}\partial_\tau\phi$ (see (8.6)) and thus

$$\begin{aligned} \partial_\tau\hat{O}\phi &= \partial_\tau\left(\frac{3}{2}\sinh(\tau)\phi + (\cosh(\tau) - 1)\partial_\tau\phi\right) = \\ &= \frac{3}{2}\cosh(\tau)\phi + \frac{1}{2}\sinh(\tau)\partial_\tau\phi + (\cosh\tau - 1)\Delta\phi = \\ &= \frac{3}{2}\cosh(\tau)\phi + \frac{1}{2}\sinh(\tau)(\cosh(\tau) - 1)\partial_t\phi + (\cosh\tau - 1)\Delta\phi. \end{aligned} \tag{8.35}$$

Then, formula (8.24) gives

$$\hat{O}\phi = \sinh(\tau - \tau_0)M_{\psi_1}(\tau - \tau_0; x) + \partial_\tau(\sinh(\tau - \tau_0)M_{\psi_0}(\tau - \tau_0; x)). \tag{8.36}$$

More explicitly, equation (8.25) gives:

$$\begin{aligned} \sinh^3\left(\frac{\tau}{2}\right)\phi(\tau, x) = &\int_{\tau_0}^{\tau}\frac{\cosh(s - \tau_0)}{8\pi}\sinh\left(\frac{s}{2}\right)\int_{S^2}\psi_0\big(\exp_x((s - \tau_0)z)\big)\,dV_2(z)ds \\ &+ \int_{\tau_0}^{\tau}\frac{\sinh(s - \tau_0)}{8\pi}\sinh\left(\frac{s}{2}\right)\int_{S^2}d\psi_0(c_z'(s - \tau_0))\,dV_2(z)ds \\ &+ \int_{\tau_0}^{\tau}\frac{\sinh(s - \tau_0)}{8\pi}\sinh\left(\frac{s}{2}\right)\int_{S^2}\psi_1\big(\exp_x((s - \tau_0)z)\big)\,dV_2(z)ds \\ &+ \sinh^3\left(\frac{\tau_0}{2}\right)\phi_0(x), \end{aligned} \tag{8.37}$$

where $c_z(t) = \exp_x(tz)$ parametrizes the geodesic with properties $c_z(0) = x$, $c_z'(0) = z$ and we used the definition (8.32) of our operator. Here and in what follows, the primed characters denote differentiation by τ. Let us analyse the decay of the solution for $\tau \to \infty$. We can start with the first term. It can be estimated by splitting the interval (τ_0, τ) in two parts. For $\tau \in (\tau_0, 2\tau_0)$:

$$\left|\int_{\tau_0}^{\tau} \cosh(s-\tau_0)\sinh\left(\frac{s}{2}\right)\int_{S^2}\psi_0\big(\exp_x((s-\tau_0)z)\big)\,dV_2(z)ds\right| \lesssim \|\psi_0\|_\infty. \tag{8.38}$$

Now, we notice that we can rewrite the integral on S^2 as an integral over the geodesic sphere using (8.15):

$$\int_{\tau_0}^{\tau}\frac{\cosh(s-\tau_0)}{8\pi\sinh(s-\tau_0)^2}\sinh\left(\frac{s}{2}\right)\int_{S(x,s-\tau_0)}\psi_0(y)dA(y)ds. \tag{8.39}$$

Therefore, for $\tau > 2\tau_0$:

$$\left|\int_{2\tau_0}^{\tau}\frac{\cosh(s-\tau_0)}{\sinh(s-\tau_0)^2}\sinh\left(\frac{s}{2}\right)\int_{S(x,s-\tau_0)}\psi_0(y)dA(y)ds\right| \lesssim \|\psi_0\|_1\int_{2\tau_0}^{\tau}e^{-\frac{s}{2}}ds \lesssim \|\psi_0\|_1. \tag{8.40}$$

Similar computations are valid for the other integrals of expression (8.37). After dividing both sides of the latter by $\sinh^3\left(\frac{\tau}{2}\right)$, we can write

$$\begin{aligned}|\phi| \lesssim e^{-\frac{3}{2}\tau}\Big(&\|\psi_0\|_1 + \|\nabla\psi_0\|_1 + \|\psi_1\|_1 + \\ &+\|\psi_0\|_\infty + \|\nabla\psi_0\|_\infty + \|\psi_1\|_\infty + \sinh^3\left(\frac{\tau_0}{2}\right)\phi_0(x)\Big) \sim e^{-\frac{3}{2}\tau} \text{ as } \tau\to\infty\end{aligned} \tag{8.41}$$

for every x if we suppose that the above norms are finite. We notice that, in general, a faster decay rate is not possible. Indeed, we can choose the initial data and a fixed point $\bar{x} \in \mathbb{H}^3$ such that $|\phi(\tau,\bar{x})| \gtrsim e^{-\frac{3}{2}\tau}$ as well. In particular, we can consider $\tau_0 > 0$ and choose the functions ϕ_0 and ϕ_1 such that $\psi_0(x) = 0$ for every x. So, from (8.34):

$$\frac{3}{2}\sinh(\tau_0)\phi_0(x) = (1-\cosh(\tau_0))\,\phi_1(x), \tag{8.42}$$

which implies

$$\psi_1(x) = A\phi_0(x) + B\Delta\phi_0(x) \tag{8.43}$$

with $A = \frac{3}{2}\left(\cosh(\tau_0) - \frac{1}{2}\sinh^2(\tau_0)\right)$, $B = \cosh(\tau_0) - 1$. Now, we fix $\bar{x} \in \mathbb{H}^3$ and choose a function ϕ_0 which is radially symmetric with respect to $\bar{x}$. In other words, the value $\phi_0(x)$ only depends on the distance of x from $\bar{x}$: $\phi_0(x) = \phi_0(\mathrm{dist}_{\mathbb{H}^3}(x,\bar{x}))$ for every $x \in \mathbb{H}^3$. We will prove that for such a choice, $|\phi(\tau,\bar{x})| \sim e^{-\frac{3}{2}\tau}$ as $\tau\to\infty$. Indeed, let us choose the following function:

$$\phi_0(\mathrm{dist}_{\mathbb{H}^3}(x,\bar{x})) = \phi_0(p) := \begin{cases} Np^3\exp\left(\frac{1}{p-1}\right), & 0 \le p \le 1 \\ 0, & p \ge 1\end{cases} \tag{8.44}$$

where $N \in \mathbb{N}$ and we denoted the distance from $\bar{x}$ by the radial variable p. Seen as a function of

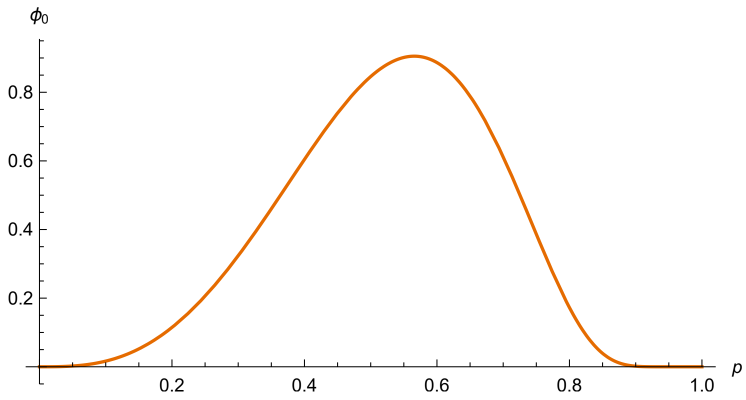

Figure 2.2: Plot of $\phi_0(p)$ along a radial direction for $N = 50$

the radial variable, we have that $\phi_0 \in C^2([0,\infty))$. In fact, it is easy to check that its derivatives

$$\phi_0'(p) = \begin{cases} Np^2 \exp\left(\frac{1}{p-1}\right) \frac{3+p(3p-7)}{(p-1)^2}, & 0 \le p \le 1 \\ 0, & p \ge 1. \end{cases}$$

$$\phi_0''(p) = \begin{cases} Np \exp\left(\frac{1}{p-1}\right) \frac{6p^4-28p^3+47p^2-30p+6}{(p-1)^4}, & 0 \le p \le 1 \\ 0, & p \ge 1. \end{cases} \tag{8.45}$$

are well-defined and continuous in $[0,\infty)$. Let us plug this function in the spherical means solution (8.37) evaluated at $\bar{x}$. Given that $\psi_0 \equiv 0$, the first two integrals of the right hand side of the spherical means solution vanish. Moreover, $\phi_0(\bar{x}) = 0$. So, the right hand side of (8.37) becomes

$$\begin{aligned}
&\int_{\tau_0}^{\tau} \frac{\sinh(s-\tau_0)}{8\pi} \sinh\left(\frac{s}{2}\right) \int_{S^2} \psi_1\big(\exp_{\bar{x}}((s-\tau_0)z)\big)\, dV_2(z) ds \\
=&\int_{\tau_0}^{\tau} \frac{1}{8\pi \sinh(s-\tau_0)} \sinh\left(\frac{s}{2}\right) \int_{S(\bar{x},s-\tau_0)} \psi_1(y) dA(y)\, ds && \text{(by (8.15))} \\
=&\int_{0}^{\tau-\tau_0} \frac{\sinh\left(\frac{p+\tau_0}{2}\right)}{8\pi \sinh(p)} \int_{S(\bar{x},p)} \big(A\phi_0(y) + B\Delta\phi_0(y)\big) dA(y)\, dp && \text{(with } p = s-\tau_0) \\
=&\frac{1}{2}\int_{0}^{\tau-\tau_0} \sinh(p) \sinh\left(\frac{p+\tau_0}{2}\right) \big(A\phi_0(p) + B\Delta\phi_0(p)\big) dp && \text{(since } \phi_0(y) = \phi_0(p)) \\
=&\frac{1}{2}\int_{0}^{\tau-\tau_0} \sinh(p) \sinh\left(\frac{p+\tau_0}{2}\right) \big(A\phi_0(p) + B\phi_0''(p) + 2B\coth(p)\phi_0'(p)\big) && (8.46)
\end{aligned}$$

using that $\Delta = \partial_p^2 + 2\coth(p)\partial_p + \operatorname{csch}(p)\Delta_S$ (see also (8.10)). By definition (8.44), we have that, up to a constant, the above integral becomes

$$\int_0^1 \sinh\left(\frac{p+\tau_0}{2}\right) \big(A\sinh(p)\phi_0(p) + B\sinh(p)\phi_0''(p) + 2B\cosh(p)\phi_0'(p)\big)\, dp \tag{8.47}$$

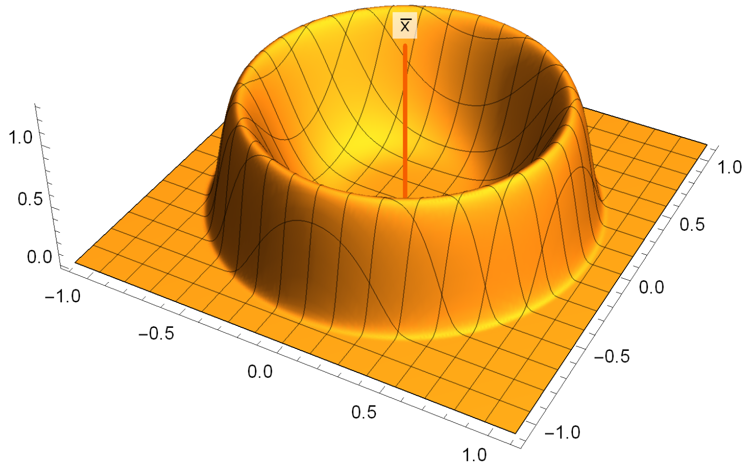

Figure 2.3: 3D plot of $\phi_0(p)$ for $N = 50$

for $\tau > \tau_0 + 1$. This gives a constant since it is the sum of three integrals of summable functions over [0, 1]. Moreover, we can choose τ_0 such that the integral is non-zero*. So, after this choice of ϕ_0 and ϕ_1, our solution (8.37) becomes

$$\phi(\tau, \bar{x}) = C \sinh^{-3}\left(\frac{\tau}{2}\right) \sim e^{-\frac{3}{2}\tau} \text{ as } \tau \to \infty. \tag{8.48}$$

Now, let us extend this construction to smooth initial data. Here, ϕ_0 cannot be differentiated more than twice in $p = 0$. However, since continuous functions on a manifold can be approximated by smooth functions (Whitney approximation theorem, see e.g. [17]), we can choose smooth initial data that approximate the chosen ϕ_0 and get the same estimate for $\phi(\tau, \bar{x})$. The above reasoning disproves the decay rate of $e^{-2\tau}$ in the conformal time variable stated by Abbasi and Craig in [1] for the case $K = -1$. The inequality proved in theorem 3.1 of their article seems indeed to hold specifically for the chosen term, but it does not hold for the last integral terms of their solutions (3.9) and (3.10).

We can also obtain an estimate in t via

$$t = \int a(\tau)d\tau = \int (\cosh(\tau) - 1)d\tau = \sinh(\tau) - \tau. \tag{8.49}$$

From the above we cannot retrieve $t(\tau)$, but we can notice that, for $\tau \to \infty$:

$$t \sim e^{\tau} \Leftrightarrow \tau \sim \log(t), \tag{8.50}$$

*For $\tau_0 = 1$ and $N = 50$, the evaluation of the integral (8.47) through Mathematica gives 0.151503. Actually, this value grows linearly with N.

whence we get $e^{-\frac{3}{2}\tau} \sim t^{-\frac{3}{2}}$ and finally

$$|\phi| \lesssim t^{-\frac{3}{2}} \text{ as } t \to \infty. \tag{8.51}$$

Moreover, there exist solutions which are well-defined at the Big Bang. We can prove it in the following way. If we keep ϕ_0 and ϕ_1 constant, the initial data (8.34) tells us that

$$\psi_0 \xrightarrow[\tau_0 \to 0]{} 0 \quad \text{and} \quad \psi_1 \xrightarrow[\tau_0 \to 0]{} \frac{3}{2}\phi_0. \tag{8.52}$$

Hence, the solution given by (8.37) for initial data $\tau_0 \to 0$ is

$$\phi\left(\tau, x\right) = \frac{1}{\sinh^3\left(\frac{\tau}{2}\right)} \int_0^\tau \frac{\sinh\left(\frac{s}{2}\right)\sinh(s)}{8\pi} \int_{S^2} \frac{3}{2}\phi_0(\exp_x(zs))dV_2(z)ds. \tag{8.53}$$

Using L'Hôpital's rule:

$$\lim_{\tau\to 0}\phi(\tau, x) = \lim_{\tau\to 0} \frac{\sinh\left(\frac{\tau}{2}\right)\sinh(\tau)}{8\pi\sinh^2\left(\frac{\tau}{2}\right)\cosh\left(\frac{\tau}{2}\right)} \int_{S^2} \phi_0(\exp_x(z\tau))dV_2(z) = \phi_0(x), \tag{8.54}$$

since $\sinh(\tau) = 2\sinh\left(\frac{\tau}{2}\right)\cosh\left(\frac{\tau}{2}\right)$ and $\exp_x(0) = x$. If we keep ψ_0 and ψ_1 constant instead of ϕ_0 and ϕ_1, we have $\psi_0 = 0$ so that the solution does not diverge at the Big Bang. Hence:

$$\begin{cases} \phi_0(x) = \frac{2(1-\cosh(\tau_0))}{3\sinh(\tau_0)}\phi_1, \\ \frac{1}{2}(\cosh(\tau_0) - 1)(2\coth(\tau_0) - \sinh(\tau_0))\phi_1 + \frac{2(1-\cosh(\tau_0)^2)}{3\sinh(\tau_0)}\Delta\phi_1 = \psi_1. \end{cases} \tag{8.55}$$

8.3 Radiation-filled Hyperbolic FLRW Model

Let us consider the case

$$a(\tau) = \sinh(\tau). \tag{8.56}$$

With this choice of the scale factor, our wave equation (4.3) becomes

$$\partial_\tau^2\phi + 2\coth(\tau)\partial_\tau\phi = \Delta\phi. \tag{8.57}$$

In order to find the operator $\hat{O} = f(\tau)\partial_\tau + g(\tau)$ such that f and g satisfy equation (8.4), i.e.

$$2(\alpha\cosh(\tau) + \cancel{\beta\sinh(\tau)})\cosh(\tau) - 2(\alpha\sinh(\tau) + \cancel{\beta\cosh(\tau)})\sinh(\tau) - 2k = 0, \tag{8.58}$$

$$\alpha - \kappa = 0, \tag{8.59}$$

we need to consider $\alpha = \kappa$. For instance: $\alpha = \kappa = 0$ and $\beta = 1$. Therefore, in order to get this solution we can consider the operator

$$\hat{O}\phi = \sinh(\tau)\phi. \tag{8.60}$$

An independent solution will also be given by the choice $\alpha = \kappa = 1$ and $\beta = 0$, i.e.

$$\hat{O}\phi = \sinh(\tau)\partial_\tau\phi + \cosh(\tau)\phi = \partial_\tau(\sinh(\tau)\phi). \tag{8.61}$$

Both operators satisfy $\partial_\tau^2\hat{O}\phi = L\hat{O}\phi$ by construction. More in general, any operator

$$\hat{O}\phi = \partial_\tau^k(\sinh(\tau)\phi) \tag{8.62}$$

will satisfy the constraint equation (4.4). This reminds us of the flat radiation-filled model analysed in section 7.4. Indeed, this model corresponds to a radiation-dominated fluid, since $a(\tau) = \sinh(\tau)$ solves the Friedmann equation (5.12) for $w = \frac{1}{3}$. Now, given initial data for the wave equation at $t = t_0$:

$$\begin{cases} \phi(t_0, x) = \phi_0(x), \\ \partial_t\phi(t_0, x) = \phi_1(x), \end{cases} \tag{8.63}$$

we have the following initial data for $\hat{O}\phi = \sinh(\tau)\phi$ at the corresponding conformal time $\tau = \tau_0$:

$$\begin{cases} \psi_0(x) = \hat{O}\phi(\tau_0, x) = \sinh(\tau_0)\phi_0(x), \\ \psi_1(x) = \partial_\tau\hat{O}\phi = \cosh(\tau_0)\phi_0(x) + \sinh^2(\tau_0)\phi_1(x), \end{cases} \tag{8.64}$$

where the above result follows from the fact that $\partial_\tau = \sinh(\tau)\partial_t$:

$$\partial_\tau\hat{O}\phi = \partial_\tau(\sinh(\tau)\phi) = \cosh(\tau)\phi + \sinh(\tau)\partial_\tau\phi = \cosh(\tau)\phi + \sinh^2(\tau)\partial_t\phi \tag{8.65}$$

The solution in terms of spherical means is given by (8.25):

$$\begin{aligned} \sinh(\tau)\phi = &\frac{\cosh(\tau-\tau_0)}{4\pi}\int_{S^2}\psi_0\big(\exp_x((\tau-\tau_0)z)\big)\,dV_2(z) + \\ &+\frac{\sinh(\tau-\tau_0)}{4\pi}\int_{S^2} d\psi_0(c_z'(\tau-\tau_0))\,dV_2(z) + \\ &+\frac{\sinh(\tau-\tau_0)}{4\pi}\int_{S^2}\psi_1\big(\exp_x((\tau-\tau_0)z)\big)\,dV_2(z). \end{aligned} \tag{8.66}$$

Let us analyse the decay of the solution as $\tau \to \infty$. We notice that, after dividing the above by $\sinh(\tau)$, the first term of the solution can be estimated as

$$\begin{aligned} |\phi| &\sim \left|\frac{\cosh(\tau-\tau_0)}{\sinh(\tau)}\int_{S^2}\psi_0\big(\exp_x((\tau-\tau_0)z)\big)\,dV_2(z)\right| \\ &= \left|\frac{\cosh(\tau-\tau_0)}{\sinh(\tau)\sinh(\tau-\tau_0)^2}\int_{S(x,\tau-\tau_0)}\psi_0(y)\,dA(y)\right| \lesssim \left(\|\psi_0\|_{L^1(\mathbb{H}^3)} + \|d\psi_0\|_{L^1(\mathbb{H}^3)}\right)e^{-2\tau}, \end{aligned} \tag{8.67}$$

using the spherical mean expression on the geodesic sphere (8.15) and assuming that the initial data have compact support. In fact, we used that:

$$\begin{aligned}\int_{S(x,R)} \psi_0(y) dA(y) &= -\int_R^{+\infty} \frac{\mathrm{d}}{\mathrm{d}r}\left(\int_{S(x,r)} \psi_0(y) dA(y)\right) dr \\ = -\int_R^{+\infty} 2\sinh(r)\cosh(r) &\left(\int_{S^2} \psi_0\left(\exp_x(rz)\right) dV_2(z)\right) dr + \\ -\int_R^{+\infty} \sinh(r)^2 &\left(\int_{S^2} d\psi_0(\dot{c}_z(r)) dV_2(z)\right) dr,\end{aligned} \tag{8.68}$$

where $c_z(r) = \exp_x(rz)$. If we write again the above integrals as integrals over the geodesic spheres and extend the domain of integration to $[0, +\infty)$ for the variable r:

$$\left|\int_{S(x,R)} \psi_0(y) dA(y)\right| \leq C \coth(R) \, \|\psi_0\|_{L^1(\mathbb{H}^3)} + \|d\psi_0\|_{L^1(\mathbb{H}^3)}, \tag{8.69}$$

which implies the desired result.

The remaining two terms have the same decay rate, which can be found after similar computations. Therefore, if the respective Sobolev norms of the initial data are finite:

$$|\phi| \lesssim e^{-2\tau}. \tag{8.70}$$

In order to get the decay rate in function of the variable t, we use that $dt = a(\tau)d\tau$ and thus

$$t = \int \sinh(\tau) d\tau = \cosh(\tau) \Leftrightarrow \tau = \operatorname{arcosh}(t). \tag{8.71}$$

Finally, if we suppose that the 1-norms of ψ_0, $\nabla\psi_0$ and ψ_1 are finite:

$$|\phi| \lesssim e^{-2\operatorname{arcosh}(t)} \sim t^{-2} \text{ as } t \to \infty, \tag{8.72}$$

where in the last step:

$$t = \cosh(\tau) \sim e^{\tau} \text{ as } \tau \to \infty \Leftrightarrow \tau \sim \log(t) \text{ as } t \to \infty. \tag{8.73}$$

If we keep ϕ_0 and ϕ_1 constant, solutions starting at the Big Bang are still well defined, as showed in (8.64):

$$\psi_0 \underset{\tau_0 \to 0}{\longrightarrow} 0 \quad \text{and} \quad \psi_1 \underset{\tau_0 \to 0}{\longrightarrow} \phi_0, \tag{8.74}$$

and thus the limiting solution is

$$\phi(\tau, x) = \frac{1}{4\pi} \int_{S^2} \phi_0\big(\exp_x(\tau z)\big) dV_2(z) \underset{\tau \to 0}{\longrightarrow} \phi_0(x) \tag{8.75}$$

by the spherical means expression. On the other hand, if we keep ψ_0 and ψ_1 constant, we can see from solution (8.66) that we need $\psi_0 = 0$ (i.e. $\phi_0 = 0$) so that the solution does not diverge. From the initial data (8.64) we must have

$$\phi_1(x) = \frac{\psi_1(x)}{\sinh^2(\tau_0)}. \tag{8.76}$$

The function ϕ_1 diverges at the Big Bang, but our solution $\phi(\tau, x)$ is well defined and $\phi(\tau, x) \to \psi_1(x)$ as $\tau \to 0$ (again by the spherical means expression).

8.4 Anti-de Sitter Space

For the vacuum energy case we have two solutions describing different spacetimes. Let us first analyse the case

$$a(\tau) = \operatorname{sech}(\tau), \tag{8.77}$$

with $\tau \in (0, \infty)$. This choice of the scale factor applied to (4.3) leads to

$$\partial_\tau^2\phi - 2\tanh(\tau)\partial_\tau\phi = \Delta\phi \tag{8.78}$$

In order to find the operator $\hat{O} = f(\tau)\partial_\tau + g(\tau)$ such that f and g satisfy equation (8.4), i.e.

$$\beta(1 + \tanh(\tau)^2) + 2\alpha\tanh(\tau) = 0, \tag{8.79}$$

we have to choose $\alpha = \beta = 0$ and, for instance, $\kappa = 1$. From the above computation it follows that suitable operators for this case must be first-order differential operators. So, we have

$$\hat{O} = \operatorname{sech}(\tau)\partial_\tau, \tag{8.80}$$

which satisfies $\partial_\tau^2\hat{O}\phi = L\hat{O}\phi$ by construction. This corresponds to an unphysical fluid model, since $a(\tau) = \operatorname{sech}(\tau)$ is a solution for the Friedmann equation (5.12) for $w = -1$. More precisely, this describes an open region in the anti-de Sitter solution. We can get the corresponding scale factor in time t:

$$t = \int \frac{d\tau}{\cosh(\tau)} = 2\arctan\left(\tanh\left(\frac{\tau}{2}\right)\right) \Leftrightarrow \tau = 2\,\operatorname{artanh}\left(\tan\left(\frac{t}{2}\right)\right) \tag{8.81}$$

which implies:

$$a(t) = \cosh\left(2\,\operatorname{artanh}\left(\tan\left(\frac{t}{2}\right)\right)\right)^{-1} = \cos(t), \tag{8.82}$$

thanks to the relations $\operatorname{artanh}(x) = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$, for $|x| < 1$, and $\frac{1-\tan^2\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)} = \cos(x)$. Since $\tau \in (-\infty, \infty)$, then $t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Therefore, we obtained the desired scale factor. We can express the solution in terms of spherical means as follows. If we choose for the wave equation the following initial data at $t = t_0$:

$$\begin{cases} \phi(t_0, x) = \phi_0(x), \\ \partial_t \phi(t_0, x) = \phi_1(x), \end{cases} \tag{8.83}$$

then we have the following initial data for $\hat{O}\phi = \operatorname{sech}(\tau)\partial_\tau \phi$ at the corresponding conformal time $\tau = \tau_0$:

$$\begin{cases} \psi_0(x) = \hat{O}\phi(\tau_0, x) = \operatorname{sech}^2(\tau_0)\phi_1(x), \\ \psi_1(x) = \partial_\tau \hat{O}\phi(\tau_0, x) = \operatorname{sech}(\tau_0)(\operatorname{sech}(\tau_0)\tanh(\tau_0)\phi_1(x) + \Delta\phi_0(x)), \end{cases} \tag{8.84}$$

where the above follows from the fact that $\partial_\tau = \operatorname{sech}(\tau)\partial_t$, $\partial_\tau^2 \phi = 2\tanh(\tau)\partial_\tau \phi + \Delta\phi$ (from (8.78)) and thus:

$$\begin{aligned} \partial_\tau \hat{O}\phi &= \partial_\tau(\operatorname{sech}(\tau)\partial_\tau \phi) = -\operatorname{sech}(\tau)\tanh(\tau)\partial_\tau \phi + \operatorname{sech}(\tau)\partial_\tau^2 \phi \\ &= \operatorname{sech}(\tau)\tanh(\tau)\partial_\tau \phi + \operatorname{sech}(\tau)\Delta\phi = \operatorname{sech}^2(\tau)\tanh(\tau)\partial_t \phi + \operatorname{sech}(\tau)\Delta\phi. \end{aligned} \tag{8.85}$$

Then, the spherical means solution is

$$\begin{aligned} \phi(\tau, x) = &\int_{\tau_0}^{\tau} \frac{\cosh(s-\tau_0)}{4\pi}\cosh(s)\int_{S^2} \psi_0\big(\exp_x((s-\tau_0)z)\big)\, dV_2(z)ds \\ &+ \int_{\tau_0}^{\tau} \frac{\sinh(s-\tau_0)}{4\pi}\cosh(s)\int_{S^2} d\psi_0(c_z'(s-\tau_0))\, dV_2(z)ds \\ &+ \int_{\tau_0}^{\tau} \frac{\sinh(s-\tau_0)}{4\pi}\cosh(s)\int_{S^2} \psi_1\big(\exp_x((s-\tau_0)z)\big)\, dV_2(z)ds \\ &+ \phi(\tau_0, x), \end{aligned} \tag{8.86}$$

which goes to a constant function as $\tau \to \infty$ for initial data with compact support. In particular, the result does not depend on x. In order to see this, we can consider two different and independent cases.

If $\boxed{\phi_1(x) \equiv 0}$, then we have that $\psi_0 \equiv 0$ and $\psi_1(x) = \operatorname{sech}(\tau_0)\Delta\phi_0(x)$. Thus:

$$\begin{aligned} \phi(\tau, x) - \phi_0(x) &= \frac{1}{4\pi\cosh(\tau_0)}\int_0^{\tau-\tau_0} \sinh(s)\cosh(s+\tau_0)\int_{S^2} \Delta\phi_0(\exp_x(sz))dV_2(z)ds \\ &= \frac{1}{4\pi\cosh(\tau_0)}\int_0^{\tau-\tau_0} \sinh(s)\cosh(s+\tau_0)\int_{S^2} (\partial_s^2 + 2\coth(s)\partial_s)\phi_0(\exp_x(sz))dV_2(z)ds, \end{aligned} \tag{8.87}$$

where we shifted the integration variable, used the expression (8.10) for the Laplacian, and the angular part of the Laplacian vanishes thanks to Stokes's theorem. Now, if we integrate by parts in the variable s the integral containing ∂_s^2, we obtain

$$\phi(\tau,x)-\phi_0(x) = \frac{1}{4\pi\cosh(\tau_0)}\left[\coth(\tau)\int_{S(x,\tau-\tau_0)}\partial_s\phi_0(y)dA(y)+\right.$$

$$+\int_0^{\tau-\tau_0}\cosh(s)\cosh(s+\tau_0)\int_{S^2}\partial_s\phi_0(\exp_x(sz))dV_2(z)ds+ \tag{8.88}$$

$$\left.-\int_0^{\tau-\tau_0}\sinh(s)\sinh(s+\tau_0)\int_{S^2}\partial_s\phi_0(\exp_x(sz))dV_2(z)ds\right]. \tag{8.89}$$

Then, we can integrate by parts the last two integrals to obtain the final result in terms of geodesic spheres:

$$\begin{aligned}\phi(\tau,x) = \frac{1}{4\pi\cosh(\tau_0)}&\left[\frac{\cosh(\tau)}{\sinh(\tau-\tau_0)}\int_{S(x,\tau-\tau_0)}\partial_s\phi_0(y)dA(y)+\right.\\ &+\frac{\cosh(\tau-\tau_0)\cosh(\tau)}{\sinh^2(\tau-\tau_0)}\int_{S(x,\tau-\tau_0)}\phi_0(y)dA(y)+\\ &\left.-\frac{\sinh(\tau)}{\sinh(\tau-\tau_0)}\int_{S(x,\tau-\tau_0)}\phi_0(y)dA(y)\right].\end{aligned} \tag{8.90}$$

Using the following formulas for the hyperbolic functions:

$$\begin{cases}\sinh(s\pm\tau_0) = \sinh(s)\cosh(\tau_0)\pm\cosh(s)\sinh(\tau_0),\\ \cosh(s\pm\tau_0) = \cosh(s)\cosh(\tau_0)\pm\sinh(s)\sinh(\tau_0),\end{cases} \tag{8.91}$$

we finally obtain

$$\begin{aligned}\phi(\tau,x) &= \frac{\cosh(\tau)\sinh(\tau-\tau_0)}{4\pi\cosh(\tau_0)}\int_{S^2}\partial_s\phi_0\big(\exp_x((\tau-\tau_0)z)\big)dV_2(z)\\ &+\frac{1}{4\pi}\int_{S^2}\phi_0\big(\exp_x((\tau-\tau_0)z)\big)dV_2(z),\end{aligned} \tag{8.92}$$

which tends to zero in the limit $\tau\to\infty$ if we assume that the initial data are compactly supported.

The case $\boxed{\phi_0\equiv 0}$ is similar: we have that $\psi_0(x) = \frac{1}{\cosh^2(\tau_0)}\phi_1(x)$ and $\psi_1(x) = \tanh(\tau_0)\psi_0(x)$. Therefore:

$$\begin{aligned}\phi(\tau,x) = \frac{1}{4\pi\cosh^2(\tau_0)}&\left[\int_0^{\tau-\tau_0}\cosh(s+\tau_0)\cosh(s)\int_{S^2}\phi_1(\exp_x(sz))dV_2(z)ds\right.\\ &+\int_0^{\tau-\tau_0}\cosh(s+\tau_0)\sinh(s)\tanh(\tau_0)\int_{S^2}\phi_1(\exp_x(sz))dV_2(z)ds\\ &\left.+\int_0^{\tau-\tau_0}\sinh(s)\cosh(s+\tau_0)\int_{S^2}\partial_s\phi_1(\exp_x(sz))dV_2(z)ds\right].\end{aligned} \tag{8.93}$$

If we apply integration by parts to the last integral, in the variable s:

$$\begin{aligned}\phi(\tau,x) = \frac{1}{4\pi\cosh^2(\tau_0)}\Bigg[\int_0^{\tau-\tau_0} &\cosh(s+\tau_0)\sinh(s)\tanh(\tau_0)\int_{S^2}\phi_1(\exp_x(sz))dV_2(z)ds \\ -\int_0^{\tau-\tau_0}&\sinh(s)\sinh(s+\tau_0)\int_{S^2}\phi_1(\exp_x(sz))dV_2(z) \\ +\sinh(\tau-\tau_0)&\cosh(\tau)\int_{S^2}\phi_1\big(\exp_x((\tau-\tau_0)z)\big)dV_2(z)\Bigg].\end{aligned} \tag{8.94}$$

The sum of the first two terms in the square brackets converges to $C\|\phi_1\|_{L^1(\mathbb{H}^3)}$ as $\tau \to \infty$. Indeed we can use the relations (8.91) and the sum of the first two integrals becomes

$$C\int_0^{\tau-\tau_0}\sinh^2(s)\int_{S^2}\phi_1(\exp_x(sz))dV_2(z)ds = C\int_0^{\tau-\tau_0}\int_{S(x,s)}\phi_1(y)dA(y)ds, \tag{8.95}$$

with $C = \frac{\sinh^2(\tau_0)}{\cosh(\tau_0)} - \cosh(\tau_0)$. Furthermore, last term of (8.94) goes to zero as $\tau \to \infty$ if we assume that the initial data have compact support. Therefore, the integral does not depend on x in the limit $\tau \to \infty$, and the limiting value is finite.

Since this result holds for the cases $\phi_0 \equiv 0$ and $\phi_1 \equiv 0$ considered separately, it also holds in general.

8.5 Hyperbolic de Sitter Space

Let us now consider the case

$$a(\tau) = -\operatorname{csch}(\tau), \tag{8.96}$$

where $\tau \in (-\infty, 0)$. This choice of the scale factor applied to (4.3) leads to

$$\partial_\tau^2\phi + 2\coth(\tau)\partial_\tau\phi = \Delta\phi \tag{8.97}$$

In order to find the operator $\hat{O} = f(\tau)\partial_\tau + g(\tau)$ such that f and g satisfy equation (8.4), i.e.

$$\alpha(1+\coth^2(\tau)) + 2\beta\coth(\tau) = 0, \tag{8.98}$$

we have to choose $\alpha = \beta = 0$ and, for instance, $\kappa = -1$. Again, it follows that suitable operators for this case must be first-order differential operators. So, we have

$$\hat{O} = -\operatorname{csch}(\tau)\partial_\tau, \tag{8.99}$$

which satisfies $\partial_\tau^2\hat{O}\phi = L\hat{O}\phi$ by construction. This corresponds to an unphysical fluid model as well, since $a(\tau) = -\operatorname{csch}(\tau)$ is a solution for the Friedmann equation (5.12) for $w = -1$. More precisely, this is the de Sitter solution, having scale factor $a(t) = \sinh(t)$ for the hyperbolic case (see also [20]). In fact, if we switch to conformal time :

$$t = -\int\frac{d\tau}{\sinh(\tau)} = -\log\left(-\tanh\left(\frac{\tau}{2}\right)\right) \Leftrightarrow \tau = 2\operatorname{artanh}(-e^{-t}) \tag{8.100}$$

which implies:

$$a(t) = -\sinh\big(2\,\mathrm{artanh}(-e^{-t})\big)^{-1} = \frac{e^{2t}-1}{2e^t} = \sinh(t), \tag{8.101}$$

thanks to the relation $\mathrm{artanh}(x) = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$, for $|x| < 1$. Since $\tau \in (-\infty, 0)$, we have that $t \in (0, \infty)$. We can express the solution in terms of spherical means as follows. If we choose for the wave equation the following initial data at $t = t_0$:

$$\begin{cases} \phi(t_0, x) = \phi_0(x), \\ \partial_t \phi(t_0, x) = \phi_1(x), \end{cases} \tag{8.102}$$

then we have the following initial data for $\hat{O}\phi = -\mathrm{csch}(\tau)\partial_\tau \phi$ at the corresponding conformal time $\tau = \tau_0$:

$$\begin{cases} \psi_0(x) = \hat{O}\phi(\tau_0, x) = \mathrm{csch}^2(\tau_0)\phi_1(x), \\ \psi_1(x) = \partial_\tau \hat{O}\phi(\tau_0, x) = -\mathrm{csch}(\tau_0)(3\,\mathrm{csch}(\tau_0)\coth(\tau_0)\phi_1(x) + \Delta\phi_0(x)), \end{cases} \tag{8.103}$$

where the above follows from the fact that $\partial_\tau = -\mathrm{csch}(\tau)\partial_t$, $\partial_\tau^2\phi = -2\coth(\tau)\partial_\tau\phi + \Delta\phi$ (from (8.97)) and thus:

$$\begin{aligned} \partial_\tau \hat{O}\phi &= -\partial_\tau(\mathrm{csch}(\tau)\partial_\tau\phi) = \mathrm{csch}(\tau)\coth(\tau)\partial_\tau\phi - \mathrm{csch}(\tau)\partial_\tau^2\phi \\ &= 3\,\mathrm{csch}(\tau)\coth(\tau)\partial_\tau\phi - \mathrm{csch}(\tau)\Delta\phi \\ &= -3\,\mathrm{csch}^2(\tau)\coth(\tau)\partial_t\phi - \mathrm{csch}(\tau)\Delta\phi. \end{aligned} \tag{8.104}$$

Then, the spherical means solution is

$$\begin{aligned} \phi(\tau, x) = &-\int_{\tau_0}^{\tau} \frac{\cosh(s-\tau_0)}{4\pi}\sinh(s)\int_{S^2}\psi_0\big(\exp_x((s-\tau_0)z)\big)\,dV_2(z)ds + \\ &-\int_{\tau_0}^{\tau} \frac{\sinh(s-\tau_0)}{4\pi}\sinh(s)\int_{S^2} d\psi_0(c_z'(s-\tau_0))\,dV_2(z)ds + \\ &-\int_{\tau_0}^{\tau} \frac{\sinh(s-\tau_0)}{4\pi}\sinh(s)\int_{S^2}\psi_1\big(\exp_x((s-\tau_0)z)\big)\,dV_2(z)ds + \\ &-\phi(\tau_0, x). \end{aligned} \tag{8.105}$$

We recall that $\tau \in (-\infty, 0)$. As $\tau \to 0$, we have that $|\phi(\tau, x)|$ is bounded by a function of the variable x, since we can repeat the steps done for the anti-de Sitter solution. The difference between the latter and the present case is that the solution now does depend on x since the interval for the conformal time is $(-\infty, 0)$ rather than $(0, \infty)$. This is also the behaviour that we would expect from the Penrose diagram of the spacetime.

Similarly to the flat case of section 7.2, here the first derivative of the solution decays exponentially: $|\partial_t\phi| \lesssim e^{-2t}$ as $t \to \infty$. Indeed, if we take the partial derivative with respect to τ to both sides of (8.105) and we write the integrals over S^2 as integrals over the geodesic spheres, we have

$$|\partial_\tau \phi| \overset{(8.101)}{=} |\sinh(t)\partial_t \phi| \lesssim \frac{\sinh(\tau-\tau_0)}{\sinh(\tau-\tau_0)^2}\sinh(\tau) \sim \frac{1-e^{-2\tau}}{e^{-\tau_0}-e^{-2\tau+\tau_0}} \text{ as } \tau \to 0^-, \tag{8.106}$$

where we are assuming that the L^1-norms of ψ_0, $d\psi_0$ and ψ_1 are finite. As $\tau \to 0^-$, the limits $e^{-2\tau} \to 1$ and $\frac{1-e^{-t}}{1+e^{-t}} \to 1$ hold. Therefore, we can estimate the numerator of our last expression as follows:

$$\begin{aligned} 1-e^{-2\tau} \sim 2\tau &\overset{(8.100)}{=} 4\,\text{artanh}(-e^{-t}) = 2\log\left(\frac{1-e^{-t}}{1+e^{-t}}\right) \sim \left(\frac{1-e^{-t}}{1+e^{-t}}\right) - 1 \\ &= -2\frac{2e^{-t}}{(1+e^{-t})^2} \sim e^{-t} \text{ as } t \to \infty. \end{aligned} \tag{8.107}$$

In order to obtain the logarithm, we used the relation $\text{artanh}(x) = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$ for $|x| < 1$. Finally, from the relation (8.106):

$$|\partial_t \phi| \lesssim e^{-2t} \text{ as } t \to \infty \tag{8.108}$$

since $\sinh(t) \sim e^t$ in the limit $t \to \infty$.

8.6 Milne Universe

We will now work with the scale factor

$$a(\tau) = e^\tau, \tag{8.109}$$

with $\tau \in (0, \infty)$. After this choice, our wave equation (4.3) becomes

$$\partial_\tau^2 \phi + 2\partial_\tau \phi = \Delta\phi. \tag{8.110}$$

The constraint equation for the operator $\hat{O} = f(\tau)\partial_\tau + g(\tau)$ simply becomes

$$\alpha = \beta. \tag{8.111}$$

We can then pick, for instance, $\alpha = \beta = 1$ and $\kappa = 0$. In such case, the operator is

$$\hat{O}\phi = e^\tau \phi \tag{8.112}$$

and satisfies $\partial_\tau^2 \hat{O}\phi = L\hat{O}\phi$ by construction. This model represents the Milne universe. We can see that from the scale factor in the variable t (see also [20]):

$$t = \int e^\tau d\tau = e^\tau \Leftrightarrow \tau = \log t, \tag{8.113}$$

i.e. $a(t) = t$, with $t \in (0, \infty)$. Now, given initial data for the wave equation at $t = t_0$:

$$\begin{cases} \phi(t_0, x) = \phi_0(x), \\ \partial_t \phi(t_0, x) = \phi_1(x), \end{cases} \tag{8.114}$$

we have the following initial data for $\hat{O}\phi = e^\tau \phi$ at the corresponding conformal time $\tau = \tau_0$:

$$\begin{cases} \psi_0(x) = \hat{O}\phi(\tau_0, x) = e^{\tau_0}\phi_0(x), \\ \psi_1(x) = \partial_\tau \hat{O}\phi(\tau_0, x) = e^{\tau_0}\left(\phi_0(x) + e^{\tau_0}\phi_1(x)\right), \end{cases} \tag{8.115}$$

using that $\partial_\tau = e^\tau \partial_t$. The solution in terms of spherical means is given by (8.25):

$$\begin{aligned} e^\tau \phi(\tau, x) = & \frac{\cosh(\tau - \tau_0)}{4\pi} \int_{S^2} \psi_0\big(\exp_x((\tau - \tau_0)z)\big)\, dV_2(z) + \\ & + \frac{\sinh(\tau - \tau_0)}{4\pi} \int_{S^2} d\psi_0(c_z'(\tau - \tau_0))\, dV_2(z) \\ & + \frac{\sinh(\tau - \tau_0)}{4\pi} \int_{S^2} \psi_1\big(\exp_x((\tau - \tau_0)z)\big)\, dV_2(z). \end{aligned} \tag{8.116}$$

We notice that the decay of the solution as $\tau \to \infty$ is the same as that of the solution (8.66) if we suppose that the appropriate norms are finite. Thus,

$$|\phi| \lesssim t^{-2} \text{ as } t \to \infty. \tag{8.117}$$

9 First Order Operators for the Spherical Case

Now, we want to consider spacetimes with spherical space sections. Let $K = 1$. The constraint equation (6.2) becomes

$$\begin{aligned} \hat{O}\left(\partial_\tau^2 + 2\frac{a'}{a}\partial_\tau\right)\phi - \partial_\tau^2 \hat{O}\phi - K\hat{O}\phi = & \left(2\frac{a'}{a}f - 2f'\right)\partial_\tau^2\phi + \\ & + \left[2\frac{a'}{a}g - 2g' - f'' + 2\left(\frac{a'}{a}\right)' f - f\right]\partial_\tau\phi - (g'' + g)\phi = 0 \end{aligned} \tag{9.1}$$

and is satisfied if and only if we assume that

$$\begin{cases} f(\tau) = \kappa a, \\ g(\tau) = \alpha \cos(\tau) + \beta \sin(\tau), \end{cases} \tag{9.2}$$

and

$$2\frac{a'}{a}g - 2g' - f'' + 2\left(\frac{a'}{a}\right)' f - f = 0, \tag{9.3}$$

with $\alpha, \beta, \kappa \in \mathbb{R}$. By plugging f and g in the above:

$$2a'(\alpha\cos(\tau) + \beta\sin(\tau)) - 2(\beta\cos(\tau) - \alpha\sin(\tau))a + \kappa a'' a - 2\kappa(a')^2 - \kappa a^2 = 0 \tag{9.4}$$

is the constraint equation we are going to use to find suitable operators.

9.1 Spherical Means in a Static Spacetime with $K = 1$

In the following, we will find the solution of the generalized wave equation (4.5) in the spacetime given by

$$g = -dt^2 + d\Sigma_3^2, \tag{9.5}$$

where now $d\Sigma_3^2$ is the line element for the sphere S^3. We will find again a Kirchhoff-like formula as for the case $K = -1$. We will consider the Cauchy problem

$$\begin{cases} \partial_t^2 \phi - L\phi = 0, \\ \phi(0, x) = g(x), \dot{\phi}(0, x) = h(x). \end{cases} \tag{9.6}$$

Here, $L\phi = \Delta\phi - \phi$ and $x = (x^1, x^2, x^3)$ are the spatial coordinates. Using geodesic polar coordinates about a point x, we know that

$$d\Sigma_3^2 = dr^2 + \sin(r)^2 d\Omega^2, \tag{9.7}$$

where $d\Omega^2$ is the line element for S^2. So:

$$ds^2 = dr^2 + \sin(r)^2 d\Omega^2 = dr^2 + \sin(r)^2 d\theta^2 + \sin(r)^2 \sin(\theta)^2 d\varphi^2. \tag{9.8}$$

Now, the definition of Laplacian is

$$\Delta = \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^i} \left(\sqrt{|g|} g^{ij} \frac{\partial}{\partial x^j} \right) \tag{9.9}$$

and we know that $\sqrt{|g|} = \sin(r)^2 \sin(\theta)$, $g_{rr} = 1$, $g_{\theta\theta} = \sin(r)^2$, $g_{\varphi\varphi} = \sin(r)^2 \sin(\theta)^2$. Thus, we can write the Laplacian as:

$$\begin{aligned} \Delta = & \frac{1}{\sin(r)^2 \sin(\theta)} \left[\frac{\partial}{\partial r} \left(\sin(r)^2 \sin(\theta) \frac{\partial}{\partial r} \right) + \frac{\partial}{\partial \theta} \left(\sin(r)^2 \sin(\theta) \frac{1}{\sin(r)^2} \frac{\partial}{\partial \theta} \right) + \right. \\ & \left. + \frac{\partial}{\partial \varphi} \left(\sin(r)^2 \sin(\theta) \frac{1}{\sin(r)^2 \sin(\theta)^2} \frac{\partial}{\partial \varphi} \right) \right] = \\ = & \frac{\partial^2}{\partial r^2} + 2\cot(r) \frac{\partial}{\partial r} + \frac{1}{\sin(r)^2} \Delta_S, \end{aligned} \tag{9.10}$$

where we used that

$$\Delta_S = \frac{1}{\sin\theta} \frac{\partial}{\partial \theta} \left(\sin(\theta) \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin(\theta)^2} \frac{\partial^2}{\partial \varphi^2} \tag{9.11}$$

is the Laplacian on S^2. The geodesic sphere $S(x, r)$ about the point x is defined as

$$S(x, r) := \exp_x \left(\{ y \in T_x S^3 \text{ such that } \|y\| = r \} \right) = \exp_x(\partial B_r(0)), \tag{9.12}$$

where $B_r(0)$ is the ball of radius r about 0 in the tangent space. Here and in the following we will assume that $r \in (0, \pi)$. Similarly to what we did in the hyperbolic case, we define the spherical mean of a function $f(t, y)$ over the geodesic sphere of radius r about x as

$$M_f(t, r, x) := \frac{1}{4\pi \sin(r)^2} \int_{S(x,r)} f(t, y) dA(y) = \frac{1}{4\pi} \int_{S^2} f(t, \exp_x(rz)) dV_2(z) \tag{9.13}$$

where dA is the area element of the geodesic sphere. We also define the spherical mean of a function $g = g(y)$ in a similar way:

$$M_g(r; x) := \frac{1}{4\pi \sin(r)^2} \int_{S(x,r)} g(y) dA(y) = \frac{1}{4\pi} \int_{S^2} g(\exp_x(rz)) dV_2(z). \tag{9.14}$$

We notice that the geodesic spheres $S(x, r)$ and $S(x, \pi - r)$ are different, even though $\sin(r) = \sin(\pi - r)$, see also figure 2.4.

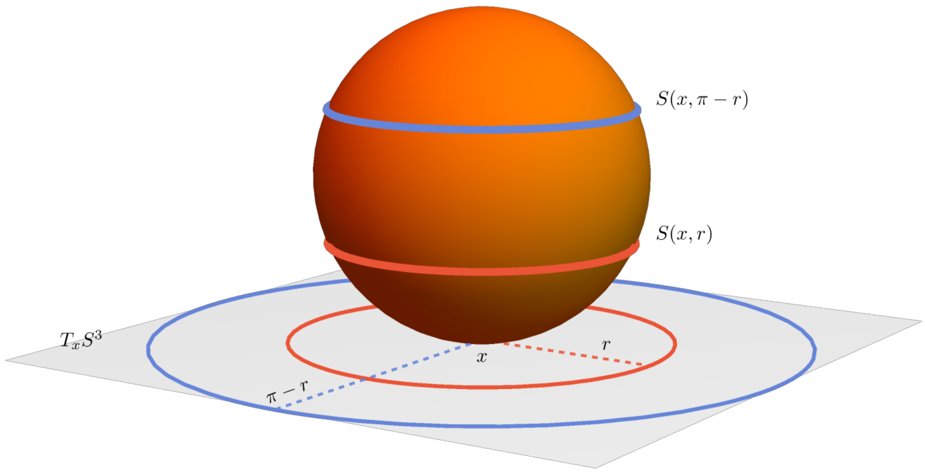

Figure 2.4: If we suppress a dimension, the geodesic spheres can be visualized as the parallels of a 2-sphere. Here, $0 < r < \frac{\pi}{2}$.

Moreover, we define $M_f(t, -r, x) = M_f(t, r, x)$ in order to extend the function to negative values of the second argument. Similarly to the hyperbolic case, we want to extend it as an even function, since we can think of any term $(-r)z$ in the last integral of (9.13) as $r(-z)$, i.e. as if we consider the vector antipodal to z, still belonging to $S^2 \subset T_x S^3$. We also define $M_f(t, k\pi + r, x) = M_f(t, r, x), k \in \mathbb{N}$, so that we can extend the domain of r. Now, given a solution ϕ for our Cauchy problem, we have:

$$\begin{aligned}
\frac{\partial^2}{\partial t^2} M_\phi(t,r,x) &= \frac{1}{4\pi}\int_{S^2} \frac{\partial^2}{\partial t^2}\phi(t,\exp_x(rz))dV_2(z) && \text{(by (9.13))}\\
&= \frac{1}{4\pi}\int_{S^2} L\phi(t,\exp_x(rz))dV_2(z) && \text{(by (9.6))}\\
&= \frac{1}{4\pi}\int_{S^2}\left[\frac{\partial^2}{\partial r^2}\phi(t,\exp_x(rz)) + 2\cot(r)\frac{\partial}{\partial r}\phi(t,\exp_x(rz)) + \right.\\
&\quad \left. + \frac{1}{\sin(r)^2}\Delta_S\phi(t,\exp_x(rz))\right] dV_2(z) + \\
&\quad - \frac{1}{4\pi}\int_{S^2}\phi(t,\exp_x(rz)dV_2(z) && \text{(by (9.10))}\\
&= \left(\frac{\partial^2}{\partial r^2} + 2\cot(r)\frac{\partial}{\partial r} - 1\right) M_\phi(t,r,x) && (9.15)
\end{aligned}$$

where in the last step we used the divergence theorem on the sphere:

$$\int_{S^2}\Delta_S\phi(t,\exp_x(rz)))dV_2(z) = \int_{S^2}\nabla_S\cdot\big(\nabla_S\phi(t,\exp_x(rz))\big)dV_2(z) = 0. \tag{9.16}$$

If g and h are the initial data for ϕ, it follows that we have initial data

$$\begin{cases} M_\phi(0,r,x) = M_g(r;x), \\ \partial_t M_\phi(0,r,x) = M_h(r;x), \end{cases} \tag{9.17}$$

for the spherical means.

Ignoring for the moment the dependence of M_ϕ on x, if $M_\phi(t,r)$ satisfies (9.15), it can be checked that $\omega(t,r) := \sin(r)M_\phi(t,r)$ satisfies

$$\partial_t^2\omega - \partial_r^2\omega = 0. \tag{9.18}$$

Given initial data $\omega(0,r) = \gamma(r) = \sin(r)M_g(r;x)$ and $\dot\omega(0,r) = \psi(r) = \sin(r)M_h(r;x)$, the solution of equation (9.18) is given by the d'Alembert formula:

$$\omega(t,r) = \frac{1}{2}\left[\gamma(r+t) + \gamma(r-t) + \int_{r-t}^{r+t}\psi(s)ds\right] \tag{9.19}$$

Since $M_\phi(t,r) = \frac{1}{\sin(r)}\omega(t,r)$ and, recovering the x-dependence:

$$M_\phi(t,r,x) = \frac{1}{2\sin(r)}\left[\gamma(r+t) + \gamma(r-t) + \int_{r-t}^{r+t}\psi(s)ds\right]. \tag{9.20}$$

In the limit $r \to 0$, we have that $M_\phi(t,r,x) \to \phi(t,x)$ (as can be checked from (9.13)) and the term in (9.20) inside the square brackets tends to 0 since γ and ψ are odd functions. Thus, taking the limit $r \to 0$ and using L'Hôpital's rule:

$$\begin{aligned}\phi(t,x) &= \lim_{r\to 0} \frac{1}{2\cos(r)} \left[\gamma'(r+t) + \gamma'(r-t) + \psi(r+t) - \psi(r-t)\right] \\ &= \gamma'(t) + \psi(t) = \partial_t(\sin(t) M_g(t;x)) + \sin(t) M_h(t;x),\end{aligned} \tag{9.21}$$

using again the symmetries of γ and ψ.

Therefore, the solution of the Cauchy problem

$$\begin{cases}\partial_t^2 \Phi - L\Phi = 0, \\ \Phi(t_0,x) = g(x),\ \dot{\Phi}(t_0,x) = h(x)\end{cases} \tag{9.22}$$

is given by

$$\Phi(t,x) = \partial_t(\sin(t-t_0) M_g(t-t_0;x)) + \sin(t-t_0) M_h(t-t_0;x). \tag{9.23}$$

Using definition (9.14) and the definition of the exponential map, the above can be written as

$$\begin{aligned}\Phi(t,x) = &\frac{\cos(t-t_0)}{4\pi} \int_{S^2} g\big(\exp_x((t-t_0)z)\big)\, dV_2(z) + \\ &+ \frac{\sin(t-t_0)}{4\pi} \int_{S^2} dg\ (\dot{c}_z(t-t_0))\ dV_2(z) + \\ &+ \frac{\sin(t-t_0)}{4\pi} \int_{S^2} h\big(\exp_x((t-t_0)z)\big)\, dV_2(z),\end{aligned} \tag{9.24}$$

where the same formalism used to obtain formula (8.25) was adopted, i.e. $c_z(t) = \exp_x(tz)$ with $c_z(0) = x$ and $\dot{c}_z(0) = z$. In terms of geodesic spheres:

$$\begin{aligned}\Phi(t,x) = &\frac{\cos(t-t_0)}{4\pi \sin(t-t_0)^2} \int_{S(x,t-t_0)} g(y) dA(y) + \\ &+ \frac{1}{4\pi \sin(t-t_0)} \int_{S(x,t-t_0)} [dg(X)](y)\, dA(y) + \\ &+ \frac{1}{4\pi \sin(t-t_0)} \int_{S(x,t-t_0)} h(y)\, dA(y),\end{aligned} \tag{9.25}$$

where X is the outward unit normal to $S(x,t-t_0)$.

9.2 Dust-filled Spherical FLRW Model

Let us now consider the case with scale factor

$$a(\tau) = 1 - \cos(\tau), \tag{9.26}$$

corresponding to the dust model. The wave equation (4.3) becomes

$$\partial_\tau^2\phi + \frac{2\sin(\tau)}{1-\cos(\tau)}\partial_\tau\phi = \Delta\phi \tag{9.27}$$

and is well defined in the interval $(0, 2\pi)$ in which the conformal time τ varies. We expect solutions to blow up in correspondence to the physical singularities at $\tau = 0$ and $\tau = 2\pi$, similarly to what happens to the solutions of the wave equation as $\tau \to 0$ for $K = 0$ and $K = -1$. In order to find the operator $\hat{O}\phi = f(\tau)\partial_\tau + g(\tau)$ such that f and g satisfy equation (9.4), i.e.

$$(2\beta - 3\kappa)(\cos(\tau) - 1) = 2\alpha\sin(\tau), \tag{9.28}$$

we can choose $\alpha = 0$, $\beta = \frac{3}{2}$, $\kappa = 1$. Using (9.2), the sought operator is therefore

$$\hat{O}\phi = f(\tau)\partial_\tau\phi + g(\tau)\phi = (1-\cos(\tau))\partial_\tau\phi + \frac{3}{2}\sin(\tau)\phi, \tag{9.29}$$

where $\tau \in (0, 2\pi)$. If we define $\gamma := \frac{\tau}{2}$, then

$$\hat{O}\phi = \frac{1}{\sin(\gamma)}\partial_\gamma((\sin\gamma)^3\phi) = \frac{2}{\sin\left(\frac{\tau}{2}\right)}\partial_\tau\left(\sin\left(\frac{\tau}{2}\right)\phi\right) \tag{9.30}$$

is the expression given in [16] that we can invert in the following steps. Given initial data for the wave equation at $t = t_0$:

$$\begin{cases}\phi(t_0, x) = \phi_0(x),\\ \partial_t\phi(t_0, x) = \phi_1(x),\end{cases} \tag{9.31}$$

we have the following initial data for $\hat{O}\phi = (1-\cos(\tau))\partial_\tau\phi + \frac{3}{2}\sin(\tau)\phi$ at $\tau = \tau_0$:

$$\begin{cases}\psi_0(x) = \hat{O}\phi(\tau_0, x) = \frac{3}{2}\sin(\tau_0)\phi_0(x) + (1-\cos(\tau_0))\phi_1(x),\\ \psi_1(x) = \partial_\tau\hat{O}\phi(\tau_0, x) = \frac{3}{2}\cos(\tau_0)\phi_0(x) + \frac{1}{2}\sin(\tau_0)(1-\cos(\tau_0))\phi_1(x)\\ \qquad\qquad + (1-\cos(\tau_0))\Delta\phi_0(x),\end{cases} \tag{9.32}$$

where we used that $\partial_\tau = (1-\cos(\tau))\partial_t$, $\partial_\tau^2 = \Delta\phi - \frac{2\sin(\tau)}{1-\cos(\tau)}\partial_\tau\phi$ (see (9.27)) and thus

$$\begin{aligned}\partial_\tau\hat{O}\phi &= \partial_\tau\left((1-\cos(\tau))\partial_\tau\phi + \frac{3}{2}\sin(\tau)\phi\right) =\\ &= \frac{5}{2}\sin(\tau)\partial_\tau\phi + \frac{3}{2}\cos(\tau)\phi + (1-\cos(\tau))\left(\Delta\phi - \frac{2\sin(\tau)}{1-\cos(\tau)}\partial_\tau\phi\right) =\\ &= \frac{3}{2}\cos(\tau)\phi + \frac{1}{2}\sin(\tau)(1-\cos(\tau))\partial_t\phi + (1-\cos(\tau))\Delta\phi.\end{aligned} \tag{9.33}$$

Finally, the spherical means formula (9.24) applied to the expression (9.30) of our operator gives:

$$\begin{aligned} \sin\left(\frac{\tau}{2}\right)\phi(\tau,x) = & \int_{\tau_0}^{\tau} \frac{\cos(s-\tau_0)}{8\pi} \sin\left(\frac{s}{2}\right) \int_{S^2} \psi_0\big(\exp_x((s-\tau_0)z)\big)\, dV_2(z) ds + \\ & + \int_{\tau_0}^{\tau} \frac{\sin(s-\tau_0)}{8\pi} \sin\left(\frac{s}{2}\right) \int_{S^2} d\psi_0(c_z'(s-\tau_0))\, dV_2(z) ds + \\ & + \int_{\tau_0}^{\tau} \frac{\sin(s-\tau_0)}{8\pi} \sin\left(\frac{s}{2}\right) \int_{S^2} \psi_1\big(\exp_x((s-\tau_0)z)\big)\, dV_2(z) ds + \\ & + \sin\left(\frac{\tau_0}{2}\right)\phi_0(x). \end{aligned} \tag{9.34}$$

In general, we expect that such a solution will diverge as $\tau \to 0$ and as $\tau \to 2\pi$. In particular, we can pick $\bar{x} \in S^3$, choose the initial data ϕ_0 and ϕ_1 such that $\psi_0 = 0$ and repeat the steps done for the dust model in the case of hyperbolic space sections (see 8.2). Indeed, if we choose the function (8.44) as ϕ_0, we get

$$\frac{3}{2}\sin(\tau_0)\phi_0(x) = (\cos(\tau_0) - 1)\phi_1(x) \tag{9.35}$$

and

$$\psi_1(x) = A\phi_0(x) + B\Delta\phi_0(x) \tag{9.36}$$

with $A = \frac{3}{2}(\cos(\tau_0) - \frac{1}{2}\sin^2(\tau_0))$, $B = 1 - \cos(\tau_0)$. Following the same steps, we finally get that the integral over ψ_1 in (9.34) gives the following contribution to $\sin\left(\frac{\tau}{2}\right)\phi(\tau,\bar{x})$ for $\tau > 1$:

$$\int_0^1 \sin\left(\frac{p+\tau_0}{2}\right)\big(A\sin(p)\phi_0(p) + B\sin(p)\phi_0''(p) + 2B\cos(p)\phi_0'(p)\big)\, dp \simeq 0.1 \tag{9.37}$$

if $\tau_0 = \frac{\pi}{4}$ and $N = 50$. The value grows linearly with N. If we divide both sides of the spherical means expression by $\sin\left(\frac{\tau}{2}\right)$, we can explicitly see the divergence of the solution as $\tau \to 2\pi$. A similar computation leads to the same result for $\tau \to 0$.

Analogously to the dust case for hyperbolic and flat space sections, also here we can construct specific solutions which do not blow up at the physical singularities. We can either choose to keep the initial data (ψ_0, ψ_1) constant, or we can choose to work with the initial data (ϕ_0, ϕ_1).

9.3 Radiation-filled Spherical FLRW Model

Now, let us pick the scale factor

$$a(\tau) = \sin(\tau), \tag{9.38}$$

which describes a universe filled with radiation. Hence, the wave equation (4.3) is

$$\partial_\tau^2\phi + 2\cot(\tau)\partial_\tau\phi = \Delta\phi, \tag{9.39}$$

which is well defined for $\tau \in (0, \pi)$. As for the dust case, we expect solutions to blow up at the physical singularities, i.e. for $\tau = 0$ and $\tau = \pi$. Now, we can find the operator $\hat{O}\phi = f(\tau)\partial_\tau\phi + g(\tau)\phi$ such that equation (9.4) is satisfied, i.e.

$$\alpha = \kappa \tag{9.40}$$

if we choose $\alpha = \kappa = 0$ and $\beta = 1$. The operator then becomes:

$$\hat{O}\phi = \sin(\tau)\phi. \tag{9.41}$$

An independent choice is given by $\beta = 0$ and $\alpha = \kappa = 1$:

$$\hat{O}\phi = \sin(\tau)\partial_\tau\phi + \cos(\tau)\phi = \partial_\tau(\sin(\tau)\phi). \tag{9.42}$$

As happens for the cases $K = 0$ and $K = -1$, the operator

$$\hat{O}\phi = \partial_\tau^k(\sin(\tau)\phi) \tag{9.43}$$

will be a solution for every $k \in \mathbb{N}$. Now, if we consider the initial data for the wave equation at $t = t_0$:

$$\begin{cases} \phi(t_0, x) = \phi_0(x), \\ \partial_t\phi(t_0, x) = \phi_1(x), \end{cases} \tag{9.44}$$

we have the following initial data for $\hat{O}\phi = \sin(\tau)\phi$ at $\tau = \tau_0$:

$$\begin{cases} \psi_0(x) = \hat{O}\phi(\tau_0, x) = \sin(\tau_0)\phi_0(x), \\ \psi_1(x) = \partial_\tau\hat{O}\phi = \sin(\tau_0)\Delta\phi_0(x) - \sin(\tau_0)\cos(\tau_0)\phi_1(x) \end{cases} \tag{9.45}$$

since $\partial_\tau = \sin(\tau)\partial_t$, $\partial_\tau^2 = \Delta\phi - 2\cot(\tau)\partial_\tau\phi$ (see (9.39)) and thus:

$$\begin{aligned} \partial_\tau\hat{O}\phi &= \partial_\tau(\sin(\tau)\partial_\tau\phi) = \cos(\tau)\partial_\tau\phi + \sin(\tau)(\Delta\phi - 2\cot(\tau)\partial_\tau\phi) \\ &= \sin(\tau)\Delta\phi - \sin(\tau)\cos(\tau)\partial_t\phi. \end{aligned} \tag{9.46}$$

The spherical means solution is given by (9.24):

$$\begin{aligned} \phi(\tau, x) = &\frac{\cos(\tau - \tau_0)}{4\pi\sin(\tau)}\int_{S^2}\psi_0\big(\exp_x((\tau - \tau_0)z)\big)\,dV_2(z) + \\ &+ \frac{\sin(\tau - \tau_0)}{4\pi\sin(\tau)}\int_{S^2} d\psi_0\left(\dot{c}_z(\tau - \tau_0)\right)\,dV_2(z) + \\ &+ \frac{\sin(\tau - \tau_0)}{4\pi\sin(\tau)}\int_{S^2}\psi_1\big(\exp_x((\tau - \tau_0)z)\big)\,dV_2(z), \end{aligned} \tag{9.47}$$

When evaluated at a generic $x \in S^3$ such that $\phi_0(x) \neq 0$ and $\phi_1(x) \neq 0$, the solution will diverge at the singularities, i.e. as $\tau \to 0$ and as $\tau \to \pi$.

Again, specific non-divergent solutions can be constructed after working with the initial data, as done for the cases $K = 0$ and $K = -1$.

10 Higher Order Operators

Two interesting cases appear when experimenting with higher order operators: the radiation-filled flat FLRW model, already discussed in section 7.4 and a family of FLRW models that includes the even-dimensional de Sitter spaces, which we will now study. We will consider the flat case.

Setting

$$a(\tau) = \tau^j, \tag{10.1}$$

equation (4.3) becomes

$$\partial_\tau^2 \phi + \frac{2}{\tau} j \partial_\tau = \Delta \phi, \tag{10.2}$$

where $\Delta = \delta^{ij} \partial_i \partial_j$ is the Laplacian on $\mathbb{R}^3$. The operator

$$\hat{O} = \tau^{-1} \partial_\tau, \tag{10.3}$$

which already appeared in section 7.2, has the property

$$\hat{O} \left(\partial_{\tau^2} + \frac{2}{\tau} j \partial_\tau \right) \phi = \left(\partial_\tau^2 + \frac{2j+2}{\tau} \partial_\tau \right) \hat{O} \phi, \tag{10.4}$$

and consequently

$$\hat{O}^k \left(\partial_\tau^2 + \frac{2}{\tau} j \partial_\tau \right) \phi = \left(\partial_\tau^2 + \frac{2j+2k}{\tau} \partial_\tau \right) \hat{O}^k \phi. \tag{10.5}$$

In particular, if ϕ is a solution of (10.2) and

$$j = -k, \tag{10.6}$$

then

$$\partial_\tau^2 \hat{O}^k \phi = \Delta \hat{O}^k \phi, \tag{10.7}$$

that is, $\hat{O}^k \phi$ is a solution of the wave equation in Minkowski space. In this case, we have

$$a(\tau) = \tau^{-k}, \tag{10.8}$$

so

$$\frac{dt}{d\tau} = a(\tau) \Leftrightarrow t = \log \tau. \tag{10.9}$$

Changing the time coordinate t to $-t$:

$$a(t) = e^t, \tag{10.10}$$

so we have the expanding flat de Sitter space.

Given initial data for the wave equation at $t = t_0$:

$$\begin{cases} \phi(t_0, x) = \phi_0(x), \\ \partial_t \phi(t_0, x) = \phi_1(x), \end{cases} \tag{10.11}$$

we can construct initial data for $\hat{O}^k \phi$ at the corresponding conformal time $\tau = \tau_0$ using equation (10.2) to convert second order time derivatives into spatial derivatives. Therefore, if the appropriate Sobolev norms of the initial data are finite, then by the standard decay of the wave equation in Minkowski space we have

$$|\hat{O}^k \phi| \lesssim 1 \Leftrightarrow |\tau^{-1} \partial_\tau (\hat{O}^{k-1} \phi)| \lesssim 1. \tag{10.12}$$

Integrating we obtain, for $\tau \leq \tau_0$,

$$|\partial_\tau (\hat{O}^{k-1}) \phi| \lesssim \tau \Rightarrow |\hat{O}^{k-1} \phi| \lesssim 1, \tag{10.13}$$

and iterating this argument leads to

$$|\hat{O} \phi| \lesssim 1 \Leftrightarrow |\partial_\tau \phi| \lesssim \tau \Leftrightarrow |\partial_t \phi| \lesssim \tau^2 = e^{-2t}, \tag{10.14}$$

in agreement with the estimate (2.36).

3
Energy Methods

11 Introduction

Energy methods are used in PDE theory to prove uniqueness and decay rates of solutions to the wave equation $\Box\phi = 0$. The energy at the time t of the solution is defined as a quantity depending on ϕ and on its first-order partial derivatives. The time derivative of this energy gives information about the behaviour of the solution in the so-called *domain of dependence*, determined by the space on which initial data are given. As we will see, this approach can be extended to a general-relativistic context, where the role of the energy is played by the 00 component of the energy-momentum tensor. Particularly, we will employ energy methods to find the decay rate of solutions of $\Box_g\phi = 0$ in some cosmological spacetimes. Although we already know some decay rates from the mode expansions of chapter 1, the energy methods constitute a general approach that can give insights about the more complicated analysis of Einstein's equations.

In the current chapter, we will use some results about Sobolev spaces and Lorentzian manifolds that we will repeat here for convenience (see also e.g. [18], [24] and chapter 5 of [20]). First of all, we want to define the space to which initial data belong.

Definition 3.1 (Sobolev norm)**.** *Given a smooth function $f : \mathbb{R}^n \to \mathbb{R}$, we define the k-th Sobolev norm of f as*

$$\|f\|^2_{H^k(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \sum_{|\alpha|\le k} (\partial^\alpha f)^2, \tag{11.1}$$

where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$, $|\alpha| = \sum_i \alpha_i$, $\partial = (\partial_{x_1}, \ldots, \partial_{x_n})$ and $\partial^\alpha = (\partial_{x_1})^{\alpha_1} \cdots (\partial_{x_n})^{\alpha_n}$. The space

$$H^k(\mathbb{R}^n) = \{f \in L^2(\mathbb{R}^n) \,|\, \partial^\alpha f \in L^2(\mathbb{R}^n) \text{ for every } \alpha \in \mathbb{N}_0^n, |\alpha| \le k\} \tag{11.2}$$

is the k-th Sobolev space.

The space $H^k(\mathbb{R}^n)$ is a Hilbert space when endowed with the Sobolev norm. From [18], theorem 8.8, where we put $p = 2$ and $m = k$, we have the following inequality:

Theorem 3.1 (Sobolev inequality)**.** *If $f \in H^k(\mathbb{R}^n)$ for $k > \frac{n}{2}$, then:*

$$\|f\|_{L^\infty(\mathbb{R}^n)} \lesssim \|f\|_{H^k(\mathbb{R}^n)}. \tag{11.3}$$

In the following, we will assume that the initial data ϕ_0 and ϕ_1 are, respectively, in $H^k(\mathbb{R}^n)$ and in $H^{k-1}(\mathbb{R}^n)$. This hypothesis lets us assume without loss of generality that the solution of our wave equation is smooth. Indeed, let us suppose that we proved some estimate for smooth solutions of the wave equation and we want to apply such a result for non-smooth solutions. Since $\phi_0 \in H^k(\mathbb{R}^n)$ and $\phi_1 \in H^{k-1}(\mathbb{R}^n)$, they can be approximated by smooth functions (see e.g. page 265 in [12]). The sequence of smooth initial data for the wave equation gives a sequence of smooth solutions (by theorem 2.4, section 5 of [20]). Thus, we can apply our estimate to the latter. Taking the limit to retrieve the original data, the estimate will hold true also for the non-smooth solution since the Cauchy problem is well-posed for $\phi_0 \in H^k(\mathbb{R}^n)$ and $\phi_1 \in H^{k-1}(\mathbb{R}^n)$.

Moreover, we want to use some notions of causality in order to define the domain of dependence in this relativistic context.

Definition 3.2 (Time-orientable spacetime)**.** *A spacetime (M, g) is time-orientable if there exists a timelike vector field X, i.e. such that $g(X, X) < 0$. Given this time orientation and some point $p \in M$, we say that $v \in T_pM$ is past-directed (or past-pointing) if $g(v, X) > 0$.*

Now, let (M, g) be a time-orientable spacetime.

Definition 3.3 (Curves in a spacetime)**.** *A curve $\lambda(t)$ in M is causal if its tangent vectors at each point of the curve are either null or timelike vectors. A curve $\lambda(t)$ is a past-directed causal curve if it is causal and its tangent vectors at each point of the curve are past-directed.*

Similar definitions hold for past-directed/future-directed timelike/null curves.

Definition 3.4 (Endpoint of a curve, extendibility)**.** *A point $p \in M$ is the past endpoint of a past-directed causal curve $\lambda(t)$ if for every neighbourhood O of p there exists an instant of time t_0 such that $\lambda(t) \in O, \forall\, t > t_0$. A past-directed curve λ is past inextendible if it has no past endpoint.*

We notice, in particular, that if λ has a past endpoint, then it must be unique. Indeed, let us assume by contradiction that $p, q \in M$ are two distinct past endpoints for λ. It means that there exist two instants of time t_1 and t_2 such that $\lambda(t) \in O_1, \forall\, t \geq t_1$, for every neighbourhood O_1 of p and $\lambda(t) \in O_2, \forall\, t \geq t_2$, for every neighbourhood O_2 of q. By definition of a smooth manifold, M is a Hausdorff topological space. So, we can choose t_1, t_2, O_1 and O_2 such that $\lambda(t) \in O_1\ \forall\, t \geq t_1$, $\lambda(t) \in O_2\ \forall\, t \geq t_2$ and $O_1 \cap O_2 = \emptyset$. If $t := \max\{t_1, t_2\}$, we have that $\lambda(s)$ belongs simultaneously to two disjoint sets for every $s > t$, which is a contradiction.

12 FLRW Metric with Flat Space Sections

In this section, we are going to obtain the decay of solutions to the wave equation in a FLRW background spacetime endowed with scale factor $a(t) = t^p$. Following the idea behind the proof of theorem 2.3 of [21], we will prove the following:

Theorem 3.2. *Let $n > 2$ and let $I \subset \mathbb{R}$ be the open interval $(t_*, \infty) \ni t_0$ for some $t_* \geq 0$. Assume that (M, g) describes an expanding universe given by the metric*

$$g = -dt^2 + t^{2p}\left(\left(dx^1\right)^2 + \ldots + (dx^n)^2\right) \tag{12.1}$$

with $p > 1$, whose flat n-dimensional space sections are given by $\mathbb{R}^n$. Suppose that ϕ is a smooth solution to the Cauchy problem

$$\begin{cases} \Box_g \phi(t, x) = 0, \\ \phi(t_0, x) = \phi_0(x), \\ \partial_t \phi(t_0, x) = \phi_1(x) \end{cases} \tag{12.2}$$

with $t \geq t_0$, $x \in \mathbb{R}^n$, $\phi_0 \in H^k(\mathbb{R}^n)$, $\phi_1 \in H^{k-1}(\mathbb{R}^n)$ and $k > \frac{n}{2} + 2$. Then:

$$\|\partial_t \phi(t, \cdot)\|_{L^\infty(\mathbb{R}^n)} \lesssim a(t)^{-2+\frac{1}{p}} = t^{1-2p}. \tag{12.3}$$

Proof. We divide the proof into two separate steps.
1) Laplacian estimate. The energy-momentum tensor associated to the wave equation is

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} \partial_\alpha \phi \partial^\alpha \phi, \tag{12.4}$$

see for instance [20]. In particular, this is the energy-momentum associated to the Lagrangian density $\mathcal{L} = \sqrt{-g}\left(\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \frac{1}{16\pi} R\right)$, which gives the wave equation as one of its equations of motion. Now, we can define the vector field

$$X := a^{2-n} \frac{\partial}{\partial t} \tag{12.5}$$

and the current

$$J_\mu := T_{\mu\nu} X^\nu. \tag{12.6}$$

The vector X is timelike (i.e. $g(X, X) < 0$) and therefore defines a time orientation. The current can also be written as

$$J = (X \cdot \phi) \text{grad}\, \phi - \frac{1}{2} g(\text{grad}\, \phi, \text{grad}\, \phi) X. \tag{12.7}$$

In the following, we will use that $g(J, X) > 0$, i.e. J is past-pointing. Indeed, we can choose spacelike and orthogonal vector fields $E_1, \dots, E_n$ such that $\{X, E_1, \dots E_n\}$ is an orthogonal basis in each tangent space. In such tangent spaces we can write grad ϕ as

$$\text{grad}\,\phi = c^0 X + c^1 E_1 + \ldots + c^n E_n. \tag{12.8}$$

Since $X \cdot \phi = d\phi(X) = g(X, \text{grad}\,\phi)$, and by linearity, we get

$$\begin{aligned} g(J, X) &\overset{(12.7)}{=} g(X, \text{grad}\,\phi)^2 - \frac{1}{2} g(\text{grad}\,\phi, \text{grad}\,\phi) g(X, X) \overset{(12.8)}{=} \\ &= \frac{c_0^2}{2} g(X, X)^2 - \frac{1}{2}\left(\left(c^1\right)^2 g(E_1, E_1) + \ldots + (c^n)^2 g(E_n, E_n)\right) g(X, X) \geq 0, \end{aligned} \tag{12.9}$$

since $g(X, X) < 0$, $g(E_i, E_i) \geq 0$ and $g(X, E_i) = 0$.

Moreover, we define the future unit normal vector field

$$N := \frac{\partial}{\partial t} \tag{12.10}$$

and the energy

$$E(t) := \int_{\{t\}\times\mathbb{R}^n} g(J, N)\sigma = \int_{\mathbb{R}^n} a^2 T_{00}\, d^n x = \frac{1}{2}\int_{\mathbb{R}^n} \left(a^2 \dot{\phi}^2 + \delta^{ij}\,\partial_i\phi\partial_j\phi\right) d^n x, \tag{12.11}$$

where the equalities follow from the fact that $g(J, N) = J_0 = a^{2-n} T_{00}$, $i^*\sigma = \sqrt{\tilde{g}} d^n x = a^n d^n x$ where $i : \mathbb{R}^n \to \{t\} \times \mathbb{R}^n$ is the inclusion map, $\tilde{g}$ is the induced metric on $\mathbb{R}^n$ and, finally, we used the definition (12.4) of the energy-momentum tensor. We will denote by

$$\Pi := \frac{1}{2}\mathcal{L}_X g \tag{12.12}$$

the deformation tensor associated to the vector field X. Using the definition of Lie derivative for a (0, 2)-tensor:

$$\mathcal{L}_X g_{\mu\nu} = g_{\mu\nu,\gamma} X^\gamma + X^\gamma{}_{,\nu} g_{\mu\gamma} + X^\gamma{}_{,\mu} g_{\gamma\nu}, \tag{12.13}$$

where commas followed by indices denote partial derivatives with respect to such indices. Then, we have

$$2\Pi_{\mu\nu} = \mathcal{L}_X g_{\mu\nu} = g_{\mu\nu,0} X^0 - 2X^0{}_{,0}\delta_\mu{}^0 \delta_\nu{}^0 \tag{12.14}$$

and by plugging the components of the metric and of X in the previous expression:

$$\Pi = (n-2)\dot{a} a^{1-n} dt^2 + \dot{a} a^{3-n} \delta_{ij}\, dx^i dx^j. \tag{12.15}$$

Since the energy-momentum is conserved, we can write

$$\nabla^\mu J_\mu \overset{(12.6)}{=} \nabla^\mu \left(T_{\mu\nu} X^\nu\right) = T_{\mu\nu} \nabla^\mu X^\nu. \tag{12.16}$$

We can then prove that $\nabla^\mu X^\nu = \Pi^{\mu\nu}$. Indeed, since the metric (12.1) is diagonal and the 0-0 component is constant, we can easily compute the following Christoffel symbol:

$$\Gamma^\nu{}_{0\mu} = \frac{1}{2} g^{\nu\delta} \left(g_{\delta\mu,0} + \cancel{g_{\delta 0,\mu}} - \cancel{g_{0\mu,\delta}} \right) = \frac{1}{2} g^{\nu\delta} g_{\delta\mu,0}. \tag{12.17}$$

On the other hand, by definition of covariant derivative:

$$\nabla^\mu X^\nu = X^{\nu,\mu} + g^{\mu\alpha}\Gamma^\nu{}_{0\alpha} X^0 = -X^0{}_{,0}\delta^\mu{}_0\delta^\nu{}_0 + \frac{1}{2} g^{\mu\alpha} g^{\nu\delta} g_{\delta\alpha,0} X^0. \tag{12.18}$$

We finally get the desired result by comparing the above with equation (12.14) and noticing that we have $g^{\mu\alpha} g^{\nu\delta} g_{\delta\alpha,0} = -g^{\mu\nu}{}_{,0}$. If we continue the computations:

$$\begin{aligned} \nabla^\mu J_\mu &= T_{\mu\nu}\,\Pi^{\mu\nu} = (n-2)\dot{a}a^{1-n}\dot{\phi}^2 + \frac{n-2}{2}\dot{a}a^{1-n}\partial_\alpha\phi\partial^\alpha\phi \\ &\quad + \dot{a}a^{-1-n}\delta^{ij}\,\partial_i\phi\partial_j\phi - \frac{n}{2}\dot{a}a^{1-n}\partial_\alpha\phi\partial^\alpha\phi \\ &= (n-1)\dot{a}a^{1-n}\dot{\phi}^2 \geq 0. \end{aligned} \tag{12.19}$$

Now, given $R > 0$, we define

$$B_{0,R} := \left\{ (t_0, x) \in I \times \mathbb{R}^n \,:\, \delta^{ij}\, x_i x_j \leq R^2 \right\} \tag{12.20}$$

and its future domain of dependence is

$$D^+(B_{0,R}) := \left\{ p \in M \;\middle|\; \begin{array}{l} \text{any past inextendible causal curve} \\ \text{through } p \text{ intersects } B_{0,R} \end{array} \right\}. \tag{12.21}$$

Finally, for $t_1 > t_0$, we define

$$\mathcal{R} := D^+(B_{0,R}) \cap \{(t, x) \in M : t_0 \leq t \leq t_1\}. \tag{12.22}$$

Using the divergence theorem (see also [20], section 5 for details about the orientation of normal vectors for Lorentzian manifolds):

$$0 \overset{(12.19)}{\leq} \int_{\mathcal{R}} \left(\nabla_\mu J^\mu\right)\epsilon = \int_{B_{0,R}} g(J, N)\sigma + \int_C g(J, n)\sigma + \int_{\tilde{B}_{1,R}} g(J, -N)\sigma, \tag{12.23}$$

where ϵ is the volume form on $\mathcal{R}$, σ is the induced volume form on $\partial\mathcal{R}$, N is the inward vector field for the spacelike surface $B_{0,R}$ and n is the normal to the null component of $\mathcal{R}$. The set $B_{1,R}$ is defined similarly to $B_{0,R}$, where t_0 is replaced with t_1, and $\tilde{B}_{1,R} = B_{1,R} \cap D^+(B_{0,R})$. The set C forms the null part of $\mathcal{R}$ (so, its normal vector is tangent to it).

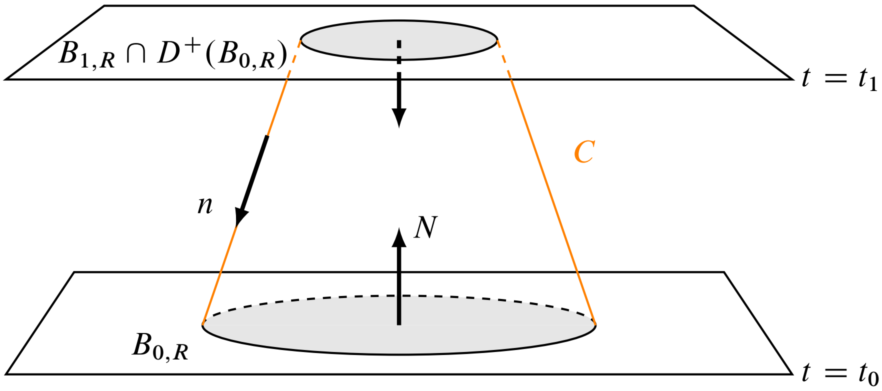

Figure 3.1: The sets $B_{0,R}$, $B_{1,R}$ and C form $\mathcal{R}$.

Now, we proved that $g(J, X) \geq 0$. Since n is past-pointing as well, it follows that $g(J, n) \leq 0$. Therefore, last expression can be written as

$$\int_{\tilde{B}_{1,R}} g(J, N)\sigma \leq \int_{B_{0,R}} g(J, N)\sigma. \tag{12.24}$$

Following the same steps pursued in (12.11):

$$\frac{1}{2}\int_{\tilde{B}_{1,R}} \left(a^2\dot{\phi}^2 + \delta^{ij}\,\partial_i\phi\partial_j\phi\right) d^n x \leq \frac{1}{2}\int_{B_{0,R}} \left(a^2\dot{\phi}^2 + \delta^{ij}\,\partial_i\phi\partial_j\phi\right) d^n x. \tag{12.25}$$

Taking the limit $R \to \infty$, we obtain that $E(t_1) \leq E(t_0)$. The procedure can be repeated for every $t_1 \geq t_0$ and therefore:

$$E(t) \leq E(t_0) < \infty \tag{12.26}$$

for every $t \geq t_0$. We have that $E(t_0) < \infty$ because $\phi_0 \in H^k(\mathbb{R}^n)$ and $\phi_1 \in H^{k-1}(\mathbb{R}^n)$ for $k > \frac{n}{2} + 2 > 3$. Moreover, due to relation (12.26) and to the positivity of the integrand terms, we can write

$$\int_{\mathbb{R}^n} \dot{\phi}(t, x)^2 d^n x \lesssim \frac{1}{a^2} \tag{12.27}$$

and

$$\int_{\mathbb{R}^n} \delta^{ij}\,\partial_i\phi(t, x)\partial_j\phi(t, x) d^n x \lesssim 1 \tag{12.28}$$

for every $t \geq t_0$. Now, if ϕ is a solution to the wave equation, then $\partial^\alpha\phi$ is a solution as well by smoothness of ϕ. Since $k = \frac{n}{2} + 2 > 3$, we can reserve two orders of derivation for the Laplacian and repeat the above procedure to get

$$\|\Delta\phi(t, \cdot)\|_{H^{k'}(\mathbb{R}^n)} \lesssim 1 \tag{12.29}$$

for every $t \geq t_0$, with $k' = k - 2 > \frac{n}{2}$. Using the Sobolev inequality (11.3), we finally get:

$$\|\Delta\phi(t, \cdot)\|_{L^\infty(\mathbb{R}^n)} \lesssim 1 \tag{12.30}$$

for every $t \geq t_0$.

2) Application of the bound on the Laplacian. By definition of the operator $\Box_g$, we can write the wave equation as $\partial_\mu \left(\sqrt{-g}\partial^\mu \phi\right) = 0$. In our case, splitting the temporal and spatial parts of the metric as done in (1.6) yields

$$\partial_\tau \left(a^{n-1}\partial_\tau \phi\right) = a^{n-1}\Delta\phi. \tag{12.31}$$

By definition (1.3) of conformal time:

$$\tau = \int_{t_0}^{t} \frac{ds}{s^p} = \frac{t^{1-p} - t_0^{1-p}}{1-p}, \tag{12.32}$$

or

$$t = \left[(1-p)\tau + t_0^{1-p}\right]^{\frac{1}{1-p}}. \tag{12.33}$$

We notice that $t \in [t_0, \infty)$ and, thus, $\tau \in (\tau_0, 0]$, where $\tau_0 = -\frac{t_0^{1-p}}{1-p}$. The scale factor can also be written as

$$a(t) = t^p = \left[(1-p)\tau + t_0^{1-p}\right]^{\frac{p}{1-p}} \tag{12.34}$$

and $a(\tau = 0) = a(t = t_0) = t_0^p$. We can now integrate (12.31) to obtain

$$a^{n-1}(0)\partial_\tau \phi(0, x) - a^{n-1}(\tau)\partial_\tau \phi(\tau, x) = \int_\tau^0 \Delta\phi(s, x)\left[(1-p)s + t_0^{1-p}\right]^{\frac{p(n-1)}{1-p}} ds \tag{12.35}$$

or, using that $\partial_\tau \phi = a\partial_t \phi$ and that $\phi_1(x) = \partial_t \phi(t_0, x)$:

$$\partial_t \phi(t, x) = a^{-n}(t)\left[a^n(t_0)\phi_1(x) + \int_0^\tau \Delta\phi(s, x)\left[(1-p)s + t_0^{1-p}\right]^{\frac{p(n-1)}{1-p}} ds\right]. \tag{12.36}$$

After taking the L^∞ norms, the bound on $\Delta\phi$ can now be used to get

$$\begin{aligned}\|\partial_t \phi(t, \cdot)\|_{L^\infty(\mathbb{R}^n)} \leq a^{-n}(t)\Bigg[& a^n(t_0)\|\phi_1\|_{L^\infty(\mathbb{R}^n)} + \\ &+ \int_0^\tau \|\Delta\phi\|_{L^\infty(\mathbb{R}^n)}\left[(1-p)s + t_0^{1-p}\right]^{\frac{p(n-1)}{1-p}} ds\Bigg] \\ \leq a^{-n}(t)&\left[a^n(t_0)\|\phi_1\|_{L^\infty(\mathbb{R}^n)} + c\left(a(t)^{n-2+\frac{1}{p}} - a(t_0)^{n-2+\frac{1}{p}}\right)\right],\end{aligned} \tag{12.37}$$

where we computed the integral:

$$\int \left[(1-p)\tau + t_0^{1-p}\right]^{\frac{p(n-1)}{1-p}} d\tau \propto \left((1-p)\tau + t_0^{1-p}\right)^{\frac{p(n-1)}{1-p}+1} + C = a(t)^{n-2+\frac{1}{p}} + C \tag{12.38}$$

and used the expression (12.34) for the scale factor. An alternative way to write the inequality (12.37) is

$$\begin{aligned}\|\partial_t\phi(t,\cdot)\|_{L^\infty(\mathbb{R}^n)} &\leq a^{-2+\frac{1}{p}}(t)\left[\frac{a^n(t_0)}{a^{n-2+\frac{1}{p}}(t)}\|\phi_1\|_{L^\infty(\mathbb{R}^n)}+c\left(1-\left(\frac{a(t_0)}{a(t)}\right)^{n-2+\frac{1}{p}}\right)\right]\\ &\overset{(n>2)}{\leq} a^{-2+\frac{1}{p}}(t)\left[\frac{a^n(t_0)}{a^{n-2+\frac{1}{p}}(t_0)}\|\phi_1\|_{L^\infty(\mathbb{R}^n)}+c\right]\\ &\lesssim a^{-2+\frac{1}{p}}(t)=t^{-2p+1},\end{aligned} \tag{12.39}$$

which is enough to prove the theorem. □

We will now consider the more involved case in which $0<p\leq 1$.

Theorem 3.3. *Let $n>2$ and let $I\subset\mathbb{R}$ be the open interval $(t_*,\infty)\ni t_0$ for some $t_*\geq 0$. Assume that (M,g) describes an expanding universe given by the metric*

$$g=-dt^2+t^{2p}\left(\left(dx^1\right)^2+\ldots+(dx^n)^2\right) \tag{12.40}$$

with $0<p\leq 1$, whose flat n-dimensional space sections are given by $\mathbb{R}^n$. Suppose that ϕ is a smooth solution to the Cauchy problem

$$\begin{cases}\Box_g\phi(t,x)=0,\\ \phi(t_0,x)=\phi_0(x),\\ \partial_t\phi(t_0,x)=\phi_1(x)\end{cases} \tag{12.41}$$

with $t\geq t_0$, $x\in\mathbb{R}^n$, $\phi_0\in H^k(\mathbb{R}^n)$, $\phi_1\in H^{k-1}(\mathbb{R}^n)$ and $k>\frac{n}{2}+2$. Then:

$$\|\phi(t,\cdot)\|_{L^\infty(\mathbb{R}^n)}\lesssim a(t)^{-\frac{n-1}{2}}=t^{-\frac{n-1}{2}p}\quad \textit{for } 0<p\leq\frac{2}{n+1} \textit{ and } p=1, \tag{12.42}$$

and for any $\delta>0$ we have

$$\|\phi(t,\cdot)\|_{L^\infty(\mathbb{R}^n)}\lesssim a(t)^{-\frac{n-1}{2}(1-\delta)}=t^{-\frac{n-1}{2}p(1-\delta)}\quad \textit{for } \frac{2}{n+1}<p<1. \tag{12.43}$$

Proof. First, let us assume that $p\neq 1$. Let τ be the conformal time variable:

$$\tau=\int_0^t\frac{ds}{s^p}=\frac{t^{1-p}}{1-p}\Longrightarrow t=[(1-p)\tau]^{\frac{1}{1-p}}, \tag{12.44}$$

which implies that $\tau\in\left[\frac{t_0^{1-p}}{1-p},\infty\right)$. The wave equation $\Box_g\phi=0$ can also be written as $\partial_\mu(\sqrt{-g}\partial^\mu\phi)=0$, or

$$\partial_\tau(a^{n-1}\partial_\tau\phi)-a^{n-1}\Delta\phi=0. \tag{12.45}$$

Now, let us define $\psi(\tau, x)$ such that

$$\phi = a^{-\frac{n-1}{2}}\psi. \tag{12.46}$$

In order to complete the proof, we need to obtain a uniform bound in τ for the $L^\infty(\mathbb{R}^n)$-norm of $\psi(\tau, \cdot)$. We notice that

$$\partial_\tau \phi = \phi' = a^{-\frac{n-1}{2}}\psi' - \frac{n-1}{2}a^{-\frac{n-1}{2}-1}a'\psi. \tag{12.47}$$

If we plug the expression of ϕ in terms of ψ in the wave equation (12.45):

$$\partial_\tau\left(a^{\frac{n-1}{2}}\psi' - \frac{n-1}{2}a^{\frac{n-1}{2}-1}a'\psi\right) - a^{\frac{n-1}{2}}\Delta\psi = 0. \tag{12.48}$$

If we apply the derivative in τ and divide both sides by $a^{\frac{n-1}{2}}$, we get

$$\psi'' - \left[\Delta + \frac{n-1}{2}\frac{a''}{a} + \frac{(n-1)(n-3)}{4}\left(\frac{a'}{a}\right)^2\right]\psi = 0. \tag{12.49}$$

We can simplify the above expression by computing the derivatives of $a(\tau) = a(t(\tau))$:

$$a(t) = t^p \overset{(12.44)}{=} [(1-p)\tau]^{\frac{p}{1-p}} = a(\tau) \tag{12.50}$$

and therefore

$$a'(\tau) = p\,[(1-p)\tau]^{\frac{p}{1-p}-1} \quad \text{and} \quad a''(\tau) = p(2p-1)\,[(1-p)\tau]^{\frac{p}{1-p}-2}. \tag{12.51}$$

It follows, then, that the (12.49) becomes

$$\left(\Box + \frac{\mu}{\tau^2}\right)\psi(\tau, x) = 0 \tag{12.52}$$

if we define

$$\mu := \frac{(n-1)p(2p-1)}{2(1-p)^2} + \frac{(n-1)(n-3)p^2}{4(1-p)^2} \tag{12.53}$$

and $\Box = -\partial_\tau^2 + \Delta$ is the d'Alembertian in Minkowski space, where we chose the signature $(-,+,+,+)$ for $\eta_{\mu\nu}$ *. The PDE that we found is a second-order, linear, hyperbolic PDE in Minkowski space. By theorem 2.5 in [20] we know that the Cauchy problem for the corresponding initial conformal time $\tau_0 = \frac{t_0^{1-p}}{1-p}$:

$$\begin{cases} \left(\Box + \frac{\mu}{\tau^2}\right)\psi(\tau, x) = 0, \\ \psi(\tau_0, x) = a(\tau_0)^{\frac{n-1}{2}}\phi_0(x), \\ \partial_\tau\psi(\tau_0, x) = a(\tau_0)^{\frac{n-1}{2}}\left(\frac{n-1}{2}a(\tau_0)^{-1}a'(\tau_0)\phi_0(x) + \phi_1(x)\right), \end{cases} \tag{12.54}$$

*With this choice of signature, the Klein-Gordon equation with positive mass and the one with imaginary mass are given, respectively, by $\left(\Box \mp m^2\right)\psi = 0$.

is well-posed. However, the solution is unstable when $\mu > 0$. In such case, indeed, the PDE describes a "Klein-Gordon" problem with time-dependent, imaginary mass. Physically speaking, this is the case in which the equation describes tachyons, whose instability can be observed by considering the differential equation in Fourier space.

We will now consider the case $\mu \leq 0$, i.e. $\boxed{0 < p \leq \frac{2}{n+1}}$. It means that the PDE (12.52) is a Klein-Gordon equation with positive and time-dependent mass. Similarly to the case $p > 1$, we can define an energy $E(\tau)$ and prove that it is non-increasing. However, we immediately notice that the energy-momentum tensor

$$T_{\mu\nu} = \partial_\mu \psi \partial_\nu \psi - \frac{1}{2}\eta_{\mu\nu}\left(\partial_\alpha \psi \partial^\alpha \psi - \frac{\mu}{\tau^2}\psi^2\right), \tag{12.55}$$

is not conserved due to the time dependence of the mass:

$$\partial^\mu T_{\mu\nu} = \cancel{\left(\Box\psi + \frac{\mu}{\tau^2}\psi\right)}\partial_\nu \psi - \frac{\mu}{\tau^3}\psi^2 \delta_\nu{}^\tau = -\frac{\mu}{\tau^3}\psi^2\delta_\nu{}^\tau \neq 0. \tag{12.56}$$

However, we can still prove that the energy is non-increasing. Let us consider the multiplier vector field

$$X := \frac{\partial}{\partial \tau} \tag{12.57}$$

and define the current as

$$J_\mu = T_{\mu\nu} X^\nu. \tag{12.58}$$

The vector field X is Killing for $\eta_{\mu\nu}$ and it is timelike: $\eta(X, X) < 0$. Moreover, the current J is past-pointing:

$$\eta(J, X) = J_\mu X^\mu = J_0 = T_{0\nu} X^\nu = T_{00} = \frac{1}{2}\left(\psi'^2 + \delta^{ij}\partial_i \psi \partial_j \psi - \frac{\mu}{\tau^2}\psi^2\right) \geq 0. \tag{12.59}$$

We can now define the energy as

$$E(\tau) := \int_{\{\tau\}\times\mathbb{R}^n} \eta(J, \partial_\tau)\sigma = \frac{1}{2}\int_{\mathbb{R}^n}\left(\psi'^2 + \delta^{ij}\partial_i \psi \partial_j \psi - \frac{\mu}{\tau^2}\psi^2\right) d^n x. \tag{12.60}$$

It is clear from the computation in (12.56) that the divergence of J is non-negative:

$$\partial_\mu J^\mu \overset{(12.58)}{=} \partial_\mu \left(T^{\mu\nu} X_\nu\right) = \left(\partial_\mu T^{\mu\nu}\right) X_\nu + \cancel{T^{\mu\nu}\partial_\mu X_\nu} \geq 0, \tag{12.61}$$

where the last term vanishes since it is a contraction between a symmetric and an antisymmetric tensor ($\partial_{(\mu} X_{\nu)} = 0$ by Killing condition). Therefore, we can define

$$B_{0,R} := \{(\tau_0, x) \in I \times \mathbb{R}^n : \delta^{ij} x_i x_j \leq R^2\} \tag{12.62}$$

for some $R > 0$. Its future domain of dependence is

$$D^+(B_{0,R}) := \left\{ p \in M \;\middle|\; \begin{array}{l} \text{any past inextendible causal curve} \\ \text{through } p \text{ intersects } B_{\tau_0,R} \end{array} \right\}. \tag{12.63}$$

Finally, we define

$$\mathcal{R} := D^+(B_{0,R}) \cap \{(\tau, x) \in M : \tau_0 \le \tau \le \tau_1\} \tag{12.64}$$

for some $\tau_1 > \tau_0$. Using the divergence theorem:

$$0 \le \int_{\mathcal{R}} (\partial_\mu J^\mu)\, \epsilon = \int_{B_{0,R}} \eta(J, \partial_\tau)\sigma + \int_C \eta(J, n)\sigma + \int_{\tilde{B}_{1,R}} \eta(J, -\partial_\tau)\sigma, \tag{12.65}$$

where ϵ is the induced volume form on $\mathcal{R}$, σ is the induced volume form on $\partial\mathcal{R}$, n is the normal to the null component of $\mathcal{R}$. The set $B_{1,R}$ is defined similarly to $B_{0,R}$ and $\tilde{B}_{1,R} = B_{1,R} \cap D^+(B_{0,R})$. The set C consists of the null part of $\mathcal{R}$. Now, since both J and n are past-pointing, we have that $\eta(J, n) < 0$. Therefore, the last expression becomes

$$\int_{\tilde{B}_{1,R}} \eta(J, \partial_\tau)\sigma \le \int_{B_{0,R}} \eta(J, \partial_\tau)\sigma. \tag{12.66}$$

Since $\eta(J, \partial_\tau)$ gives the integrand of the energy:

$$\frac{1}{2}\int_{\tilde{B}_{1,R}} \left(\psi'^2 + \delta^{ij}\,\partial_i\psi\partial_j\psi - \frac{\mu}{\tau_1^2}\psi^2\right) d^n x \le \frac{1}{2}\int_{B_{0,R}} \left(\psi'^2 + \delta^{ij}\,\partial_i\psi\partial_j\psi - \frac{\mu}{\tau_0^2}\psi^2\right) d^n x. \tag{12.67}$$

Taking the limit $R \to \infty$, we obtain $E(\tau_1) \le E(\tau_0) < \infty$. This is true for every $\tau_1 > \tau_0$, hence

$$E(\tau) \le E(\tau_0) < \infty \tag{12.68}$$

for every $\tau > \tau_0$. The fact that $E(\tau_0) < \infty$ follows from the hypotheses on the initial data.

We can use this result on the energy to prove that $\|\psi(\tau, \cdot)\|_{L^\infty(\mathbb{R}^n)} \le C$ for every τ, where the constant C does not depend on the conformal time τ. This implies the sought result

$$\|\phi(\tau, \cdot)\|_{L^\infty(\mathbb{R}^n)} \le C a^{-\frac{n-1}{2}}(\tau) \tag{12.69}$$

So, let us prove that the L^2 norm of $\psi(\tau, \cdot)$ is uniformly bounded. If we write ψ in spherical coordinates and perform an integration by parts:

$$\int_0^\infty \psi^2 dr = \cancel{\left[r\psi^2\right]_{r=0}^{r=\infty}} - 2\int_0^\infty r\psi(\partial_r\psi)dr, \tag{12.70}$$

where we are assuming that $r\psi^2$ vanishes at infinity, which is always true if the initial data have compact support. We notice that the right hand side is positive. If we now apply the Peter-Paul inequality:

$$\int_0^\infty \psi^2 dr \le \epsilon\int_0^\infty \psi^2 dr + \frac{1}{\epsilon}\int_0^\infty (\partial_r\psi)^2\, r^2 dr < +\infty \tag{12.71}$$

for every $\epsilon > 0$, where the finiteness of the right hand side comes from the compact support of ψ. If we choose ϵ small enough, we have

$$\int_0^\infty \psi^2 dr \le C\int_0^\infty (\partial_r\psi)^2 r^2 dr. \tag{12.72}$$

It is useful to notice that, if we restrict the interval of integration:

$$\int_0^1 \psi^2 \frac{1}{r^2} r^2 dr = \int_0^1 \psi^2 dr \leq \int_0^\infty \psi^2 dr \leq C \int_0^\infty (\partial_r \psi)^2 r^2 dr. \tag{12.73}$$

We can now integrate over the spheres of radius r centred in r_0 and:

$$\begin{aligned}\|\psi(\tau,\cdot)\|^2_{L^2(B_1(r_0))} \overset{(r<1)}{\leq} \left\| \frac{\psi}{r} \right\|^2_{L^2(B_1(r_0))} &= \int_0^1 \int_{S(r_0,r)} \psi^2 \frac{1}{r^2} r^2 dr d\omega \\ &\leq c \int_{\mathbb{R}^n} (\partial_r \psi)^2 d^n x \leq c E(\tau) \leq c E(t_0) < +\infty, \end{aligned} \tag{12.74}$$

where we used the energy condition (12.68). Moreover, if ψ is a solution for the PDE (12.52), then any spatial derivative of ψ is a solution as well. Therefore, we can repeat the above procedure for such derivatives and get

$$\|\psi(\tau,\cdot)\|_{H^k(B_1(r_0))} \leq C \tag{12.75}$$

for every τ. By the Sobolev inequality, it follows that

$$\|\psi(\tau,\cdot)\|_{L^\infty(B_1(r_0))} \leq \|\psi(\tau,\cdot)\|_{H^k(B_1(r_0))} \leq C \tag{12.76}$$

So, we obtained a bound for the L^∞ norm of ψ in every ball of radius 1 in $\mathbb{R}^n$. Since the obtained bound is independent of the position of the ball and it is uniform in τ:

$$\|\psi(\tau,\cdot)\|_{L^\infty(\mathbb{R}^n)} \leq C. \tag{12.77}$$

This result implies that

$$\|\phi(\tau,\cdot)\|_{L^\infty(\mathbb{R}^n)} \leq C a^{-\frac{n-1}{2}}(\tau), \tag{12.78}$$

which concludes the proof for the case $\mu \leq 0$.

Now, we want to consider the case $\boxed{\frac{2}{n+1} < p < 1}$. When p belongs to this interval, μ (as defined in (12.53)) is bigger than 0. This implies that the energy for ψ is not positive definite and the procedure that we have just completed is not valid anymore. As a possible solution to this problem, we could define ψ such that

$$\phi = a^{-\frac{n-1}{2}+\epsilon} \psi, \tag{12.79}$$

for some $\epsilon > 0$ and, again, try to obtain a bound on the $L^\infty(\mathbb{R}^n)$-norm of $\psi(\tau,\cdot)$. Actually, it can be observed that the value of ϵ which lets us obtain the best estimate is given by $\epsilon = \frac{n-1}{2}$. Therefore, it is convenient to work with the original equation $\Box_g \phi = 0$. The latter can be written as

$$\Box \phi - (n-1) \frac{p}{(1-p)\tau} \phi' = 0, \tag{12.80}$$

where $\Box = -\partial_\tau^2 + \Delta$. We can consider our PDE as a wave equation in Minkowski with a damping term (which depends on the scale factor). Let us then define the energy in conformal time as

$$E(\tau) := \frac{1}{2}\int_{\mathbb{R}^n} \left(\phi'^2 + \delta^{ij}\partial_i\phi\partial_j\phi\right) d^n x. \tag{12.81}$$

Due to the damping term, the energy will decay for large times. This can be seen as follows. The energy-momentum tensor for our equation (12.80) is

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\eta_{\mu\nu}\partial_\alpha\phi\partial^\alpha\phi. \tag{12.82}$$

If we define

$$b(\tau) := (n-1)\frac{p}{(1-p)\tau} = (n-1)\frac{a'(\tau)}{a(\tau)} \geq 0, \tag{12.83}$$

then the 0-th component of

$$\partial^\mu T_{\mu\nu} = \partial_\nu\phi\Box\phi = b(\tau)\partial_\nu\phi\phi' \tag{12.84}$$

is positive. As done for the previous case of the proof, we can consider

$$X = \frac{\partial}{\partial\tau} \quad \text{and} \quad J_\mu = T_{\mu\nu} X^\nu. \tag{12.85}$$

The vector field X is Killing for the Minkowski metric and we have that $\eta(X, X) < 0$. Moreover, J is past-pointing also in this case:

$$\eta(J, X) = J_\mu X^\mu = J_0 = T_{0\nu} X^\nu = T_{00} = \frac{1}{2}\left(|\phi'|^2 + |\nabla\phi|^2\right) \geq 0. \tag{12.86}$$

We then notice that the energy (12.81) can be expressed as

$$E(\tau) = \int_{\{\tau\}\times\mathbb{R}^n} \eta\left(J, \partial_\tau\right)\sigma. \tag{12.87}$$

The divergence of the current J is positive due to (12.84) and to the fact that X is Killing:

$$\partial_\mu J^\mu \overset{(12.85)}{=} \partial_\mu\left(T^{\mu\nu}X_\nu\right) = \left(\partial_\mu T^{\mu\nu}\right)X_\nu + \cancel{T^{\mu\nu}\partial_\mu X_\nu} \geq 0. \tag{12.88}$$

The sets $B_{0,R}$, $D^+(B_{0,R})$ and $\mathcal{R}$ are defined as in (12.62) and following equations. By the divergence theorem, then:

$$0 \leq \int_{\mathcal{R}} \left(\partial_\mu J^\mu\right)\epsilon = \int_{B_{0,R}} \eta(J, \partial_\tau)\sigma + \int_C \eta(J, n)\sigma + \int_{\tilde{B}_{1,R}} \eta(J, -\partial_\tau)\sigma, \tag{12.89}$$

where n is the normal to the null component of $\mathcal{R}$. Both J and n are past-pointing and thus $\eta(J, n) < 0$. Therefore:

$$\int_{\mathcal{R}} \left(\partial_\mu J^\mu\right)\epsilon \leq \int_{B_{0,R}} \eta(J, \partial_\tau)\sigma - \int_{\tilde{B}_{1,R}} \eta(J, \partial_\tau)\sigma. \tag{12.90}$$

The term $\eta(J, \partial_\tau)$ gives the integrand of the energy, and the divergence of the current is nothing else than the 0-th component of (12.84). We also notice that the choice of τ_1 is arbitrary, so the above result holds for every $\tau > \tau_0$. Therefore, if we take the limit as $R \to \infty$ of last expression, we obtain

$$\int_{\tau_0}^{\tau} \left(b(s) \int_{\mathbb{R}^n} |\phi'|^2 d^n x \right) ds \leq E(\tau_0) - E(\tau) \tag{12.91}$$

for every $\tau > \tau_0$. In order to gain more information from this inequality, let us assume for the moment that the ratio between kinetic energy and potential energy for the damped wave equation (12.80) tends to 1 as $\tau \to \infty$. It means that $\forall\, \delta > 0\, \exists\, M > 0$ such that:

$$\left| \frac{\|\phi'\|_{L^2}^2}{\|\nabla\phi\|_{L^2}^2} - 1 \right| < \delta \quad \forall\, \tau > M \tag{12.92}$$

and therefore

$$(1-\delta)\|\nabla\phi\|_{L_2}^2 < \|\phi'\|_{L_2}^2 < (1+\delta)\|\nabla\phi\|_{L_2}^2 \tag{12.93}$$

Thus, coming back to (12.91):

$$E(\tau_0) - E(\tau) \geq \int_{\tau_0}^{\tau} \left(\frac{b(s)}{2} \int_{\mathbb{R}^n} \left(|\phi'|^2 + |\phi'|^2\right) d^n x \right) ds \geq (1-\delta) \int_{\tau_0}^{\tau} b(s) E(s) ds \tag{12.94}$$

for $\tau > M$. If we multiply the last relation by -1, we can use the (integral) Grönwall inequality and the definition (12.83) of b to get:

$$E(\tau) \leq E(\tau_0) \exp\left(-(1-\delta) \int_{\tau_0}^{\tau} (n-1) \frac{a'(s)}{a(s)} ds \right) \lesssim a(\tau)^{-(n-1)(1-\delta)}, \tag{12.95}$$

where $E(\tau_0) < +\infty$ due to the regularity of the initial data. From the proof of the case $p \in \left(0, \frac{2}{n+1}\right]$, we know that

$$\|\phi(\tau, \cdot)\|_{L^2(B_1(r_0))}^2 \lesssim E(\tau) \tag{12.96}$$

for every $\tau \geq \tau_0$ (see (12.74)), where the estimate does not depend on the center r_0 of the ball. This is true also in this case, since our energy is positive definite. The estimate can be applied to the spatial derivatives of ϕ, since they are still solutions for the PDE (12.80). We can then obtain $\|\phi(\tau, \cdot)\|_{H^k(B_1(r_0))}^2 \leq E(\tau)$. By Sobolev inequality, the bound can be extended to the L^∞ norm, and noticing that the estimate does not depend on r_0:

$$\|\phi(\tau, \cdot)\|_{L^\infty(\mathbb{R}^n)}^2 \leq E(\tau) \leq a(\tau)^{-(n-1)(1-\delta)}. \tag{12.97}$$

Thus:

$$\|\phi(\tau, \cdot)\|_{L^\infty(\mathbb{R}^n)} \lesssim t^{-\frac{n-1}{2} p(1-\delta)}. \tag{12.98}$$

What is missing from the above reasoning is that the ratio between kinetic and potential energy for ϕ indeed goes to 1 as $\tau \to \infty$. We will prove it through a Fourier analysis, similarly to what is done in the preamble of [10]. The Fourier transform of an integrable function ϕ is

$$\hat{\phi}(t,\xi) = \int_{\mathbb{R}^n} \phi(t,x) e^{-ix\cdot\xi} d^n x. \tag{12.99}$$

So, the Cauchy problem

$$\begin{cases} -\partial_t^2 \phi(t,x) + \Delta\phi(t,x) - b(t)\partial_t \phi(t,x) = 0, \\ \phi(t_0,x) = \phi_0(x), \\ \partial_t \phi(t_0,x) = \phi_1(x), \end{cases} \tag{12.100}$$

where $b(t) = (n-1)\frac{p}{(1-p)t} =: \frac{q}{t}$, can be equivalently written as

$$\begin{cases} \partial_t^2 \hat{\phi}(t,\xi) + |\xi|^2 \hat{\phi}(t,\xi) + b(t)\partial_t \hat{\phi}(t,\xi) = 0, \\ \hat{\phi}(t_0,\xi) = \hat{\phi}_0(\xi), \\ \partial_t \hat{\phi}(t_0,\xi) = \hat{\phi}_1(\xi). \end{cases} \tag{12.101}$$

and it has solution

$$\hat{\phi}(t,\xi) = t^{-\alpha}\Big[c_1(\xi) J_\alpha(|\xi|t) + c_2(\xi) Y_\alpha(|\xi|t)\Big], \tag{12.102}$$

where we defined $\alpha = \frac{1}{2}(q-1)$, $c_1(\xi)$ and $c_2(\xi)$ are functions given by the initial data and J, Y are the Bessel functions of the first and second kind, respectively defined in (2.19) and (2.20). In particular, this means that

$$\partial_t \hat{\phi}(t,\xi) = t^{-\alpha}|\xi|\Big[c_1(\xi) J_{\alpha+1}(t|\xi|) + c_2(\xi) Y_{\alpha+1}(t|\xi|)\Big], \tag{12.103}$$

due to the relations (see e.g. [2])

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}z} J_\alpha(z) = \frac{1}{2} J_{\alpha-1}(z) - \frac{1}{2} J_{\alpha+1}(z), \\ \frac{\mathrm{d}}{\mathrm{d}z} Y_\alpha(z) = \frac{1}{2} Y_{\alpha-1}(z) - \frac{1}{2} Y_{\alpha+1}(z). \end{cases} \tag{12.104}$$

In particular, this implies that the coefficients c_1 and c_2 are given by

$$c_1(\xi) = t_0^\alpha \frac{\hat{\psi}_0(\xi) + \hat{\psi}_1(\xi)|\xi|^{-1} Y_\alpha Y_{\alpha+1}^{-1}}{J_\alpha - J_{\alpha+1} Y_\alpha Y_{\alpha+1}^{-1}}, \tag{12.105}$$

where we omitted the argument $|\xi|t_0$ of the Bessel functions, and

$$c_2(\xi) = -t_0^\alpha |\xi|^{-1} \hat{\psi}_1(\xi) Y_{\alpha+1}^{-1} - t_0^\alpha J_{\alpha+1} \frac{\hat{\psi}_0(\xi) + \hat{\psi}_1(\xi)|\xi|^{-1} Y_\alpha Y_{\alpha+1}^{-1}}{J_\alpha Y_{\alpha+1} - J_{\alpha+1} Y_\alpha}. \tag{12.106}$$

Since the initial data have compact support, c_1 and c_2 are compactly supported as well. Now, we want to analyse the behaviour of $\hat{\phi}$ and $\partial_t\hat{\phi}$ for large times, since they enter in the definition of kinetic and potential energy. By looking at the estimates (2.22) and (2.23) and noticing that $\alpha > 0$, we have

$$c_1(\xi)J_\alpha(t|\xi|) + c_2(\xi)Y_\alpha(t|\xi|) = \sqrt{\frac{2}{\pi t|\xi|}}\left[c_1(\xi)\cos\left(t|\xi| - \frac{\pi}{4}(2\alpha+1) + \cancel{O(t^{-2})}\right) + \right.$$
$$\left. + c_2(\xi)\sin\left(t|\xi| - \frac{\pi}{4}(2\alpha+1) + \cancel{O(t^{-2})}\right)\right] + O\left(t^{-\frac{3}{2}}\right) \tag{12.107}$$

as $t \to +\infty$, where the terms of order t^{-2} can be reabsorbed in $O(t^{-\frac{3}{2}})$ after using trigonometric formulas. The latter expression constitutes part of the asymptotic behaviour of $\hat{\phi}$. As regards $\partial_t\hat{\phi}$, due to (12.103), we need to analyse the behaviour of $c_1J_{\alpha+1} + c_2Y_{\alpha+1}$, which is the same of (12.107), except for an additional term $-\frac{\pi}{2}$ in the trigonometric functions. Now, by definition of potential and kinetic energy and using Parseval's theorem:

$$P(t) = \frac{1}{2}\int_{\mathbb{R}^n} |\nabla\phi|^2 d^n x = \frac{1}{2}\int_{\mathbb{R}^n} |\xi|^2|\hat{\phi}|^2 d^n\xi \tag{12.108}$$

and

$$K(t) = \frac{1}{2}\int_{\mathbb{R}^n} |\partial_t\phi|^2\, d^n x = \frac{1}{2}\int_{\mathbb{R}^n} |\partial_t\hat{\phi}|^2 d^n\xi. \tag{12.109}$$

Using the previously obtained asymptotic behaviours, we have:

$$P(t) = \frac{t^{-2\alpha-1}}{\pi}\int_{\mathbb{R}^n} |\xi|\left[c_1(\xi)\cos\left(t|\xi| - \frac{\pi}{4}(2\alpha+1)\right) + \right.$$
$$\left. + c_2(\xi)\sin\left(t|\xi| - \frac{\pi}{4}(2\alpha+1)\right)\right]^2 d^n\xi + O(t^{-2\alpha-2}) \tag{12.110}$$

and

$$K(t) = \frac{t^{-2\alpha-1}}{\pi}\int_{\mathbb{R}^n} |\xi|\left[c_1(\xi)\cos\left(t|\xi| - \frac{\pi}{4}(2\alpha+1) - \frac{\pi}{2}\right) + \right. \tag{12.111}$$
$$\left. + c_2(\xi)\sin\left(t|\xi| - \frac{\pi}{4}(2\alpha+1) - \frac{\pi}{2}\right)\right]^2 d^n\xi + O(t^{-2\alpha-2}).$$

This implies that equipartition of energy holds:

$$\lim_{t\to\infty}\frac{P(t)}{K(t)} = 1. \tag{12.112}$$

Indeed, it can be seen in the following way. Let us consider the kinetic energy. The integral over $\mathbb{R}^n$ can be divided into two parts, distinguishing between the cases $|\xi| \leq \frac{\pi}{2t}$ and $|\xi| > \frac{\pi}{2t}$. Using the change of variables $\xi = \frac{\pi}{2t}\eta$, the integral can be written as

$$\frac{C}{t^{n+1}} \int_{B_1(0)} |\eta| \Big[c_1 \left(\frac{\pi\eta}{2t}\right) \cos\left(\frac{\pi}{2}|\eta| - \frac{\pi}{4}(2\alpha+1) - \frac{\pi}{2}\right) + \\ + c_2 \left(\frac{\pi\eta}{2t}\right) \sin\left(\frac{\pi}{2}|\eta| - \frac{\pi}{4}(2\alpha+1) - \frac{\pi}{2}\right)\Big]^2 d^n\eta \tag{12.113}$$

for a constant C. Due to expressions (12.105) and (12.106), and to the regularity of the solution (12.102), this term will be ignored as $t \to +\infty$ because it decays faster than the other terms that we will consider. As regards the second integral, indeed, we can introduce a change of variables $\xi = \frac{\eta}{|\eta|}\left(|\eta| + \frac{\pi}{2t}\right)$, i.e. $\eta = \frac{\xi}{|\xi|}\left(|\xi| - \frac{\pi}{2t}\right)$. This is enough to remove the term $\frac{\pi}{2}$ in the trigonometric functions of the integral. In fact, for $|\xi| > \frac{\pi}{2t}$ we obtain

$$\int_{\mathbb{R}^n} (1 + O(t^{-1}))\left(|\eta| + \frac{\pi}{2t}\right) \Big[c_1 \left(\frac{\eta}{|\eta|}(|\eta| + \frac{\pi}{2t})\right) \cos\left(t|\eta| - \frac{\pi}{4}(2\alpha+1)\right) + \\ + c_2 \left(\frac{\eta}{|\eta|}(|\eta| + \frac{\pi}{2t})\right) \sin\left(t|\eta| - \frac{\pi}{4}(2\alpha+1)\right)\Big]^2 d^n\eta, \tag{12.114}$$

where we used that the Jacobian associated to the change of variables is

$$\prod_{i=1}^{n}\left(1 - \frac{\pi}{2t}\frac{1}{|\eta|}\left(1 - \frac{\eta_i^2}{|\eta|^2}\right)\right) = 1 + O(t^{-1}). \tag{12.115}$$

After comparing the potential energy (12.108) with the obtained kinetic energy

$$K(t) = \frac{t^{-2\alpha-1}}{\pi}\Bigg(\int_{\mathbb{R}^n} |\eta| \Big[c_1 \left(\frac{\eta}{|\eta|}\left(|\eta| + \frac{\pi}{2t}\right)\right) \cos\left(t|\eta| - \frac{\pi}{4}(2\alpha+1)\right) + \\ + c_2 \left(\frac{\eta}{|\eta|}\left(|\eta| + \frac{\pi}{2t}\right)\right) \sin\left(t|\eta| - \frac{\pi}{4}(2\alpha+1)\right)\Big]^2 d^n\eta + O(t^{-1})\Bigg), \tag{12.116}$$

we can conclude that equipartition of energy holds.

Lastly, we can do a similar reasoning for the case $p = 1$. Since

$$\tau = \int_{t_0}^{t} \frac{ds}{s} = \log\frac{t}{t_0} \tag{12.117}$$

and

$$a(t) = t = t_0 e^{\tau}, \tag{12.118}$$

we have that the PDE for $\psi = a^{\frac{n-1}{2}}\phi$ is a Klein-Gordon equation:

$$\left(\Box - \frac{(n-1)^2}{4}\right)\psi = 0. \tag{12.119}$$

We can define the energy as

$$E(\tau) := \int_{\mathbb{R}^n} \left(\psi'^2 + \delta^{ij}\, \partial_i \psi \partial_j \psi + \frac{(n-1)^2}{4} \psi^2 \right) d^n x. \tag{12.120}$$

Following the steps starting at equation (12.57) of the case $p \in \left(0, \frac{2}{n+1}\right]$, we can prove that such an energy is decreasing and the same estimates hold. □

13 Morawetz Estimate in the Friedmann Universe

In Minkowski spacetime, in dimension 3+1, the total energy associated to a solution of the wave equation is conserved. However, the energy restricted to a compact subset of $\mathbb{R}^3$ decreases. In particular, the energy of the wave restricted to a ball of radius R is integrable in time and such an integral is dominated from above by a constant which depends on the radius of the ball. This result is given by the so-called Morawetz estimate.

The aim of this section is to express this idea rigorously and to prove the analogue of the Morawetz estimate in an expanding Friedmann universe. First of all, we want to prove some properties about the energy of the wave equation in a Friedmann universe. In the following, we will consider the metric

$$g = -dt^2 + t^{2p} \left(\left(dx^1\right)^2 + \left(dx^2\right)^2 + \left(dx^3\right)^2 \right) \tag{13.1}$$

$$= -dt^2 + t^{2p} \left(dr^2 + r^2 d\theta^2 + r^2 \sin^2(\theta) d\varphi^2 \right) \tag{13.2}$$

with $0 < p \leq 1$ (we are interested in the case in which the causal structure contains a null infinity). In particular, the scale factor is given by $a(t) = t^p$.

Proposition 3.1 (Decay of the energy in the Friedmann universe)**.** *Let (M, g) be a Friedmann universe, where the metric g is given by (13.1). Let*

$$K(t) = \frac{1}{2} \int_{\mathbb{R}^3} a^2(t) |\partial_t \phi|^2 d^3 x \tag{13.3}$$

and

$$P(t) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla \phi|^2 d^3 x \tag{13.4}$$

be, respectively, the kinetic energy and the potential energy associated to a solution ϕ of the wave equation

$$\Box_g \phi = 0. \tag{13.5}$$

Then:

$$\lim_{t \to \infty} K(t) = \lim_{t \to \infty} P(t) = 0 \tag{13.6}$$

and for the total energy we have

$$E(t) = K(t) + P(t) \sim \frac{1}{a^2(t)} \text{ as } t \to \infty. \tag{13.7}$$

Proof. The wave equation is (see (1.8))

$$-\partial_t^2\phi - 3\frac{p}{t}\partial_t\phi + \frac{1}{a^2}\Delta_{\mathbb{R}^3}\phi = 0. \tag{13.8}$$

Through a Fourier transform (defined as in (12.99)):

$$\partial_t^2\hat{\phi} + \frac{3p}{t}\partial_t\hat{\phi} + \frac{|\xi|^2}{t^{2p}}\hat{\phi} = 0, \tag{13.9}$$

and we are going to use the notation

$$\begin{cases} \hat{\phi}(0,\xi) = \phi_0(\xi), \\ \partial_t\hat{\phi}(0,\xi) = \phi_1(\xi), \end{cases} \tag{13.10}$$

for the initial data. We found the solution of the above already in (2.18):

$$\hat{\phi}(t,\xi) = t^{\frac{1}{2}(1-3p)}\left[C_1(\xi)J_\nu\left(\frac{|\xi|t^{1-p}}{1-p}\right) + C_2(\xi)Y_\nu\left(\frac{|\xi|t^{1-p}}{1-p}\right)\right], \tag{13.11}$$

where $\nu = \frac{1-3p}{2(p-1)}$ and the functions $C_1(\xi)$ and $C_2(\xi)$ depend on the initial data:

$$C_1(\xi) = t_0^{\frac{1}{2}(3p-1)}\phi_0(x)J_\nu^{-1} - \frac{\pi}{2(1-p)}Y_\nu t_0^{\frac{1}{2}(1+p)}\left(t_0^p\phi_1(\xi) + |\xi|\phi_0(\xi)J_{\nu+1}J_\nu^{-1}\right) \tag{13.12}$$

and

$$C_2(\xi) = \frac{\pi}{2(1-p)}t_0^{\frac{1}{2}(1+p)}\left(t_0^p\phi_1(\xi)J_\nu + |\xi|\phi_0(\xi)J_{\nu+1}\right), \tag{13.13}$$

where we used that

$$Y_{\nu+1} - Y_\nu J_\nu^{-1}J_{\nu+1} = -\frac{2(1-p)}{\pi|\xi|t_0^{1-p}}J_\nu^{-1} \tag{13.14}$$

and we dropped the argument $\frac{|\xi|t_0^{1-p}}{1-p}$ of the Bessel functions for a better readability. Using the relations (12.104), we can compute

$$\partial_t\hat{\phi}(t,\xi) = -|\xi|t^{\frac{1}{2}(1-5p)}\left(C_1(\xi)J_{\nu+1} + C_2(\xi)Y_{\nu+1}\right). \tag{13.15}$$

The asymptotic behaviours of $\hat{\phi}$ and $\partial_t\hat{\phi}$ as $t\to\infty$ can be obtained through (2.29):

$$|\hat{\phi}| \sim |\xi|^{-\frac{1}{2}}t^{-p}\sin\left(\frac{\pi p + 2|\xi|t^{1-p}}{2-2p}\right)\cos\left(\frac{\pi p - 2|\xi|t^{1-p}}{2-2p}\right)(C_1(\xi) + DC_2(\xi)) \tag{13.16}$$

and

$$\begin{aligned}|\partial_t\hat{\phi}| \sim |\xi|^{\frac{1}{2}}t^{-2p}\Bigg(C_2(\xi)\cos\left(\frac{\pi p + 2|\xi|t^{1-p}}{2-2p}\right) + \\ +(DC_1(\xi) + EC_2(\xi))\cos\left(\frac{\pi - 2|\xi|t^{1-p}}{2-2p}\right)\Bigg),\end{aligned} \tag{13.17}$$

since the slowest term of the first derivative has the exponent $-2p > -p-1$. Thus, using Parseval's theorem:

$$K(t) = \frac{1}{2}\int_{\mathbb{R}^3} a^2|\partial_t\phi|^2 d^3x = \frac{1}{2}\int_{\mathbb{R}^3} a^2|\partial_t\hat{\phi}|^2 d^3\xi \sim \frac{1}{2a^2}\int_{\mathbb{R}^3} C(\xi)d^3\xi \sim \frac{1}{a^2}, \tag{13.18}$$

for some function $C(\xi)$ determined by (13.17). In particular, we are assuming that the initial data are smooth and decay sufficiently fast at infinity, therefore $C(\xi)$ is integrable in $\mathbb{R}^3$. Similarly:

$$P(t) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla\phi|^2 d^3x = \frac{1}{2}\int_{\mathbb{R}^3} |\xi|^2|\hat{\phi}|^2 d^3\xi \sim \frac{1}{2a^2}\int_{\mathbb{R}^3} \tilde{C}(\xi)d^3\xi \sim \frac{1}{a^2}. \tag{13.19}$$

Thus, kinetic and potential energy share the common limit 0 as $t \to \infty$ and it follows that

$$E(t) = K(t) + P(t) \sim \frac{1}{a^2}. \tag{13.20}$$

□

Now, we will prove the Morawetz estimate for the FLRW case. A proof for the Minkowski case can be found in [3]. We will use spherical coordinates and the notation

$$|\not\nabla\phi|^2 = \frac{1}{r^2}(\partial_\theta\phi)^2 + \frac{1}{r^2\sin^2(\theta)}\left(\partial_\varphi\phi\right)^2 \tag{13.21}$$

to denote the angular derivatives. In this set of coordinates, the total energy given by the sum of (13.3) and (13.4) can be written as

$$E(t) = \frac{1}{2}\int_{\mathbb{R}^3}\left(a^2(t)\,(\partial_t\phi(t,x))^2 + (\partial_r\phi(t,x))^2 + \left|\not\nabla\phi(t,x)\right|^2\right)d^3x. \tag{13.22}$$

Theorem 3.4 (Morawetz estimate in the Friedmann universe)**.** *Let ϕ be a solution of the wave equation $\Box_g\phi = 0$. Let us denote the spatial hypersurfaces of the Friedmann universe by $\Sigma_t = \{t\}\times\mathbb{R}^3$ for $t \geq t_0 > 0$ and let $\mathcal{R}(\tau) = \cup_{t\in[t_0,\tau]}\Sigma_t$. Then, for every $R > 0$ and for every $\tau > t_0$:*

$$\int_{\mathcal{R}(\tau)\cap\{r\leq R\}} a(t)\left(a^2(t)\,(\partial_t\phi)^2 + (\partial_r\phi)^2 + |\not\nabla\phi|^2\right)r^2 dt\,dr\,d\Omega \leq C_R\,a(t_0)E(t_0), \tag{13.23}$$

for some constant C_R depending on R, where $d\Omega = \sin(\theta)d\theta d\varphi$.

Proof. The idea consists of defining the multiplier vector field as

$$X = f(r)\partial_r \tag{13.24}$$

for some arbitrary function f of the radial variable, and use it to construct the current J associated to the solution of the wave equation. We will then use the divergence theorem on $\nabla^\mu J_\mu$ on the set $\mathcal{R}(\tau)$. In the previous sections, we used the divergence theorem on some compact set C for a multiplier vector field directed along the time direction and we obtained information on the boundaries of C. More precisely, we used the positivity of the current to investigate the behaviour of the energy on the boundary of C (to understand, e.g., if the energy was conserved or non-increasing). Now, the multiplier vector field is directed along the radial direction. This implies that the opposite process will occur: we will use the boundary terms to gain some information on the divergence term $\nabla^\mu J_\mu$. Let us start in the following way. The energy-momentum tensor is defined as usual:

$$\begin{aligned} T_{\mu\nu} &= \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}\partial_\alpha\phi\partial^\alpha\phi \\ &= \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}\left(-(\partial_t\phi)^2 + \frac{1}{a^2(t)}(\partial_r\phi)^2 + \frac{1}{a^2(t)}|\nabla\!\!\!/\phi|^2\right) \end{aligned} \tag{13.25}$$

The current is constructed as

$$J_\mu = T_{\mu\nu}X^\nu \tag{13.26}$$

and we define

$$K := T_{\mu\nu}\Pi^{\mu\nu} = \nabla^\mu J_\mu\,, \tag{13.27}$$

where the last equality is proved in theorem 3.2 (and the proof is valid in the current case as well). The first step consists of computing the deformation tensor

$$\Pi = \frac{1}{2}\mathcal{L}_X g. \tag{13.28}$$

We have

$$\Pi_{rr} = a^2 f', \quad \Pi_{\theta\theta} = a^2 f r, \quad \Pi_{\varphi\varphi} = a^2 f r\sin^2(\theta) \tag{13.29}$$

and, using the inverse metric:

$$\Pi^{rr} = \frac{f'}{a^2}, \quad \Pi_{\theta\theta} = \frac{f}{r^3a^2}, \quad \Pi^{\varphi\varphi} = \frac{f}{r^3a^2\sin^2(\theta)}. \tag{13.30}$$

Therefore:

$$\begin{aligned} K &= T_{rr}\Pi^{rr} + T_{\theta\theta}\Pi^{\theta\theta} + T_{\varphi\varphi}\Pi^{\varphi\varphi} \\ &= \left(\frac{f'}{2} + \frac{f}{r}\right)(\partial_t\phi)^2 + \frac{1}{a^2}\left(\frac{f'}{2} - \frac{f}{r}\right)(\partial_r\phi)^2 - \frac{1}{a^2}\frac{f'}{2}|\nabla\!\!\!/\phi|^2. \end{aligned} \tag{13.31}$$

We want this divergence to be positive when we estimate the derivatives of ϕ, but there exists no function f that makes all three coefficients of K as such. Therefore, we will also work with a slightly different current:

$$\tilde{J}_\mu := J_\mu + h_1(r)\phi\partial_\mu\phi + h_2(r)\psi^2(\partial_\mu w) \tag{13.32}$$

for some arbitrary functions h_1, h_2 and w to be chosen. The natural choice is $h_2 = 1$, $h_1 = 2G$ and $w = -G$ for some function G, indeed it makes the divergence of the respective current positive definite:

$$\begin{aligned}\tilde{K} &:= \nabla^\mu \tilde{J}_\mu \\ &= K + h_1 \left(\partial^\mu \phi \partial_\mu \phi\right) + \cancel{\left(\partial^\mu h_1 + 2h_2 \partial^\mu w\right) \phi \partial_\mu \phi} + \left(\partial_\mu h_2 \partial^\mu w + h_2 \Box_g w\right) \phi^2. \end{aligned} \tag{13.33}$$

If we keep going with the computations:

$$\begin{aligned}\tilde{K} = &\left(\frac{f'}{2} + \frac{f}{r} - 2G\right) \left(\partial_t \phi\right)^2 + \frac{1}{a^2}\left(\frac{f'}{2} - \frac{f}{r} + 2G\right) \left(\partial_r \phi\right)^2 + \\ &+ \frac{1}{a^2}\left(-\frac{f'}{2} + 2G\right) |\nabla\!\!\!/ \phi|^2 - \left(\Box_g G\right) \phi^2. \end{aligned} \tag{13.34}$$

We want to remove the terms going like $\frac{1}{r}$ since they are in general problematic in the integration for r small. So, a wise choice for G is

$$G = \frac{f'}{4} + \frac{f}{2r}. \tag{13.35}$$

Then, we can easily compute the d'Alembertian of G (given by (1.8) and (3.10), but many terms vanish since $G = G(r)$):

$$\Box_g G = \left(\frac{1}{a^2}{\partial_r}^2 + \frac{2}{ra^2}\partial_r\right)\left(\frac{f'}{4} + \frac{f}{2r}\right) = \frac{1}{a^2}\left(\frac{f'''}{4} + \frac{f''}{r}\right). \tag{13.36}$$

Finally:

$$\tilde{K} = \frac{f'}{a^2}\left(\partial_r \phi\right)^2 + \frac{f}{ra^2}|\nabla\!\!\!/ \phi|^2 - \frac{1}{a^2}\left(\frac{f'''}{4} + \frac{f''}{r}\right)\phi^2. \tag{13.37}$$

We will use the current $\tilde{J}$ to estimate the radial and angular derivatives of ϕ, whereas we are going to use J to estimate the time derivative.

1) Estimate on the angular derivatives. Let us consider the current $\tilde{J}$ and set $f \equiv 1$. Thus, we have $G = \frac{1}{2r}$ and

$$\tilde{K} \overset{(13.37)}{=} \frac{1}{ra^2}|\nabla\!\!\!/ \phi|^2. \tag{13.38}$$

Due to the term $\frac{1}{r}$, we want to apply the divergence theorem first in $\mathcal{R}(\tau) \cap \{r \geq \delta\}$ for some $\delta > 0$:

$$\int_{\mathcal{R}(\tau)\cap\{r\geq\delta\}} \tilde{K} = \int_{\Sigma_{t_0}\cup\Sigma_\tau} \tilde{J}_\mu n^\mu + \int_{\{r=\delta\}} \tilde{J}_\mu {n_\delta}^\mu, \tag{13.39}$$

where the normals are $n_{\Sigma_{t_0}} = \partial_t = -n_{\Sigma_\tau}$ and $n_\delta = -\partial_r$, and we ultimately want to take the limit as $\delta \to 0$. The integral on the timelike hypersurface $\{r = \delta\}$ can be computed in spherical coordinates:

$$\int_{\{r=\delta\}} \tilde{J}_\mu n_\delta{}^\mu = \int_0^\tau \int_{S^2} \left(-J_r - \frac{1}{\delta}\phi\partial_r\phi - \frac{1}{\delta^2}\phi^2 \right) \delta^2 \sqrt{-g}\, d\Omega dt$$

$$\xrightarrow[\delta\to 0]{} -C \int_0^\tau \phi^2(r=0)a^3(t)dt, \qquad (13.40)$$

for a positive constant C. Therefore, we will bring this term to the left hand side of (13.39). The integral over $\Sigma_{t_0} \cup \Sigma_t$ can be estimated as follows. First, we notice that

$$\tilde{J}_t \overset{(13.32)}{=} J_t + \frac{1}{r}\phi\partial_t\phi \overset{(13.26)}{=} \partial_t\phi\partial_r\phi + \frac{1}{r}\phi\partial_t\phi, \qquad (13.41)$$

where we used that $f \equiv 1$ in our case. Therefore:

$$\int_{\Sigma_{t_0}\cup\Sigma_\tau} \tilde{J}_\mu n^\mu = \int_{\mathbb{R}^3} a^3(t_0) \left(\partial_t\phi(t_0,x)\partial_r\phi(t_0,x) + \frac{1}{r}\phi(t_0,x)\partial_t\phi(t_0,x) \right) d^3x +$$

$$- \int_{\mathbb{R}^3} a^3(\tau) \left(\partial_t\phi(\tau,x)\partial_r\phi(\tau,x) + \frac{1}{r}\phi\partial_t\phi(\tau,x) \right) d^3x$$

$$\lesssim a(t_0)E(t_0), \qquad (13.42)$$

where we used the Young's inequality and the fact that the energy decays as a^{-2}. In particular, the first integral is dominated by $ca(t_0)E(t_0)$ and the terms of the second integral can be controlled using the Hardy inequality in (12.74) and the decay of the energy proved in proposition 3.1. For instance:

$$\int_{\mathbb{R}^3} a^3 \frac{\phi}{r}\partial_t\phi \lesssim \int_{\mathbb{R}^3} a^2 \left(\frac{\phi^2}{r^2} + a^2 \left(\partial_t\phi\right)^2 \right) \lesssim c. \qquad (13.43)$$

The expression given by the divergence theorem then becomes

$$\int_{\mathcal{R}(\tau)} \frac{1}{ra^2}|\nabla\!\!\!/\phi|^2 \le \int_{\mathcal{R}(\tau)} \frac{1}{ra^2}|\nabla\!\!\!/\phi|^2 + c\int_0^\tau \phi^2(r=0)a^3(t)dt \le ca(t_0)E(t_0). \qquad (13.44)$$

So, if we restrict the radial coordinate:

$$\int_{\mathcal{R}(\tau)\cap\{r\le R\}} \frac{1}{a^2}|\nabla\!\!\!/\phi|^2 \le C_R a(t_0)E(t_0). \qquad (13.45)$$

2) Estimate on the radial derivative. We consider again the current $\tilde{J}$, but now we choose the function

$$f(r) = -\frac{1}{r+1}. \qquad (13.46)$$

Thus, $G \overset{(13.35)}{=} \frac{1}{4(r+1)^2} - \frac{1}{2r(r+1)}$. We have:

$$\tilde{K} \overset{(13.37)}{=} \frac{1}{a^2}\frac{1}{(r+1)^2}\left(\partial_r\phi\right)^2 - \frac{1}{a^2}\frac{1}{r(r+1)}\left|\nabla\!\!\!/\phi\right|^2 + \frac{1}{a^2}\gamma(r)\phi^2, \qquad (13.47)$$

where $\gamma(r) > 0$, $\gamma(r) \sim r^{-4}$ for large r. Since $G \sim r^{-1}$ as $r \to 0$, the estimates done for the angular derivatives in the previous case can be repeated for the radial derivative (and we can use that the integral of $|\nabla\!\!\!/ \phi|^2$ has already been bounded). In particular, we get:

$$\int_{\mathcal{R}(\tau)\cap\{r\leq R\}} \frac{1}{a^2}\left(\partial_r \phi\right)^2 \leq C_R a(t_0) E(t_0). \tag{13.48}$$

3) Estimate on the time derivative. Since the coefficient for the term $(\partial_t \phi)^2$ can be potentially positive in (13.31), we can work with the current J and choose

$$f(r) = \frac{1}{r+1}. \tag{13.49}$$

In fact, we have:

$$K = \frac{r+2}{2r(r+1)^2}\left(\partial_t \phi\right)^2 + \frac{\alpha_1(r)}{a^2(t)}\left(\partial_r \phi\right)^2 + \frac{1}{a^2(t)}\frac{1}{2(r+1)^2}\left|\nabla\!\!\!/ \phi\right|^2, \tag{13.50}$$

where $\alpha_1(r) \sim r^{-2}$ for large r and $\alpha_1(r) \sim r^{-1}$ for small r. Again, we can repeat the estimates used in the previous cases and obtain

$$\int_{\mathcal{R}(\tau)\cap\{r\leq R\}} \left(\partial_t \phi\right)^2 \leq C_R a(t_0) E(t_0). \tag{13.51}$$

After summing the three terms we obtain the desired estimate. □

14 de Sitter Universe

The same method used to deal with homogeneous and isotropic universes with scale factor $a(t) = t^p$ can also be exploited for vacuum energy solutions. The decay rates for the wave equation in de Sitter space were already obtained in [21] for the flat form of the metric. Here we will extend the result both to de Sitter solutions with spherical space sections and to those with hyperbolic space sections.

Theorem 3.5. *Let $n > 2$ and let $I \subset \mathbb{R}$ be the open interval $(t_*, \infty) \ni t_0$ for some $t_* > 0$. Assume that (M, g) describes the de Sitter universe given by the metric*

$$g = -dt^2 + \begin{Bmatrix} \sinh^2(t) \\ e^{2t} \\ \cosh^2(t) \end{Bmatrix} \left(d\chi^2 + \begin{Bmatrix} \sinh^2(\chi) \\ \chi^2 \\ \sin^2(\chi) \end{Bmatrix} d\Omega^2 \right), \tag{14.1}$$

where the curly brackets contain the three possible cases: hyperbolic, flat and spherical space sections of dimension n, respectively. We suppose that ϕ is a smooth solution to the Cauchy *problem*

$$\begin{cases} \Box_g \phi(t,x) = 0, \\ \phi(t_0,x) = \phi_0(x), \\ \partial_t \phi(t_0,x) = \phi_1(x), \end{cases} \tag{14.2}$$

with $t \geq t_0$*,* $x \in \Sigma$*,* $\phi_0 \in H^k(\Sigma)$*,* $\phi_1 \in H^{k-1}(\Sigma)$ *and* $k > \frac{n}{2} + 2$*, where* $\Sigma = \mathbb{H}^n, \mathbb{R}^n, S^n$*. Then:*

$$\|\partial_t \phi(t,\cdot)\|_{L^\infty(\Sigma)} \lesssim a(t)^{-2} \sim e^{-2t}. \tag{14.3}$$

Proof. The proof is similar to the one for theorem 3.2. We will distinguish again among several steps.

Laplacian estimate. Since $t_0 \geq 0$, the first step of the proof of theorem (3.2) is enough to prove that

$$\|\phi\|_{H^k(\Sigma)} \lesssim 1 \tag{14.4}$$

if ϕ satisfies the wave equation. Now, in the flat case we can say that if ϕ is a solution to the wave equation, then $\partial^\alpha \phi$ is a solution as well. More generally, $K \cdot \phi$ is a solution if K is a Killing vector field on Σ for the metric g (i.e. K is the infinitesimal generator of an isometry). The reason behind this is that $\Box_g$ is invariant under isometries, since an isometry ψ is a map $\psi : M \to M$ such that $\psi_* g_{\mu\nu} = g_{\mu\nu}$ and the d'Alembertian can be entirely written in terms of the metric and of its derivatives. As pointed out in [13], we also have that

$$\Delta \phi = \sum_{i=1}^{N} (-1)^{p_i} K_i \cdot (K_i \cdot \phi) \tag{14.5}$$

for some Killing vector fields $K_1, \ldots, K_N$ and for some $p_1, \ldots, p_N \in \{0,1\}$. Therefore, the Laplacian of a solution ϕ for the wave equation is still a solution and

$$\|\Delta \phi\|_{H^{k'}(\Sigma)} \lesssim 1 \tag{14.6}$$

for $k' = k - 2 > \frac{n}{2}$. By the Sobolev inequality (11.3):

$$\|\Delta \phi\|_{L^\infty(\Sigma)} \lesssim 1. \tag{14.7}$$

Now, we have

$$\Box_g \phi = 0 \Leftrightarrow \partial_\mu \left(\sqrt{-g} \partial^\mu \phi\right) = 0. \tag{14.8}$$

More explicitly:

$$\partial_\tau \left(a^{n-1} \partial_\tau \phi\right) = a^{n-1} \Delta \phi, \tag{14.9}$$

where the conformal time variable τ depends on the scale factor of the case that we want to consider. Let us start with the flat case.

Application of the bound on the Laplacian for the flat case. When $\Sigma = \mathbb{R}^n$, the conformal time is given by

$$\tau = \int_{t_0}^{t} e^{-s} ds = e^{-t_0} - e^{-t} \Leftrightarrow t = -\log\big(e^{-t_0} - \tau\big). \tag{14.10}$$

Thus, the scale factor becomes

$$a(t) = e^t = \frac{1}{e^{-t_0} - \tau}. \tag{14.11}$$

The time coordinate t ranges in $[t_0, +\infty)$ and therefore $\tau \in [0, e^{-t_0})$. In particular: $a(t = t_0) = a(\tau = 0) = e^{t_0}$ is the scale factor at the initial time. If we integrate the relation (14.9):

$$a^{n-1}(\tau)\partial_\tau\phi(\tau, x) - a^{n-1}(t_0)\partial_\tau\phi(0, x) = \int_0^{\tau} \Delta\phi(s, x)\frac{1}{(e^{-t_0} - s)^{n-1}} ds. \tag{14.12}$$

Since $\partial_\tau = a\partial_t$ and $\phi_1(x) = \partial_t\phi(t_0, x)$:

$$\partial_t\phi(t, x) = a^{-n}(t)\left[a^n(t_0)\phi_1(x) + \int_0^{\tau} \Delta\phi(s, x)\frac{1}{(e^{-t_0} - s)^{n-1}} ds\right]. \tag{14.13}$$

After taking the L^∞ norms:

$$\begin{aligned}\|\partial_t\phi(t, \cdot)\|_{L^\infty(\mathbb{R}^n)} &\le a^{-n}(t)\left[a^n(t_0)\|\phi_1\|_{L^\infty(\mathbb{R}^n)} + \int_0^{\tau} \|\Delta\phi\|_{L^\infty(\mathbb{R}^n)}\frac{1}{(e^{-t_0} - s)^{n-1}} ds\right]\\ &= a^{-n}(t)\left[a^n(t_0)\|\phi_1\|_{L^\infty(\mathbb{R}^n)} + c\left((e^{-t_0} - \tau)^{2-n} - (e^{-t_0})^{2-n}\right)\right].\end{aligned} \tag{14.14}$$

Using the expression (14.11) for the scale factor, we get

$$\begin{aligned}\|\partial_t\phi(t, \cdot)\|_{L^\infty(\mathbb{R}^n)} &= a^{-n}(t)\left[a^n(t_0)\|\phi_1\|_{L^\infty(\mathbb{R}^n)} + c\left(a^{n-2}(t) - a^{n-2}(t_0)\right)\right]\\ &= a^{-2}(t)\left[\frac{a^n(t_0)}{a^{n-2}(t)}\|\phi_1\|_{L^\infty(\mathbb{R}^n)} + c\left(1 - \frac{a^{n-2}(t_0)}{a^{n-2}(t)}\right)\right] \overset{(n>2)}{\lesssim} a^{-2}(t),\end{aligned} \tag{14.15}$$

which concludes the proof for the flat case.

Application of the bound on the Laplacian for the spherical case. When $\Sigma = S^n$, the conformal time is given by

$$\tau = \int_{t_0}^{t} \frac{ds}{\cosh(s)} = 2\left(\arctan\left(e^t\right) - \arctan\left(e^{t_0}\right)\right) \Leftrightarrow t = \log\left(\tan\left(\frac{\tau}{2} + \arctan\left(e^{t_0}\right)\right)\right), \tag{14.16}$$

since the formula $2\arctan\left(\tanh\left(\frac{x}{2}\right)\right) = 2\arctan(e^x) - \frac{\pi}{2}$ holds. Thus, the scale factor becomes

$$a(t) = \cosh(t) = \frac{1}{\sin(\tau + 2\arctan\left(e^{t_0}\right))} = \frac{1}{\sin(\tau + b)}, \tag{14.17}$$

using $b := 2\arctan\left(e^{t_0}\right)$. Since $t \in [t_0, +\infty)$, we have $\tau \in [0, \pi - 2\arctan\left(e^{t_0}\right))$. In particular: $a(t = t_0) = a(\tau = 0) = \cosh(t_0)$ is the scale factor at the initial time. If we integrate the relation (14.9):

$$a^{n-1}(\tau)\partial_\tau\phi(\tau, x) - a^{n-1}(t_0)\partial_\tau\phi(0, x) = \int_0^\tau \Delta\phi(s, x)\frac{1}{\sin(s+b)^{n-1}}ds. \tag{14.18}$$

Since $\partial_\tau = a\partial_t$ and $\phi_1(x) = \partial_t\phi(t_0, x)$:

$$\partial_t\phi(t, x) = a^{-n}(t)\left[a^n(t_0)\phi_1(x) + \int_0^\tau \Delta\phi(s, x)\frac{1}{\sin(s+b)^{n-1}}ds\right]. \tag{14.19}$$

After taking the L^∞ norms:

$$\begin{aligned}\|\partial_t\phi(t,\cdot)\|_{L^\infty(S^n)} &\le a^{-n}(t)\left[a^n(t_0)\|\phi_1\|_{L^\infty(S^n)} + \left|\int_0^\tau \|\Delta\phi\|_{L^\infty(S^n)}\frac{1}{\sin(s+b)^{n-1}}ds\right|\right]\\ &= a^{-n}(t)\Bigg[a^n(t_0)\|\phi_1\|_{L^\infty(S^n)} +\\ &\quad + c\sin(\tau+b)^{2-n}\left|F\left(1, \frac{3-n}{2}, \frac{1+n}{2}, e^{2i(\tau+b)}\right)\right| - d\,a^{n-2}(t_0)\Bigg],\end{aligned} \tag{14.20}$$

where c, d constants and F is the Gauss hypergeometric series already encountered in (3.21). In the last step, in particular, we computed the integral

$$\int \frac{dx}{\sin(x+b)^{n-1}} = c\sin(x+b)^{2-n}F\left(1, \frac{3-n}{2}, \frac{1+n}{2}, e^{2i(x+b)}\right) + d. \tag{14.21}$$

We notice that the Gauss hypergeometric series converges absolutely when evaluated at $e^{2i(x+b)}$ since $\frac{1+n}{2} - \frac{3-n}{2} - 1 > 0$ for $n > 2$ (see e.g. [2]). Therefore $\left|F\left(1, \frac{3-n}{2}, \frac{1+n}{2}, e^{2i(x+b)}\right)\right| \le c$. Using the expression (14.17) for the scale factor we obtain:

$$\begin{aligned}\|\partial_t\phi(t,\cdot)\|_{L^\infty(S^n)} &\le a^{-n}(t)\left[a^n(t_0)\|\phi_1\|_{L^\infty(S^n)} + c\left(a^{n-2}(t) - a^{n-2}(t_0)\right)\right]\\ &= a^{-2}(t)\left[\frac{a^n(t_0)}{a^{n-2}(t)}\|\phi_1\|_{L^\infty(S^n)} + c\left(1 - \frac{a^{n-2}(t_0)}{a^{n-2}(t)}\right)\right] \overset{(n>2)}{\lesssim} a^{-2}(t),\end{aligned} \tag{14.22}$$

which is enough to prove the spherical case. We will now examine the hyperbolic expression of the metric.

Application of the bound on the Laplacian for the hyperbolic case. When $\Sigma = \mathbb{H}^n$, the conformal time is given by

$$\begin{aligned}\tau = \int_{t_0}^t \frac{ds}{\sinh(s)} = \log\left(\frac{\tanh\left(\frac{t}{2}\right)}{\tanh\left(\frac{t_0}{2}\right)}\right) \Leftrightarrow t = 2\,\mathrm{artanh}\left(e^\tau\tanh\left(\frac{t_0}{2}\right)\right) =\\ = \log\left(\frac{1+\tanh\left(\frac{t_0}{2}\right)e^\tau}{1-\tanh\left(\frac{t_0}{2}\right)e^\tau}\right),\end{aligned} \tag{14.23}$$

using that $\log\frac{1+x}{1-x} = 2\,\mathrm{artanh}(x)$ for $|x| < 1$, since $\tau \in \left[0, -\log\left(\tanh\left(\frac{t_0}{2}\right)\right)\right)$. Thus, the scale factor becomes

$$a(t) = \sinh(t) = \frac{1}{2}\left(e^t - e^{-t}\right) = -\frac{1}{\sinh\left(\tau + \log\left(\tanh\left(\frac{t_0}{2}\right)\right)\right)} = -\frac{1}{\sinh(\tau + b)}, \qquad (14.24)$$

using $b := \log\left(\tanh\left(\frac{t_0}{2}\right)\right) < 0$. Since $t \in [t_0, +\infty)$, we have $\tau \in [0, -b)$. In particular: $a(t = t_0) = a(\tau = 0) = \sinh(t_0)$ is the scale factor at the initial time. If we integrate the relation (14.9):

$$a^{n-1}(\tau)\partial_\tau\phi(\tau, x) - a^{n-1}(t_0)\partial_\tau\phi(0, x) = \int_0^\tau \Delta\phi(s, x)\,[-\sinh(s + b)]^{1-n}\,ds. \qquad (14.25)$$

Since $\partial_\tau = a\partial_t$ and $\phi_1(x) = \partial_t\phi(t_0, x)$:

$$\partial_t\phi(t, x) = a^{-n}(t)\left[a^n(t_0)\phi_1(x) + \int_0^\tau \Delta\phi(s, x)\,[-\sinh(s + b)]^{1-n}\,ds\right]. \qquad (14.26)$$

After taking the L^∞ norms:

$$\begin{aligned}\|\partial_t\phi(t,\cdot)\|_{L^\infty(\mathbb{H}^n)} &\le a^{-n}(t)\left[a^n(t_0)\|\phi_1\|_{L^\infty(\mathbb{H}^n)} + \int_0^\tau \|\Delta\phi\|_{L^\infty(\mathbb{H}^n)}\sinh(s+b)^{1-n}ds\right]\\ &= a^{-n}(t)\Big[a^n(t_0)\|\phi_1\|_{L^\infty(\mathbb{H}^n)} +\\ &\quad + ce^{\tau-b}\sinh(\tau+b)^{2-n}F\left(1, \frac{3-n}{2}, \frac{1+n}{2}, e^{2(\tau+b)}\right) - d\,a^{n-2}(t_0)\Big],\end{aligned} \qquad (14.27)$$

where c, d are constants and F is again the Gauss hypergeometric series. In the last step, we computed the integral

$$\int \sinh(x+b)^{1-n}dx = ce^{\tau+b}\sinh(\tau+b)^{2-n}F\left(1, \frac{3-n}{2}, \frac{1+n}{2}, e^{2(\tau+b)}\right) + d. \qquad (14.28)$$

We notice that the Gauss hypergeometric series converges absolutely when evaluated at $e^{2(\tau+b)} \le 1$, since its circle of convergence is given by the unit ball centred at 0 and since $\frac{1+n}{2} - \frac{3-n}{2} - 1 > 0$ for $n > 2$ (see e.g. [2]). Therefore $F\left(1, \frac{3-n}{2}, \frac{1+n}{2}, e^{2(\tau+b)}\right) \le c$. Using the expression (14.24) for the scale factor, we obtain:

$$\begin{aligned}\|\partial_t\phi(t,\cdot)\|_{L^\infty(\mathbb{H}^n)} &\le a^{-n}(t)\left[a^n(t_0)\|\phi_1\|_{L^\infty(\mathbb{H}^n)} + c\left(a^{n-2}(t) - a^{n-2}(t_0)\right)\right]\\ &= a^{-2}(t)\left[\frac{a^n(t_0)}{a^{n-2}(t)}\|\phi_1\|_{L^\infty(\mathbb{H}^n)} + c\left(1 - \frac{a^{n-2}(t_0)}{a^{n-2}(t)}\right)\right] \overset{(n>2)}{\lesssim} a^{-2}(t),\end{aligned} \qquad (14.29)$$

which concludes the proof. □

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