Ph.D. Thesis

On the future stability of cosmological solutions of the Einstein-nonlinear scalar field system

Artur Carlos Ferreira Alho

November 12, 2012

Acknowledgments

First of all, I would like to express my deep gratitute to Professors Filipe Carteado Mena, José Maciel Natário, João Lopes Costa and Juan Valiente-Kroon. Without them this thesis would definitely not exist.

I would also like to thank all staff of the Master course in Cosmology and Particle Physics at the University of Salamanca, in particular Professors Marc Mars, Chus Martin and Eduardo Ruiz; the Relativity Group at the School of Mathematical Sciences, Queen Mary, University of London, in particular Christian Lubbe and Thomas Backdahl; and the CAMGSD at Instituto Superior Técnico, University Técnica de Lisboa in particular Pedro Girão and Jorge Silva. At the University of Minho I would like to thank the Mathematical Physics Group and specially Filipe Moura and Alfonso Garcia-Parrado.

I appreciate the financial support by the FCT grant SFRH/BD/48658/2008, projects PTDC/MAT/108921/2008, CERN/FP/116377/2010 CERN/FP/123609/2011 and, CMAT University of Minho, through FCT plurianual funding.

Abstract

This thesis concerns with the future asymptotic stability of cosmological solutions to the Einstein-Field-Equations with a minimally coupled nonlinear scalar field. To do this, we shall take two distinct routes which are the main reason for the structure of this thesis.

More precisely, in a first part we shall consider perturbations of (flat) spatially homogeneous and isotropic spacetimes. This will consist in an analysis of covariant and gauge-invariant linear scalar perturbations, and the exponential decay of fully nonlinear perturbations. In the linear stability analysis, we start by deducing the set of evolution equations for density inhomogeneities when multiple interacting nonlinear scalar fields are present. Then, using the decomposition into scalar harmonics we study the system qualitatively by applying techniques from the theory of dynamical systems. The cases of a single scalar field with exponential, quadratic and quartic potentials are studied in detail. In particular we prove a cosmic no-hair result for power-law inflation and show that homogenization occurs in models of chaotic and new inflation. We then extend the analysis to two scalar fields, and the particular situation when each scalar field has independent exponential potentials is analysed, resulting in a linear cosmic no-hair result for assisted power-law inflation. A frame representation is then used to derive a first-order quasi-linear symmetric hyperbolic system for the Einstein-nonlinear scalar field system. This procedure is inspired by similar evolution equations introduced by Friedrich to study the Einstein-Euler system, and it is the first step for studying the exponential decay of nonlinear perturbations for which we derive preliminary results.

The second part of the thesis concerns with the simplest inhomogeneous scalar field solutions, namely within the class of spherically symmetric spacetimes. We generalize Christodoulou's framework, developed to study the Einstein-scalar field equations with vanishing cosmological constant Λ, by introducing $\Lambda > 0$. As a first step towards the Einstein-scalar field equations with positive cosmological constant, and in order to gain insight into its nonlinearity, we start by solving the wave equation in de Sitter spacetime. We obtain an integro-differential evolution equation which we solve by taking initial data on a null cone. As a corollary we obtain elementary derivations of expected properties of linear waves in de Sitter spacetime: boundedness in terms of (characteristic) initial data, and a Price law establishing uniform exponential decay, in Bondi time, to a constant. Afterwards we study the Einstein-scalar field system with positive cosmological constant and spherically symmetric characteristic initial data given on a truncated null cone. We prove well-posedness, global existence and exponential decay in (Bondi) time, for small data. From this, it follows that initial data close enough to de Sitter data evolves to a causally geodesically complete spacetime (with boundary), which approaches a region of de Sitter asymptotically at an exponential rate; this is a non-linear stability result for de Sitter within the class under consideration, as well as a realization of the cosmic no-hair conjecture.

Resumo

Nesta tese debruçamo-nos sobre a estabilidade assimptótica no futuro de soluções cosmológicas das equações de campo de Einstein com um campo escalar não-linear minimamente acopolado. A tese divide-se em duas partes.

Numa primeira parte consideramos perturbações de espaços-tempo espacialmente (planos) homogéneos e isotrópicos, que consiste na análise de perturbações lineares escalares covariantes e invariantes de gauge e no decaimento exponencial de perturbações não-lineares. Na estabilidade linear, começamos por deduzir as equações de evolução de perturbações na densidade de matéria para N campos escalares que interagem entre si. De seguida, usando a decomposição em harmónicos escalares, estudamos qualitativamente o sistema de equações resultante usando métodos da teoria de sistemas dinâmicos. Os casos de um único campo escalar com potencial exponencial, quadrático e quártico são então estudados em detalhe. Em particular, provamos um resultado da conjectura cósmica sem cabelo, para inflação do tipo lei da potência e mostramos que existe homogenização em modelos de inflação caótica e nova inflação. A análise é então alargada para o caso de dois campos, e em particular, na situação em que cada um dos campos tem um potentical exponencial, provando-se um resultado linear da conjectura cósmica sem cabelo para inflação assistida do tipo lei da potência. Por fim, utilizamos uma representação de referenciais ortonormados para deduzir um sistema de primeira ordem simétrico e hiperbólico para o sistema de Einstein-campo escalar não-linear. Este procedimento é inspirado em equações de evolução introduzidas por Friedrich no estudo do sistema de Einstein-Euler, e é o primeiro passo em direção ao estudo do decaimento exponencial de peturbações não-lineares, para o qual obtemos resultados preliminares.

Na segunda parte da tese consideramos as soluções não-homogéneas mais simples com um campo escalar, nomeadamente a classe de espaços-tempo esfericamente simétricos. Para tal, generalizamos o método de Christodoulou para o sistema de Einstein-campo escalar sem constante cosmológica Λ, introduzindo $\Lambda > 0$. Como primeiro passo em direção ao estudo do sistema de Einstein-campo escalar com constante cosmológica positiva, e de modo a entender as não-lineariedades, começamos por resolver a equação de onda no espaço-tempo de de Sitter. Obtemos uma equação de evolução integro-diferencial que resolvemos dando dados iniciais num cone de luz futuro. Como corolário obtemos as seguintes propriedades elementares de ondas lineares em de Sitter: a solução é limitada em termos dos dados iniciais (característicos) e verifica-se uma lei de Price estabelecendo decaimento uniforme, na coordenada temporal de Bondi, para uma constante. De seguida estudamos o sistema de Einstein-campo escalar com constante cosmológica positiva em simetria esférica, com dados iniciais característicos num cone de luz truncado. Provamos que o problema é bem posto, existência global e decaimento exponencial (no tempo de Bondi) de soluções para dados pequenos. Destes resultados, tem-se então que para dados iniciais suficientemente perto dos dados de de Sitter, estes dão origem a um espaço-tempo geodesicamente completo (com fronteira), que se aproxima de de Sitter a uma taxa exponencial; Este é um resultado de estabilidade não-linear para de Sitter dentro da classe de soluções esfericamente simétricas, assim como uma demonstração da conjectura cósmica sem cabelo.

Contents

List of Tables

List of Figures

Part I

Topics on General Relativity and Cosmology

Chapter 1

Introduction

By 1916, almost a century ago, Einstein had already finished writing the general theory of relativity and the *Einstein-Field-Equations* (EFEs) which incorporate the relativistic classical field theory with the effects of gravitation. Ten years before, Einstein had made a strong impact in the scientific community with his special theory of relativity, in which old Galilean postulates as absolute time and space were abandoned to give place to a more complex theory where no absolute observer exists. From a mathematical point of view, the theory of General Relativity is rather complex, being a geometric theory in nature, where the physical system is described by the spacetime structure. Direct consequences of the theory are the predictions of black holes and the expansion of the universe.

While in the Newtonian theory of gravitation, the gravitational field is described by the *Poisson equation*

$$\triangle\varphi = 4\pi G\rho \tag{1.1}$$

with the suitable condition that the *Newtonian potential* φ vanishes at infinity. Here ρ represents the *mass-density* of the matter and $\triangle$ is the Laplace operator. In the theory of General Relativity, the gravitational field is described by a non-degenerate bilinear form called metric $\boldsymbol{g}$ naturally living in a four-dimensional[1] connected smooth manifold $\mathcal{M}$. The difference from traditional Riemannian geometry lies in that the metric has Lorentzian signature $(-,+,+,+)$ see e.g.. [98, 146, 142]. The simplest form of the metric is the analogue of the Euclidean space in Riemannian geometry, namely the Minkowski space $(\mathbb{R}^{3+1}, \mathbf{g}_{\mathrm{M}})$, where in global Cartesian coordinates the metric reads

$$\mathbf{g}_{\mathrm{M}} = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$$

and is the space of *special relativity* for which there is no gravitation. Lorentzian manifolds representing physical systems are solutions of the Einstein-Field-Equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}\ , \quad \kappa = \frac{8\pi G}{c^4} \tag{1.2}$$

where $R_{\mu\nu}$, $R = g^{\mu\nu}R_{\mu\nu}$ are the *Ricci tensor* and *scalar* of the metric $\mathbf{g}$, Λ the so-called cosmological constant. $T_{\mu\nu}$ is a symmetric tensor of rank 2 known as the *energy-momentum tensor* and it describes the matter, being a function of the matter field. Through this work the Einstein summation convention is understood (unless otherwise specified), Greek indices run from $\alpha, \beta, \mu, ... = 0, 1, 2, 3$, and we shall make use of units such that the velocity of light $c = 1$ and the Gravitational constant $8\pi G = 1$, i.e. $\kappa = 1$.

The EFEs can also be cast into the form

$$R_{\mu\nu} = T_{\mu\nu} - \frac{1}{2}T_{\sigma}{}^{\sigma} g_{\mu\nu}, \tag{1.3}$$

[1] Although many modern physical theories predict the existence of higher dimensional spacetimes, in this thesis we shall only focus in 4-dimensional world models.

where the Ricci tensor is given in terms of the Christoffel symbols through

$$R_{\alpha\beta} = \partial_\mu \Gamma^\mu_{\alpha\beta} - \partial_\alpha \Gamma^\mu_{\mu\beta} + \Gamma^\nu_{\alpha\beta}\Gamma^\mu_{\nu\mu} - \Gamma^\nu_{\mu\beta}\Gamma^\mu_{\nu\alpha} \tag{1.4}$$

and the Christoffel symbols are given through the metric by

$$\Gamma^\mu_{\nu\sigma} = \frac{1}{2} g^{\mu\lambda} \left(\partial_\nu g_{\sigma\lambda} + \partial_\sigma g_{\lambda\nu} - \partial_\lambda g_{\nu\sigma}\right) \,, \tag{1.5}$$

where $\partial_\mu = \frac{\partial}{\partial x^\mu}$. Due to the twice contracted differential Bianchi identity, the left hand side of the EFEs (1.2) satisfy the identity

$$\nabla^\alpha \left(R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R \right) = 0. \tag{1.6}$$

with ∇_α the covariant derivative of the metric $\mathbf{g}$. Then (1.2) imply that the energy-momentum tensor of the matter sources satisfies the conservation laws

$$\nabla_\alpha T^{\alpha\beta} = 0. \tag{1.7}$$

There are a variety of matter models describing different physical situations, which can have different mathematical nature (scalar, vector, spinor, etc...) and satisfy different equations of motion, given by (1.7). For example, in Cosmology, the medium is often described by a perfect fluid

$$T_{\mu\nu} = (\rho + p)\, u_\mu u_\nu + p g_{\mu\nu} \tag{1.8}$$

with linear equation of state

$$p = (\gamma - 1)\, \rho, \tag{1.9}$$

where ρ is the energy density, p the pressure and u^μ the four-velocity of the fluid. Usually, *dust* $\gamma = 1$ can be used to model galaxies (or clusters of galaxies) in an expanding universe and *radiation* $\gamma = \frac{4}{3}$ is important in the early universe. Standard Cosmology, or the hot big-bang model, is constructed under three fundamental Cosmological observations: The fact that, the universe is expanding as discovered by Hubble in the late twenties [102], due to the redshift effect on the luminosity of distant galaxies; The measurement of the cosmic microwave background radiation (CMB), first detected by Penzias and Wilson [152] and predicted by Gamow, Alpher and Herman [87, 88, 9, 10] in their work on nucleosynthesis and processes formation of light elements abundances seconds after the big-bang; The third fundamental observation it is the abundance pattern of the light elements, particularly the Helium.

However, many questions remain(ed) open, as the fact that the universe is flat and isotropic, or the nature and origin of the primordial density perturbations that give rise to all structure formation. In order to explain these and other pathologies in the hot big-bang scenario, the inflationary paradigm was introduced in its simpler form by Guth [93], where during a period of supercooling cosmological phase transition, the universe expands exponentially, in a false vacuum state, i.e. as if it had no matter, but contained a large positive cosmological constant. More recently, and due to the discovery that the Universe is actually accelerating, as confirmed by supernovae observations in the late nineties [153, 178, 168] (a discovery which was indeed awarded with the 2011 Nobel prize in physics), Cosmological solutions exhibiting accelerated expansion have become of particular physical relevance. One of the biggest puzzles in modern cosmology is the discrepancy of present value of Λ and the value predicted by particle physics theory of the early universe.

From the mathematical point of view acccelerated expanding Cosmologies present various important mathematical challenges. When studying initial value problems for the Einstein-matter field equations with positive cosmological constant, a general framework is provided by the *cosmic no-hair conjecture*, which states that generic ever expanding solutions of Einstein's field equations with a positive cosmological constant approach the de Sitter solution asymptotically into the future. This conjecture has been rigorously proved, in the class of Spatially Homogeneous models which do not recollapse (Bianchi types I-VIII) containing matter satisfying the strong and dominant energy conditions, by Wald [201], showed the exponential decay at late times for the scalar curvature, the shear and the energy density. By the same time, formal series expansions for late time behaviour of vacuum spacetimes with a positive cosmological constant without symmetries or having a perfect fluids with $\gamma \in [1, 2)$ were written down by Starobinskii in [183], see [162] for more details. The first non-linear stability result for the EFEs without symmetry assumptions, was the non-linear stability of de Sitter spacetime, within the class of vacuum solutions with a positive cosmological constant, obtained in the celebrated work of Friedrich [83]. This result is based on the conformal method, developed by Friedrich, which avoids the difficulties of establishing global existence of solutions to a system of non-linear hyperbolic differential equations, but seems to be difficult to generalize to Einstein-matter systems. In [160], Rendall showed the uniqueness of the Starobinskii series, and, in the vacuum case using the results of Friedrich, that in an open set of data close enough to de Sitter data, all solutions have asymptotic expansions of that form.

A more "sophisticated" way of producing accelerated expansion is by using nonlinear scalar field models [162]. In general, we shall consider N minimally coupled scalar fields $\phi_A \in C^\infty(\mathcal{M})$ with smooth self-interacting potentials $\mathcal{V}_A = \mathcal{V}(\phi_A)$ and a smooth interaction potential between the scalar fields $\mathcal{W} = \mathcal{W}(\phi_1, ..., \phi_N)$. The total energy-momentum tensor is assumed to be the sum of each individual energy-momentum tensor plus the interaction potential between the scalar fields

$$T_{\mu\nu} = \sum_{A=1}^{N} T^A_{\mu\nu} - \mathcal{W} g_{\mu\nu}, \tag{1.10}$$

where, from the classical field theory, each individual energy-momentum tensor is given by

$$T^A_{\mu\nu} = \nabla_\mu \phi_A \nabla_\nu \phi_A - \left(\frac{1}{2} \nabla_\lambda \phi_A \nabla^\lambda \phi_A + \mathcal{V}_A \right) g_{\mu\nu}. \tag{1.11}$$

The divergenceless of each energy momentum tensor gives a system of N evolution equations for the scalar fields (nonlinear wave-equations)

$$\Box_{\mathbf{g}} \phi_A = \frac{d\mathcal{V}_A}{d\phi_A} + \frac{\partial \mathcal{W}}{\partial \phi_A} \quad A = 1, .., N, \tag{1.12}$$

where $\Box$ is the usual box operator. Moreover, there exists a big family of cosmological inflationary solutions having nonlinear scalar fields, and it is thus important to understand different models of early inflation (see e.g. [130] and references therein), or quintessence (dynamical dark energy), responsible for the present acceleration of the universe [48]. The simplest of these models is when the potential has a strictly positive lower bound, and it is the straightforward generalisation of the cosmological constant, resembling it at late times. The more general classes of Bianchi types I-VIII with other matter models satisfying the dominant and strong energy conditions, (and therefore which cannot produce accelerated expansion by their own) were studied in detail by Rendall [159], see also [21], where a type of Wald's theorem was given. In [159], Rendall also presented the asymptotic dynamics when the matter is an untilted perfect fluid with linear equation of state or Vlasov matter. Roughly, Rendall has shown, under mild conditions, that as $t \to +\infty$, the scalar field converges to a critical point of the potential $\mathcal{V}(\phi_\infty) = \mathcal{V}_\infty > 0$, $\mathcal{V}'(\phi_\infty) = 0$, and the Hubble function H converges exponentially to $\sqrt{\mathcal{V}_\infty/3}$, where $\mathcal{V}_\infty$ is interpreted as an effective cosmological constant [159]. In the subsequent works, Rendall considered potentials with zero lower bound when ϕ is either infinite [161] or finite [163]. For the former class of solutions, it was shown that if $\frac{d\mathcal{V}}{d\phi}/\mathcal{V}$ satisfies an upper bound which rules out too rapid exponential decay, accelerated expansion is expected to exist indefinetely and has a dynamical behavious between power-law [94, 32] and exponential type, commonly termed as intermediate inflation [14]. This restriction could be seen already from the power-law inflationary solution due to an exponential potential found by Halliwell using a phase plane analysis [94]. In that paper, it was shown that, for $\lambda < \sqrt{2}$, with λ the slope parameter, the solution exhibits accelerated expansion and is the stable future attractor for all classes of Friedman-Lemaitre models. The models of Bianchi type I, III and Kantowski-Sachs were studied by Burd and Barrow [32]. Bianchi type I models were also studied by Lidsey [127] and Aguirregabiria et al. [1], and Bianchi models of types III and VI were studied by Feinstein and Ibanez [77, 78]. The case when matter satisfying the standard energy conditions is included was studied by Kitada and Maeda [114, 113]. The formal series description of Starobinskii [183] was also applied to power-law inflation in [140]. Later, the first nonlinear stability result for scalar field cosmologies (without symmetry assumptions) was given by Heinzle and Rendall for power-law inflation [99], using Kaluza-Klein reductions and the methods of Anderson [11] —the latter, in turn are inspired by Friedrich's analysis of the stability of the de Sitter spacetime [83].

In [163], Rendall considered potentials as the harmonic type. Isotropic models were first studied by Belinskii et al. [17, 15, 16] (see also [139, 123, 100, 158]). These potentials are very usefull for studying the early inflationary stage of the universe, in models of chaotic inflation (see again [130]).

A new, more flexible approach for stability was recently developed by Ringström [170, 171] who showed that small perturbations, of the initial data of scalar field cosmological solutions to the EFEs with accelerated expansion, have maximal globally hyperbolic developments that are future causally geodesically complete. In particular, in [170], exponential decay of nonlinear perturbations was shown for potentials $\mathcal{V}(\phi)$, satisfying $\mathcal{V}(0) > 0$, $\mathcal{V}'(0) = 0$ and $\mathcal{V}''(0) > 0$. Later, these methods were applied for the power-law solution by Ringström himself in [171]. In the meantime, based on Ringström's breakthrough, Rodnianski and Speck [174], and later Speck [181], proved the non-linear stability of Friedman-Lemaitre solutions with flat toroidal space within the Einstein-Euler system satisfying the equation of state $1 < \gamma < 4/3$. The exponential decay of solutions close to the Friedman-Lemaitre was also established therein. In the same context, by generalizing Friedrich's conformal method to pure radiation matter models, Lübbe and Kroon [131] were able to extend Rodnianski and Speck's non-linear stability result to the pure radiation fluids case, $\gamma = 4/3$, see also [182].

In this thesis we investigate the future asymptotic stability of cosmological solutions to the Einstein-Field-Equations with a minimally coupled nonlinear scalar field.

In Chapter 3 we consider perturbations of (flat) spatially homogeneous and isotropic spacetimes by making an analysis of covariant and gauge-invariant linear scalar perturbations. Using the harmonic decomposition, we study the system qualitatively by applying techniques from the theory of dynamical systems. The cases of a single scalar field with exponential, quadratic and quartic potentials are studied in detail. In particular we prove a cosmic no-hair result for power-law inflation and show that homogenization occurs in models of chaotic and new inflation. We then extend the analysis to two scalar fields, and the particular situation when each scalar field has independent exponential potentials is analysed, resulting in a linear cosmic no-hair result for assisted power-law inflation.

In Chapter 4 a frame representation is used to derive a first-order quasi-linear symmetric hyperbolic system for the Einstein-nonlinear scalar field system. This procedure is inspired by similar evolution equations introduced by Friedrich to study the Einstein-Euler system, and it is the first step for studying the exponential decay of nonlinear perturbations of Friedman-Lemaitre backgrounds, for which we derive preliminary results.

In Chapter 5 we generalize Christodoulou's framework, developed to study the Einstein-scalar field equations with vanishing cosmological constant Λ, by introducing $\Lambda > 0$. In Chapter 6, and as a first step towards the Einstein-scalar field equations with positive cosmological constant, we solve the wave equation in de Sitter spacetime, obtaining an integro-differential evolution equation which we solve by taking initial data on a null cone. As a corollary we obtain elementary derivations of expected properties of linear waves in de Sitter spacetime: boundedness in terms of (characteristic) initial data, and a Price law establishing uniform exponential decay, in Bondi time, to a constant.

In the final chapter of this thesis we study the Einstein-scalar field system with positive cosmological constant and spherically symmetric characteristic initial data given on a truncated null cone. We prove well-posedness, global existence and exponential decay in (Bondi) time, for small data. From this, it follows that initial data close enough to de Sitter data evolves to a causally geodesically complete spacetime (with boundary), which approaches a region of de Sitter asymptotically at an exponential rate; this is a non-linear stability result for de Sitter within the class under consideration, as well as a realization of the cosmic no-hair conjecture.

The results present in this thesis are included in the following published articles and preprints: Chapter 3: [2, 4, 5] ; Chapter 4 [6, 7] Chapter 6 [3, 49, 51] ; Chapter 7 [50].

Chapter 2

Cosmology

In this chapter, we review important concepts about Cosmology and properties of the underlying spacetimes. We start by introducing de Sitter space and the class of spatially homogeneous and isotropic spacetimes given by the Robertson-Walker (RW) metric. These two solutions play a major role in the next two parts of this thesis. Given the RW metric and N real nonlinear scalar fields, we present the set of evolution and constraint equations obtained through the EFEs, which we shall call the *Friedman-Lemaitre-nonlinear scalar fields models.* We then briefly discuss its simplest solutions and commonly used potentials in the physics literature. Afterwards, we introduce the so-called *Hubble-normalized-variables* (HNV), which allow us to obtain a reduced dynamical system and study its properties on a compact state space. The existence of future/past attractors in such models, their stability and inflationary character is reviewed while providing a more general framework which shall make the analysis of Chapter 3 more straightforward by not fixing a priori any particular potential.

2.1 The de Sitter and Robertson-Walker metrics

The classical theory of relativistic Cosmology dates back to the works of Albert Einstein and Willem de Sitter, just after the formulation of the Theory of General Relativity in 1916. In his 1917 paper [66] "*Cosmological considerations on the General Theory of Relativity*", Einstein formulated the first model of the universe as whole in his static picture. In order to obtain a reasonable solution of the field equations which was compatible with Mach's principle of inertia and the principle of relativity, Einstein introduced the *cosmological constant* Λ in his field equations. Thus he was able to obtain a finite, yet unbounded static universe model. In the same year de Sitter published "*On Einstein's Theory of Gravitation, and its Astronomical Consequences*" with his maximally symmetric vaccum-Λ solution of positive constant scalar curvature [57]. In the following years, these two models were the very core of cosmology [56, 55, 57, 59, 58, 60, 68, 67, 64, 203, 204], and subject to several discussions between Einstein and de Sitter. It was not until Felix Klein [116, 115] shown that the de Sitter solution was fully regular, that Einstein acknowledge it as a solution of his field equations. For more on these issues see e.g. [107, 143, 169, 18, 108, 74]. The de Sitter spacetime can be seen as a 4-dimensional hyperboloid

$$-(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 = \mathcal{H}^{-2}$$

embedded in a (4+1)-dimensional Minkowski spacetime, with $\mathcal{H}^{-1} = \sqrt{\frac{3}{\Lambda}}$ the radius of the hyperboloid. One can introduce global coordinates on the hyperboloid $(\tilde{t}, \chi, \theta, \varphi) \in \mathbb{R} \times \mathbb{S}^3$

$$\begin{aligned}
x^0 =& \mathcal{H}^{-1} \sinh(\mathcal{H}\tilde{t}) \\
x^1 =& \mathcal{H}^{-1} \cosh(\mathcal{H}\tilde{t}) \cos(\chi) \\
x^2 =& \mathcal{H}^{-1} \cosh(\mathcal{H}\tilde{t}) \sin(\chi) \cos(\theta) \\
x^3 =& \mathcal{H}^{-1} \cosh(\mathcal{H}\tilde{t}) \sin(\chi) \cos(\theta) \\
x^4 =& \mathcal{H}^{-1} \cosh(\mathcal{H}\tilde{t}) \sin(\chi) \sin(\theta) \sin(\varphi)
\end{aligned}$$

for which the metric takes the form

$$\mathrm{d}s^2 = -\mathrm{d}\tilde{t}^2 + \mathcal{H}^{-2} \cosh(\mathcal{H}t) \left\{ d\chi^2 + \sin^2(\chi) \left(\mathrm{d}\theta^2 + \sin(\theta) \mathrm{d}\varphi^2 \right) \right\}. \tag{2.1}$$

We can also introduce coordinates $(\hat{t}, r, \theta, \varphi)$

$$\begin{aligned}
x^0 =& \sqrt{\mathcal{H}^{-2} - r^2} \sinh(\mathcal{H}\hat{t}) \\
x^1 =& \sqrt{\mathcal{H}^{-2} - r^2} \cosh(\mathcal{H}\hat{t}) \\
x^2 =& r \cos(\theta) \\
x^3 =& r \sin(\theta) \sin(\varphi) \\
x^4 =& r \sin(\theta) \cos(\varphi)
\end{aligned}$$

in which the metric is time independent

$$\mathbf{g}_{\mathrm{dS}} = -\left(1 - \mathcal{H}^2 r^2\right) \mathrm{d}\hat{t}^2 + \left(1 - \mathcal{H}^2 r^2\right)^{-1} \mathrm{d}r^2 + r^2 \left(\mathrm{d}\theta^2 + \sin(\theta) \mathrm{d}\varphi^2 \right). \tag{2.2}$$

In 1922 and 1924 Aleksandr Friedman [81, 82] showed that the existing two models were the only possibilities for static universes and discovered solutions of the field equations with expansion, for which the matter is described by dust. Later, Lemaitre [125] extended Friedman's work to the case of a perfect fluid. The metric of such models can be infered by what is known as the *Copernican Principle* which basically states that we do not occupy a privileged position in the universe, and that local physical laws are the same at every points in spacetime. This Principle leads to the class of spatially homogeneous and isotropic spacetimes whose metric, in Cartesian coordinates, can be written as

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \left(\frac{a(t)}{\omega} \right)^2 \delta_{ij} \mathrm{d}x^i \mathrm{d}x^j, \tag{2.3}$$

with $a(t)$ being the scale factor, and

$$\omega = 1 + \frac{K}{4} \delta_{ij} x^i x^j, \qquad \partial_i \omega = K x_i.$$

where t shall be refered as *cosmic time* and the constant $K < 0, K = 0, K > 0$ is the curvature of the spatial hypersurfaces. Transforming to spherical polar coordinates (R, θ, φ) and, redefining the radial coordiante R by

$$\frac{R}{1 + \frac{K}{4} R^2} = r \tag{2.4}$$

gives the well-known Robertson-Walker metric [172, 202]

$$\mathbf{g}_{\mathrm{RW}} = -\mathrm{d}t^2 + a^2(t) \left[\frac{\mathrm{d}r^2}{1 - kr^2} + r^2 \mathrm{d}\Omega^2 \right], \tag{2.5}$$

with k having values $+1, 0, -1$ and, $K = k/a^2$.

2.2 Friedman-Lemaitre-nonlinear scalar fields models

Given the energy momentum tensor (1.10) for N interacting scalar fields, the symmetry of the RW metric reduces the EFEs to a set of nonlinear ordinary differential equations (ODEs), the Friedman-Lemaitre-nonlinear scalar fields models. On such spacetimes the scalar fields ϕ_A are functions of cosmic time t only, and defining the momentum density variable $\psi_A := \dot{\phi}_A$, we can write the EFEs as a first-order system of evolutions equations

$$\begin{aligned}
\dot{\phi}_A &= \psi_A \\
\dot{\psi}_A &= -3H\psi_A - \frac{d\mathcal{V}_A}{d\phi_A} - \frac{\partial \mathcal{W}}{\partial \phi_A} \\
\dot{H} &= -H^2 - \frac{1}{3}\sum_{A=1}^{N}\psi_A^2 + \frac{1}{3}\sum_{A=1}^{N}\mathcal{V}_A + \frac{1}{3}\mathcal{W},
\end{aligned} \tag{2.6}$$

subject to the Friedman constraint

$$6H^2 = \sum_{A=1}^{N}\psi_A^2 + 2\sum_{A=1}^{N}\mathcal{V}_A + 2\mathcal{W} - {}^3R, \tag{2.7}$$

where the dot denotes differentiation with respect to cosmic time t, H is the Hubble function and ${}^3R$ the Ricci scalar of the spatial metric given, respectively, by

$$H = \frac{\dot{a}}{a} \quad , \quad {}^3R = \frac{6k}{a^2}. \tag{2.8}$$

An important quantity is the deceleration parameter given by

$$q := -\frac{\ddot{a}a}{\dot{a}^2} = -\left[1 + \frac{\dot{H}}{H^2}\right] . \tag{2.9}$$

A solution has acccelererated expansion, $\ddot{a} > 0$, if and only if, $q < 0$. The simplest solution to the system (2.6) is the *flat massless scalar field*, obtained by setting $\mathcal{V}(\phi) = 0$. In this case

$$a(t) = t^{\frac{1}{3}} \quad , \quad H(t) = \frac{1}{3t} , \tag{2.10}$$

$$\phi = \sqrt{\frac{2}{3}}\ln\left(\frac{C}{t}\right) \quad , \quad \psi = -\sqrt{\frac{2}{3}}\frac{1}{t} , \tag{2.11}$$

with $C > 0$ some real constant. One of the most well understood cosmological models with scalar fields, is the one containing an exponential potential

$$\mathcal{V}(\phi) = \Lambda e^{\lambda\phi}; , \tag{2.12}$$

with λ and Λ positive constants. Luchin and Matarrese [132] were the first to realize that solutions with the scale factor having a power-law dependence on time and exhibiting accelerated expansion could be obtained by exponentials potentials. In fact, the unique solution of the FL-scalar field equations (2.6) with $k = 0$ and the above exponential potential is given by

$$a(t) = t^p \quad , \quad H(t) = \frac{p}{t} ,$$

and

$$\phi(t) = \frac{\sqrt{2p}}{2}\ln\left(\frac{p(3p-1)}{\Lambda t^2}\right) \quad , \quad \psi(t) = -\frac{\sqrt{2p}}{t} ,$$

where $p > 1/3$ and $\lambda^2 p = 2$. This solution has accelerated expansion if and only if

$$0 < \lambda < \sqrt{2} \Leftrightarrow p > 1 \; . \tag{2.13}$$

In [94], Halliwell showed, using a phase plane analysis, that the *flat power-law inflation* (i.e. for $\lambda < \sqrt{2}$) is the stable future attractor of all FL-scalar field models. When more scalar fields endowed with independent exponential potentials are present

$$\mathcal{V}_A = \Lambda e^{\lambda_A \phi_A} \tag{2.14}$$

a scaling inflationary solution was found in [126], and termed as *assisted inflation*, see also [136, 47]. This solution has

$$p = 2 \sum_{A=1}^{N} \frac{1}{\lambda_A^2} \; , \quad \lambda_A \phi_A = \lambda_B \phi_B \; ; 1 \leq A \neq B \leq N \; . \tag{2.15}$$

Scalar fields with harmonic type potentials are very usefull in explaining the early inflationary stage of the universe, see eg. [144, 117, 118, 130]. Such potentials have the general form

$$\mathcal{V}(\phi) = C \frac{\left(\phi^2 - v^2\right)^n}{2n} \; , \tag{2.16}$$

where $C > 0$ and $v \geq 0$ are constants. Usually, models with $v = 0$ are called *chaotic inflation* and with $v > 0$ *new inflation*, see e.g.[130, 112].

2.3 Hubble-normalized state space

A procedure which has proved to be very usefull when using techniques from dynamical systems‘ theory applied to Cosmology is the reduction of the original system of equations, by using the *Hubble-normalized-variables* [199, 198, 200]. For scalar field cosmologies such variables were defined by Coley et al. [46] in the context of a single scalar field with an exponential potential. There, the spatially homogeneous Bianchi type models I-VIII were studied in detail, and previous results in the literature [127, 1, 77] were treated in an unified way. In particular, they were able to test whether a given model inflates and/or isotropizes at late times and thus test the validity of the cosmic no-hair conjecture. More precisely, it was shown that the flat isotropic power-law inflationary solution is an attractor for all initially expanding models except for the subclass of Bianchi type IV which recollapses [195]. The flat FL model with exponential potential coupled to matter was studied in [22], and the case of several independent exponential potentials in [47]. For more details see Coley [45] and references therein.

Recently, Hubble-normalized variables as in [46], have also been used in the study of harmonic type potentials by Ureña-López and Reyes-Ibarra in [194, 167], see also [111, 112]. Contrary to the exponential potential situation, in this case the ODE system does not decouple from the Raychauduri equation, and the introduction of a new expansion-normalized variable is necessaryThe phase space analysis in [167] showed the existence of attractor trajectories in the phase-space which coincide with the slow-roll approximation, for some values of the new variable, and are thus inflationary. Moreover, it was also shown that for small values, the new variable can be treated as a potential parameter governing the stability of the so-called *quasi-attractors* of the reduced dynamical system.

We will use the *expansion normalized variables* for scalar field cosmologies defined by

$$\Psi_A := \frac{\psi_A}{\sqrt{6}H} \quad , \quad \Phi_A := \left(\frac{\mathcal{V}_A}{3H^2}\right)^{\frac{1}{2n}} \quad , \quad \Theta := \left(\frac{\mathcal{W}}{3H^2}\right)^{\frac{1}{2n}} \quad , \quad K := -\frac{{}^3R}{6H^2} \; , \tag{2.17}$$

where $n \in \mathbb{N}$ take values for specific potentials. For instance, in the case of an exponential potential, $n = 1$, and the variables coincide with those of Coley et al. [46]. For the harmonic type potentials see [194, 167, 111, 112]. We will also make use of the logarithmic time variable τ

$$\frac{d\tau}{dt} = H \quad , \quad H' = -(1+q)H, \tag{2.18}$$

so that $\tau \to -\infty$ as $t \to 0^+$, and denote differentiation with respect to τ by a prime. Using these variables, the system of ODEs governing the background dynamics becomes

$$\begin{aligned} \Psi'_A &= (q-2)\Psi_A - n\sqrt{6}\left[\Phi_A^{2n-1}\frac{d\Phi_A}{d\phi_A} + \Theta_A^{2n-1}\frac{d\Theta}{d\phi_A}\right] \\ \Phi'_A &= \frac{1}{n}(q+1)\Phi_A + \sqrt{6}\frac{d\Phi_A}{d\phi_A}\Psi_A \\ \Theta' &= \frac{1}{n}(q+1)\Theta + \sqrt{6}\sum_{A=1}^{N}\frac{\partial\Theta}{\partial\phi_A}\Psi_A \end{aligned} \tag{2.19}$$

subject to the Friedman constraint

$$K = 1 - \sum_{A=1}^{N}\Psi_A^2 - \sum_{A=1}^{N}\Phi_A^{2n} - \Theta^{2n} \tag{2.20}$$

and with

$$q = 2\sum_{A=1}^{N}\Psi_A^2 - \sum_{A=1}^{N}\Phi_A^{2n} - \Theta^{2n}. \tag{2.21}$$

These equations will allow us to treat in a unified way various families of scalar field potentials, without fixing, a priori, a specific potential. In general, we shall be interested in scalar fields which do not interact with each other $\Theta = 0$, in flat FL models. In that case, the dynamical system state space reduces to the set

$$\left\{(\Psi_1, .., \Psi_N, \Phi_1, .., \Phi_N) \in [-1,0]^N \times [0,1]^N \;:\; \sum_{A=1}^{N}\Psi_A^2 + \sum_{A=1}^{N}\Phi_A^{2n} = 1\right\} \tag{2.22}$$

with

$$q = 2 - 3\sum_{A=1}^{N}\Phi_A^{2n} = 3\sum_{A=1}^{N}\Psi_A^2 - 1,$$

and it is straightforward to get:

Lemma 1. *For N non-interacting scalar fields in flat background $\Theta = K = 0$ the system (2.19) fixed points $\mathcal{P}$ and $\mathcal{Q}$ are given by*

$$\mathcal{P}: \quad \frac{\Phi_A^{2n}}{\sum_{B=1}^{N}\Phi_B^{2n}}\frac{d\Phi_A}{d\phi_A} = -\frac{\sqrt{6}}{2n}\Psi_A\Phi_A \quad \textit{and} \quad \frac{\Psi_A^2}{\sum_{B=1}^{N}\Psi_B^2}\frac{d\Phi_A}{d\phi_A} = -\frac{\sqrt{6}}{2n}\Psi_A\Phi_A$$

and

$$\mathcal{Q}: \quad \Phi_A = \Psi_A = 0$$

with $A = 1, .., N$.

The fixed points $\mathcal{P}$ correspond to solutions depending on the potential and are the physical solutions satisfying the flat Friedman constraint, while the point $\mathcal{Q}$ is unphysical. For simplicity, we restrict now to the single scalar field case. Then, the constraint reads

$$\Psi^2 + \Phi^{2n} = 1 \tag{2.23}$$

and the linearised matrix of the system (2.19) at $\mathcal{P}$ is

$$\begin{pmatrix} 9\Psi_{\mathcal{P}}^2 - 3 & \frac{\Phi_{\mathcal{P}}^{2n}}{\Psi_{\mathcal{P}}\Phi_{\mathcal{P}}}\left(3(2n-1)\Psi_{\mathcal{P}}^2 + \frac{2n^2}{\Phi_{\mathcal{P}}}\left(\frac{d^2\Phi}{d\phi^2}\right)_{\mathcal{P}}\right) \\ \frac{3}{n}\Psi_{\mathcal{P}}\Phi_{\mathcal{P}} & \frac{3}{n}\Psi_{\mathcal{P}}^2 - \frac{2n}{\Phi_{\mathcal{P}}}\left(\frac{d^2\Phi}{d\phi^2}\right)_{\mathcal{P}} \end{pmatrix} \tag{2.24}$$

with characterictic polynomial

$$\omega^2 - \left\{\left(9 + \frac{3}{n}\right)\Psi_{\mathcal{P}}^2 - 3 - 2n\frac{1}{\Phi_{\mathcal{P}}}\left(\frac{d^2\Phi}{d\phi^2}\right)_{\mathcal{P}}\right\}\omega + \frac{18}{n}\Psi_{\mathcal{P}}^2\left\{1 - (n+1)\Phi_{\mathcal{P}}^{2n} - \frac{2n^2}{3\Phi_{\mathcal{P}}}\left(\frac{d^2\Phi}{d\phi^2}\right)_{\mathcal{P}}\right\}$$

so that the eigenvalues of the matrix in (2.24) are

$$\begin{aligned}\omega_{\mathcal{P}}^{\pm} =& \frac{3}{2}\left[\left(3 + \frac{1}{n}\right)\Psi_{\mathcal{P}}^2 - 1 - \frac{2n}{3\Phi}\left(\frac{d^2\Phi}{d\phi^2}\right)_{\mathcal{P}}\right] \pm \\ &\pm \frac{3}{2}\sqrt{\left[1 + \frac{2n}{3\Phi}\left(\frac{d^2\Phi}{d\phi^2}\right)_{\mathcal{P}}\right]^2 + 2\Psi^2\left[\left(1 - \frac{1}{n}\right) - \frac{1}{3}\left(\frac{2}{n} - \frac{1}{n^2} - 1\right)\Psi_{\mathcal{P}}^2 + \frac{2}{3}(n-1)\frac{2n}{3\Phi}\left(\frac{d^2\Phi}{d\phi^2}\right)_{\mathcal{P}}\right]}\,.\end{aligned}$$

If we denote the respective eigenvectors by $(\delta\Psi\ \delta\Phi)_{\pm}^T$, the general solution to the perturbations around $\mathcal{P}$ reads

$$\begin{pmatrix}\delta\Psi \\ \delta\Phi\end{pmatrix} = C_- \begin{pmatrix}\delta\Psi \\ \delta\Phi\end{pmatrix}_- e^{\omega_- \tau} + C_+ \begin{pmatrix}\delta\Psi \\ \delta\Phi\end{pmatrix}_+ e^{\omega_+ \tau}.$$

However, (2.23) implies to linear order that

$$\Psi_{\mathcal{P}}\delta\Psi + n\Phi_{\mathcal{P}}^{2n-1}\delta\Phi = 0 \Leftrightarrow \begin{pmatrix}\Psi_{\mathcal{P}} & n\Phi_{\mathcal{P}}^{2n-1}\end{pmatrix}\begin{pmatrix}\delta\Psi \\ \delta\Phi\end{pmatrix} = 0$$

and the evolution of linear perturbations around the fixed points $\mathcal{P}$ reduces to a single equation

$$\delta\Phi' = \left(-3\Phi^{2n} + \frac{3}{n}\Psi^2 - \frac{2n}{\Phi_{\mathcal{P}}}\left(\frac{d^2\Phi}{d\phi^2}\right)_{\mathcal{P}}\right)\delta\Phi. \tag{2.25}$$

Then, there is a single eigenvalue solution

$$\omega_{\mathcal{P}}^- = -3\left(1 + \frac{1}{n}\right)\Phi_{\mathcal{P}}^{2n} + \frac{3}{n} - \frac{2n}{\Phi_{\mathcal{P}}}\left(\frac{d^2\Phi}{d\phi^2}\right)_{\mathcal{P}} \tag{2.26}$$

which will be proportional to the eigenvector $(\delta\Psi\ \delta\Phi)_-^T$. Moreover, at $\mathcal{P}$, the solutions are inflationary if and only if

$$\Phi_{\mathcal{P}}^{2n} > \frac{2}{3} \Leftrightarrow \Psi_{\mathcal{P}}^2 < \frac{1}{3}\,. \tag{2.27}$$

We shall now recall some results for exponential and polynomial potentials, using the above result. This will be usefull in the stability analysis of Chapter 3.

2.3.1 Exponential Potentials

Using the above framework, we now review the flat (assisted) power-law solutions due to exponential potentials (2.14) discussed in Section 2.2. The analysis is done separately for the single scalar field and the two scalar fields cases. For such potentials, the Hubble-normalized-variables are defined with $n = 1$, and the zero curvature invariant set (2.22) is a higher dimensional sphere $\mathbb{S}^N$, and

$$\frac{d^p\Phi_A}{d\phi_A^p} = \left(\frac{\lambda_A}{2}\right)^p \Phi_A\,. \tag{2.28}$$

Power-law Inflation:

If only one scalar field is present, then Lemma 1 implies

$$\mathcal{P}: \quad \Phi\left(\Psi + \frac{\lambda}{\sqrt{6}}\right) = 0 \tag{2.29}$$

satisfying (2.23). Therefore, there are two fixed points $(\Psi_{\mathcal{P}}, \Phi_{\mathcal{P}}) \in [-1,0] \times [0,1]$ in $\mathbb{S}^1$

$$\mathcal{P}_0: \quad (\Psi, \Phi) = (-1, 0) \tag{2.30}$$

$$\mathcal{P}_1: \quad (\Psi, \Phi) = \left(-\frac{\lambda}{\sqrt{6}}, \frac{\sqrt{6-\lambda^2}}{\sqrt{6}}\right) \quad \text{with} \quad 0 < \lambda < \sqrt{6}. \tag{2.31}$$

Also, at $\mathcal{P}$, we have

$$\left(\frac{d^2\Phi}{d\phi^2}\right)_{\mathcal{P}} = -\frac{\sqrt{6}\lambda}{4}\Psi\Phi,$$

so that, from (2.26), the eigenvalues are

$$\omega^-_{\mathcal{P}_0} = 3 + \frac{\sqrt{6}}{2}\lambda \quad \text{and} \quad \omega^-_{\mathcal{P}_1} = -\frac{6-\lambda^2}{2}. \tag{2.32}$$

The point $\mathcal{P}_0$ corresponds to the well-known massless scalar field solution, which is the early time attractor and is a *source* for all values of $\lambda \in (0, \sqrt{6})$. The point $\mathcal{P}_1$ is a *sink*, with the deceleration parameter given by

$$q_{\mathcal{P}_1} = \frac{\lambda^2 - 2}{2} \; .$$

The solution being inflationary if and only if $q_{P_2} < 0$, i.e. for

$$0 < \lambda < \sqrt{2} \; ,$$

which corresponds to the flat homogeneous and isotropic power-law inflationary solution found by Halliwell in [94]. This solution is known to be an attractor for initially expanding Bianchi models (except those recollapsing). See e.g. [45] for details and related references.

Assisted Power-law Inflation:

For two scalar fields with independent exponential potentials, Lemma 1 implies

$$\mathcal{P}: \quad \Phi_A\Psi_A\left(\frac{\Psi_A}{\Psi^2}\frac{\lambda_A}{2} + \frac{\sqrt{6}}{2}\right) = 0 \quad \text{and} \quad \Phi_A\left(\frac{\Phi_A^2}{\Phi^2}\frac{\lambda_A}{2} + \frac{\sqrt{6}}{2}\Psi_A\right) = 0, \quad A = 1, 2$$

which leads to the four fixed points

$$\mathcal{P}_0: \quad (\Psi_1, \Psi_2, \Phi_1, \Phi_2) = \left(-\Psi_0, -\sqrt{1-\Psi_0^2}, 0, 0\right) \; , \quad 0 \leq \Psi_0 \leq 1 \tag{2.33}$$

$$\mathcal{P}_1: \quad (\Psi_1, \Psi_2, \Phi_1, \Phi_2) = \left(-\frac{\lambda_1}{\sqrt{6}}, 0, \frac{\sqrt{6-\lambda_1^2}}{\sqrt{6}}, 0\right) \; , \quad 0 < \lambda_1 < \sqrt{6} \tag{2.34}$$

$$\mathcal{P}_2: \quad (\Psi_1, \Psi_2, \Phi_1, \Phi_2) = \left(0, -\frac{\lambda_2}{\sqrt{6}}, 0, \frac{\sqrt{6-\lambda_2^2}}{\sqrt{6}}\right) \; , \quad 0 < \lambda_2 < \sqrt{6} \tag{2.35}$$

$$\mathcal{P}_3: \quad (\Psi_1, \Psi_2, \Phi_1, \Phi_2) = \left(-\frac{\lambda^2}{\sqrt{6}\lambda_1}, -\frac{\lambda^2}{\sqrt{6}\lambda_2}, \frac{\sqrt{\lambda^2(6-\lambda^2)}}{\sqrt{6}\lambda_1}, \frac{\sqrt{\lambda^2(6-\lambda^2)}}{\sqrt{6}\lambda_2}\right) \; , \quad \frac{1}{\lambda^2} = \frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} \tag{2.36}$$

and the linearised matrix of the system (2.19) at $\mathcal{P}$ is

$$\begin{pmatrix} 9\Psi_1^2+3\left(\Psi_2^2-1\right) & 6\Psi_1\Psi_2 & -\sqrt{6}\lambda_1\Phi_1 & 0 \\ 6\Psi_1\Psi_2 & 9\Psi_2^2+3\left(\Psi_1^2-1\right) & 0 & -\sqrt{6}\lambda_2\Phi_2 \\ \left(\frac{\sqrt{6}}{2}\lambda_1+6\Psi_1\right)\Phi_1 & 6\Psi_2\Phi_1 & \frac{\sqrt{6}}{2}\lambda_1\Psi_1+3\left(\Psi_1^2+\Psi_2^2\right) & 0 \\ 6\Psi_1\Phi_2 & \left(\frac{\sqrt{6}}{2}\lambda_2+6\Psi_2\right)\Phi_2 & 0 & \frac{\sqrt{6}}{2}\lambda_2\Psi_2+3\left(\Psi_1^2+\Psi_2^2\right) \end{pmatrix}_{\mathcal{P}}$$

with eigenvalues

$$\omega(\mathcal{P}_0)=6\ ,\ 0\ ,\ 3-\frac{\sqrt{6}}{2}\lambda_1\Psi_0\ ,\ 3-\frac{\sqrt{6}}{2}\lambda_2\sqrt{1-\Psi_0^2}\quad \text{where}\quad 0\leq\Psi_0\leq 1 \tag{2.37}$$

$$\omega(\mathcal{P}_{1,2})=\frac{\lambda_{1,2}^2}{2}\ ,\ \lambda_{1,2}^2\ ,\ -\frac{\left(6-\lambda_{1,2}^2\right)}{2}\ ,\ -\frac{\left(6-\lambda_{1,2}^2\right)}{2} \tag{2.38}$$

$$\omega(\mathcal{P}_3)=\lambda^2\ ,\ \frac{\lambda^2-6}{2}\ ,\ \frac{1}{4}\left\{(\lambda^2-6)\pm\sqrt{(\lambda^2-6)+8\lambda^2(\lambda^2-6)}\right\}\ . \tag{2.39}$$

Thus, in $\mathbb{S}^2$, the point $\mathcal{P}_0$ is a local source and corresponds to the massless scalar field solution which is the early time attractor. The points $\mathcal{P}_{1,2}$ are saddles which correspond to single power law solutions where either ϕ_1 dominates over ϕ_2 or the opposite. As before, the Friedman constraint can be used to eliminate the unphysical radial direction along which corresponds the positive eigenvalue of point $\mathcal{P}_3$. Therefore, $\mathcal{P}_3$ is the stable late-time attractor and corresponds to the assisted power law solution.

2.3.2 Harmonic Potentials and the quasi-attractor formalism

Recently, the expansion-normalized variables defined in [194, 111, 167, 112] were used to study the dynamical properties of scalar field cosmologies with potentials given by (2.16). Contrary to the exponential potentials, where the reduced dynamical system (2.19) is autonomous, for these kind of potentials one gets a non-autonomous system, in general. In order to turn the system autonomous, a new Hubble normalised variable must be introduced. An appropriate choice is

$$\mathcal{M}:=(6n)^{\frac{n-1}{2n}}\,n^{\frac{1}{2}}\left(\frac{C}{H^2}\right)^{\frac{1}{2n}}\ , \tag{2.40}$$

which, in ever expanding models, is a monotone and growing function obeying the evolution equation

$$\mathcal{M}'=\frac{3}{n}\Psi^2\mathcal{M}\ . \tag{2.41}$$

From (2.16) and (2.40) it follows that

$$\frac{d\Phi}{d\phi}=\frac{\mathcal{M}}{\sqrt{6}n^2}\frac{\sqrt{n^2\Phi^2+\mathcal{M}^2\mathcal{N}^2}}{\Phi}\quad\text{and}\quad\frac{d^2\Phi}{d\phi^2}=-\left(\frac{\mathcal{M}}{\sqrt{6}n}\right)^2\frac{\mathcal{M}^2\mathcal{N}^2}{n\Phi^3}\ , \tag{2.42}$$

where

$$\mathcal{N}:=\frac{v}{\sqrt{6}}\ . \tag{2.43}$$

To get a better picture of the state space of the new dynamical system, it is usefull to make a change of variable and turn the above system into a 2-dim system. This was done for $n=1,2$ and $v=0$ in [167] where the new variable Υ was defined as

$$\Psi=\cos\left(\Upsilon\right)\quad,\quad\Phi=|\sin\left(\Upsilon\right)|^{\frac{1}{n}}\ , \tag{2.44}$$

with $(\Upsilon, \mathcal{M}) \in [\frac{\pi}{2}, \pi] \times [0, +\infty)$. The dynamical system (2.19), coupled to (2.41) then reads

$$\begin{aligned} \Upsilon' &= \frac{|\sin(\Upsilon)|^{\frac{2(n-1)}{n}}}{\sin(\Upsilon)} \left[3\cos(\Upsilon)|\sin(\Upsilon)|^{\frac{2}{n}} + \frac{\mathcal{M}}{n}\sqrt{n^2|\sin(\Upsilon)|^{\frac{2}{n}} + \mathcal{M}^2\mathcal{N}^2} \right] \\ \mathcal{M}' &= \frac{3}{n}\mathcal{M}\cos^2(\Upsilon) \; . \end{aligned} \tag{2.45}$$

The fixed points of this system are located at $\Upsilon = \frac{\pi}{2}, \pi$ and $\mathcal{M} = 0$, and are independent of n. The point $(0, \pi)$ corresponds to the massless scalar field solutions $\Psi = -1$ and is unstable, while the point $(0, \frac{\pi}{2})$ is a saddle and corresponds to the potential dominated solutions $\Phi = 1$. There exists also heteroclinic curves connecting the unstable point with the saddle point along the stable direction i.e., at the $\mathcal{M} = 0$ axis. Along the unstable direction of the saddle point departs a kind of heteroclinic curve, which although is not connected to any point in the limit $\tau \to +\infty$, for small values of $\mathcal{M}$, acts as an attractor trajectory in the phase space, see [167] for details when $\mathcal{N} = 0$ (see also Figs. 2.1, 2.2, 2.3, 2.4). The approach of [194, 112] consists in

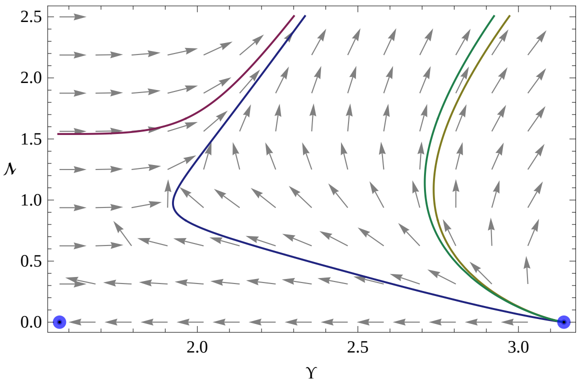

Figure 2.1: Phase space for quadratic potential $n = 1$ and $\mathcal{N} = 0$.

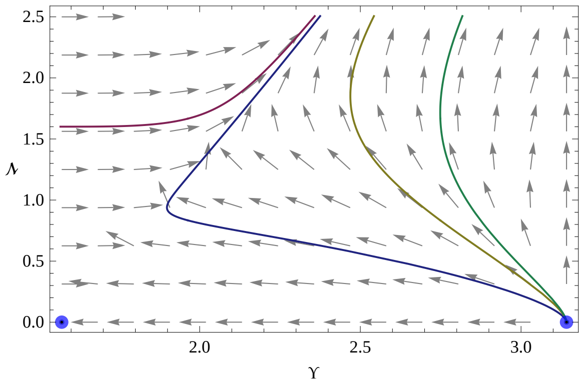

Figure 2.2: Phase space for quartic potential $n = 2$ and $\mathcal{N} = 0$.

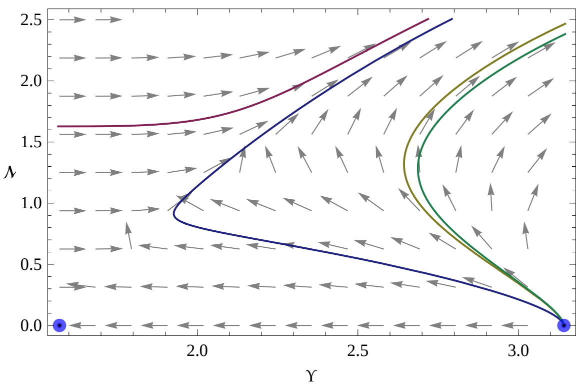

Figure 2.3: Phase space for new inflation $n = 2$ and $\mathcal{N} = 1$.

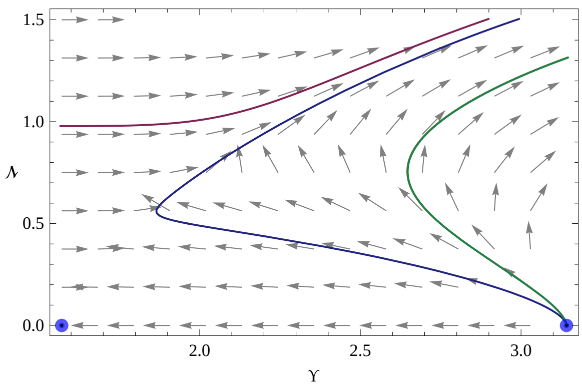

Figure 2.4: Phase space for new inflation $n = 2$ and $\mathcal{N} = 4$.

reducing the 3-dim system, obtained by coupling (2.19) with (2.41), into a 2-dim system with state vector (Ψ, Φ), by considering $\mathcal{M}$ as a control parameter.

Chaotic Inflation $\mathcal{N}=0$:

For potentials having $\mathcal{N}=0$, Lemma 1 and (2.42) give

$$\mathcal{P}: \quad \Psi\Phi = -\frac{\mathcal{M}}{3} \quad \text{with} \quad \frac{d^2\Phi}{d\phi^2} = 0 \tag{2.46}$$

subject to the Friedman constraint (2.23). Then (2.26) gives

$$\omega_- = -3\left(1+\frac{1}{n}\right)\Phi_{\mathcal{P}}^{2n} + \frac{3}{n} \tag{2.47}$$

and the fixed points $\mathcal{P}$ are stable if $\omega_- < 0$, i.e. if

$$\Phi_{\mathcal{P}}^{2n} > \frac{1}{1+n}\ , \tag{2.48}$$

which for all n contains the inflationary solutions (2.27).

- **Quadratic Potential:**

For a quadratic potential, $n=1$, and (2.40) reads

$$\mathcal{M} := \frac{m}{H}. \tag{2.49}$$

where $C = m$. In this case, the fixed points $(\Psi_{\mathcal{P}}, \Phi_{\mathcal{P}}) \in [-1,0]\times[0,1]$ in $\mathbb{S}^1$ are given by condition (2.46) subject to (2.23) as

$$\mathcal{P}_0: \quad (\Psi\ ,\ \Phi) = \left(-\sqrt{\frac{1}{2}\left(1+\sqrt{1-\frac{4}{9}\mathcal{M}^2}\right)}\ ,\ \sqrt{\frac{1}{2}\left(1-\sqrt{1-\frac{4}{9}\mathcal{M}^2}\right)}\right) \tag{2.50}$$

$$\mathcal{P}_1: \quad (\Psi\ ,\ \Phi) = \left(-\sqrt{\frac{1}{2}\left(1-\sqrt{1-\frac{4}{9}\mathcal{M}^2}\right)}\ ,\ \sqrt{\frac{1}{2}\left(1+\sqrt{1-\frac{4}{9}\mathcal{M}^2}\right)}\right) \tag{2.51}$$

with eigenvalues (2.47)

$$\omega^-_{\mathcal{P}_0} = 3\sqrt{1-\frac{4}{9}\mathcal{M}^2} \quad , \quad \omega^-_{\mathcal{P}_1} = -3\sqrt{1-\frac{4}{9}\mathcal{M}^2}\ . \tag{2.52}$$

Thus, in this case, the fixed points exist in the unitary circunference for $0 \le \mathcal{M} \le \frac{3}{2}$. For $\mathcal{M} < \frac{3}{2}$, $\mathcal{P}_0$ is the local source which, at $\mathcal{M}=0$, represents the massless scalar field early attractor, and $\mathcal{P}_1$ the future attractor. At $(\Psi,\Phi) = (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ the fixed points have zero eigenvalues. Moreover, as shown in [194], the future attractor $\mathcal{P}_1$ is inflationary if and only if

$$\mathcal{M} < \sqrt{2}\ . \tag{2.53}$$

For $\mathcal{M} = \sqrt{2}$, the attractor point ceases to be inflationary and the value of ϕ at this point (which represents the end of inflation) corresponds to that of the slow-roll formalism.

- **Quartic Potential:**

For a quartic potential, $n = 2$,

$$\mathcal{M} = \frac{\sqrt{2\sqrt{12}}\lambda}{\sqrt{H}}$$

with $C = \lambda^4$, and the fixed points, given by condition (2.46) subject to the flat Friedman constraint (2.23)

$$\Psi^2 + \Phi^4 = 1$$

are the solutions of the cubic polynomial

$$f(\Psi^2) = \Psi^6 - \Psi^4 + \left(\frac{\mathcal{M}}{3}\right)^4 = 0 \ .$$

The discrimant of this polynomial is

$$\Delta = \frac{1}{9^3}\left(\frac{\mathcal{M}^4}{3}\right)\left(\frac{\mathcal{M}^4}{12} - 1\right)$$

so that, for $0 < \mathcal{M}^4 < 12$, it follows that $\Delta < 0$ and there are three distinct real roots. For $\mathcal{M}^4 = 12$, then $\Delta = 0$ and there is a repeated real root, otherwise there are two complex roots. Now, setting

$$(\mathcal{M})^4 = 12\sin^2(\chi) \quad , \quad 0 < \chi < \frac{\pi}{2} \tag{2.54}$$

the three distinct roots are explicitly given by

$$\mathcal{P}: \Psi^2 = \frac{1}{3}\left(1 + 2\cos\left(\frac{2}{3}\chi + \frac{2}{3}\pi\, l\right)\right), \quad l = 0, \pm 1 \ . \tag{2.55}$$

The $l = 1$ solution is unphysical, since $\Psi^2 < 0$. If we denote by $\mathcal{P}_0$ the $l = 0$ solution, and $\mathcal{P}_1$ the $l = -1$ solution, then we have from (2.47) that

$$\omega^-(\mathcal{P}_0) = -\frac{3}{2} + 3\cos\left(\frac{2}{3}\chi\right) \quad , \quad \omega^-(\mathcal{P}_1) = -\frac{3}{2} + 3\cos\left(\frac{2}{3}(\chi - \pi)\right). \tag{2.56}$$

For all values of $\mathcal{M}$ for which there are fixed points, $\mathcal{P}_0$ is a source and at $\mathcal{M} = 0 \Leftrightarrow \chi = 0$ the solution represents that of a massless scalar field with $\Psi^2_{\mathcal{P}_0} = 1$. In turn, $\mathcal{P}_1$ is a sink and, at $\mathcal{M} = 0$, represents a potential dominated solution $\Phi^4_{\mathcal{P}_1} = 1$.

When $\mathcal{M}^4 = 12$, then the discrimant of the cubic equation is zero and its solutions coincide having $\Psi^2 = \frac{2}{3}$, which by (2.56) gives a saddle. Moreover, $\mathcal{P}_1$ are inflationary whenever (2.27) is satisfied, which gives

$$\chi < \frac{\pi}{4} \quad \Leftrightarrow \mathcal{M}^4 < 6 \ . \tag{2.57}$$

New Inflation $\mathcal{N} > 0$:

For potentials having $\mathcal{N} > 0$, Lemma 1 and (2.42) give the fixed points

$$\mathcal{P}: \ \Psi\Phi = -\frac{\mathcal{M}}{3n}\frac{\sqrt{n^2\Phi^2 + \mathcal{M}^2\mathcal{N}^2}}{\Phi} \quad \text{and} \quad \frac{d^2\Phi}{d\phi^2} = \frac{\mathcal{M}}{2n^3}\frac{\Psi\mathcal{M}^2\mathcal{N}^2}{\Phi\sqrt{n^2\Phi^2 + \mathcal{M}^2\mathcal{N}^2}} \tag{2.58}$$

subject to the Friedman constraint (2.23). Since $\mathcal{N} > 0$, we can write

$$\left(\frac{d^2\Phi}{d\phi^2}\right)_{\mathcal{P}} = -\frac{3}{2n^2}\Psi^2\Phi + \frac{\mathcal{M}^2}{6n^2\Phi} \tag{2.59}$$

and then (2.26) gives

$$\omega_- = -3\left(1+\frac{2}{n}\right)\Phi_{\mathcal{P}}^{2n} + \frac{6}{n} - \frac{\mathcal{M}^2}{3n\Phi^2} \tag{2.60}$$

so that the fixed points $\mathcal{P}$ are stable if $\omega_- < 0$ which, as in the case of chaotic inflation, contains the inflationary solutions

$$\omega_- < -2\left(1+\frac{1}{n}\right) - \frac{\mathcal{M}^2}{3n\Phi^2}\ . \tag{2.61}$$

The case $n = 2$, $C = \lambda^{\frac{1}{4}}$, was studied in [112]. The dynamical system is then given by

$$\begin{aligned} \Psi' &= 3\Psi^3 - 3\Psi - \Phi^2\mathcal{M}\sqrt{\Phi^2 + \mathcal{N}^2\mathcal{M}^2} \\ \Phi\Phi' &= \frac{1}{2}\left(3\Psi^2\Phi^2 + \Psi\mathcal{M}\sqrt{\Phi^2 + \mathcal{N}^2\mathcal{M}^2}\right) \end{aligned} \tag{2.62}$$

subject to the flat Friedman constraint

$$\Psi^2 + \Phi^4 = 1 \tag{2.63}$$

As in the quadratic case we can consider the state vector (Ψ, Φ) with control parameter $\mathcal{M}$. The fixed points are are solutions of

$$\begin{aligned} 3\Psi^2 - 3\Psi - \Phi^2\mathcal{M}\sqrt{\Phi^2 + \mathcal{N}^2\mathcal{M}^2} &= 0 \\ 3\Psi\Phi^2 + \mathcal{M}\sqrt{\sqrt{1-\Psi^2} + \mathcal{N}^2\mathcal{M}^2} &= 0 \end{aligned} \tag{2.64}$$

$$3\Psi\sqrt{1-\Psi^2} + \mathcal{M}\sqrt{\sqrt{1-\Psi^2} + \mathcal{N}^2\mathcal{M}^2} = 0 \tag{2.65}$$

It is not possible, in principle, to find explicitly the fixed points for this system. However we shall make a numerical stability analysis in Section 3.4.3.

Part II

Applications of Cosmological perturbation theory to FL-nonlinear scalar field models

Chapter 3

Covariant and Gauge-Invariant Linear Scalar Perturbations

Linear perturbation theory is of the most importance in physical theories and in particular in General Relativity, where exact solutions are, most often, idealizations of the natural phenomena. Using perturbation theory we try to find approximate solutions of the field equations, where perturbations are regarded as small deviations from known exact solutions. Among many applications of perturbation theory in General Relativity, Cosmology is one of great interest since the FL cosmological models describe with great accuracy the large-scale behaviour of the universe and local features are described by small linearised perturbations, in which case the models refered as *almost-FL*. Applications of these models include the anisotropy of the 3Â°K background radiation as well as the Sachs-Wolfe, Rees-Sciama and the Sunyaev-Zeldovich effects [175, 155, 179, 151, 186, 187, 188, 149, 189, 150, 205, 164, 35, 106, 119, 80], inhomogeneity in primordial element production [180, 104, 96, 90, 75, 13] and black hole formation in the early universe [34, 33, 148]. For reviews see [156, 157].

The behaviour of small perturbations is also intimately related to the question of stability of solutions to the EFEs and, in particular, it can give insights into the cosmic no-hair conjecture.

Perturbations of Newtonian universes were investigated by Bonnor, Savedoff and Vila [23, 176] and Irvine [105] in the Newtonian limit of General Relativity. Relativistic cosmological perturbation theory was born in the pioneer work of Lifshitz [128, 129] and, since then, it is well known that perturbation theory in General Relativity has the problem of *gauge-invariance*.

Roughly speaking, to define perturbations we need a *point-identification map* in order to relate points in the perturbed spacetime (representing the inhomogeneous universe), to points in the background spacetime. Then, at any point the perturbation, $\delta\psi$ is defined as the difference of the value of ψ in the perturbed manifold and the value of $\mathring{\psi}$ in the background. A choice of such map is called a *choice of gauge*, see e.g. [185, 184]. It is then useful to consider perturbation variables which are independent of the gauge-choice, i.e. which are *gauge-invariant*.

The *covariant and gauge-invariant formalism* was put forward by Ellis and Bruni [70], based on earlier works of Hawking [97], Olson [145] (see also [209]) and Stewart and Walker [185]. This more geometrical approach consists in starting from exact non-linear equations using the *1+3 covariant formalism* which, in view of the fundamental Lemma of Stewart and Walker [185] are then linearised about exact FL models.

The advantage of this approach relies on the perturbations variables having a clear geometric and physical interpretation [28]. Exact evolution equations for linear gauge-invariant perturbations of FL with a perfect-fluid as matter source were given in [72, 71] and the extension for an imperfect-fluid by Hwang and Vishniac [103]. The imperfect-fluid case was also applied to describe perturbations in a multi-component fluid by Dunsby et al. [61, 63] using the methods of King and Ellis [110] to charged multifluids [137], magnetized cosmologies [190, 191] and, more recently, to Kantowski-Sachs models with a positive cosmological constant [79, 25]. For a good survey see [192].

The $1+3$ covariant formalism applied to scalar fields has been discussed in a series of works by Madsen and Ellis [134, 133, 73] and recently by Vernizi and Langlois [197, 124]. This approach has also been applied to minimally coupled scalar-fields by Bruni et al. [29, 28] and Zimdhal [211]. In what follows, we shall derive the system of equations governing the evolution of linear scalar perturbations of Friedman-Lemaitre models with multiple interacting scalar fields generalizing the works of [29, 63, 211]. In particular, our construction uses the so-called $\mathcal{M}^2$-normalization of [193].

3.1 Kinematic variables and source terms in cosmological models

The kinematical quantities associated with a timelike congruence were first introduced by Ehlers [65] and Ellis [69]. Given $\boldsymbol{u}$, the unique tensors

$$h_{\alpha\beta} := g_{\alpha\beta} + u_\alpha u_\beta$$

$$U_{\alpha\beta} = -u_\alpha u_\beta$$

project, at each point, tensors orthogonal and parallel to $\boldsymbol{u}$, respectively. We will use the following notation:

$$\dot{f} = u^\sigma \nabla_\sigma f \quad , \quad D_\alpha f = h_\alpha^{\ \beta} \nabla_\beta f$$

$$T_{<\alpha_1 \ldots \alpha_p>} = h_{\alpha_1}^{\beta_1} \ldots h_{\alpha_p}^{\beta_p} T_{\beta_1 \ldots \beta_p}$$

so that the covariant derivative of a scalar field is decomposed into

$$\nabla_\alpha f = -u_\alpha \dot{f} + D_\alpha f \, .$$

The covariant derivative of $\boldsymbol{u}$ can also be decomposed into its irreducible parts

$$\nabla_\alpha u_\beta = D_\alpha u_\beta - u_\alpha \dot{u}_\beta = \frac{1}{3}\theta h_{\alpha\beta} + \sigma_{\alpha\beta} + w_{\alpha\beta} - u_\alpha \dot{u}_\beta$$

where

$$\sigma_{\alpha\beta} = \sigma_{(\alpha\beta)}; \sigma_\alpha^{\ \alpha} = 0; \ \ \sigma^2 = \frac{1}{2}\sigma_{\alpha\beta}\sigma^{\alpha\beta}; \ \ \sigma_{\alpha\beta}u^\beta = 0; \ \ \omega_{\alpha\beta} = \omega_{[\alpha\beta]}; \ \ \omega_{\alpha\beta}u^\beta = 0, \tag{3.1}$$

the curly (resp. squared) brackets denote symmetrization (resp. anti-symmetrization) of a tensor and

$$\begin{aligned} \theta &= \nabla_\alpha u^\alpha \\ \dot{u}_\alpha &= \nabla_\beta u_\alpha u^\beta \\ \sigma_{\alpha\beta} &= \nabla_{(\beta} u_{\alpha)} - \frac{1}{3}\theta h_{\alpha\beta} + \dot{u}_{(\alpha} u_{\beta)} \\ w_{\alpha\beta} &= \nabla_{[\beta} u_{\alpha]} + \dot{u}_{[\alpha} u_{\beta]}. \end{aligned} \tag{3.2}$$

The tensor $\omega_{\alpha\beta}$ is called the vorticity, $\sigma_{\alpha\beta}$ the shear, and θ the expansion. One can define $\omega_{\alpha\beta}$ as

$$w^\alpha = \frac{1}{2}\eta^{\alpha\beta\mu\nu} u_\beta w_{\mu\nu} \, , \tag{3.3}$$

where $\eta^{\alpha\beta\mu\nu}$ is the totally skew pseudo-tensor defined by $\eta^{\alpha\beta\mu\nu} = \eta^{[\alpha\beta\mu\nu]}$ and $\eta^{0123} = (-\det(g))^{-1/2}$. The magnitude of w_α is given by

$$w^2 = w^\alpha w_\alpha = \frac{1}{2} w^{\alpha\beta} w_{\alpha\beta}. \tag{3.4}$$

The vector field $\boldsymbol{u}$ is hypersurface forming if $\omega_{\alpha\beta} = 0$. The expansion tensor is defined as

$$\theta_{\alpha\beta} = \sigma_{\alpha\beta} + \frac{1}{3}\theta h_{\alpha\beta}. \tag{3.5}$$

In the following we shall use the *Hubble function* defined by

$$H = \frac{1}{3}\theta. \tag{3.6}$$

To deduce the propagation equations for the gauge-invariant perturbation variables it is useful to recall the following relations between commutators of spatial and time derivatives acting on scalars [71]

$$D_\alpha \dot{f} = \frac{1}{3}\theta D_\alpha f + \left(\sigma_\alpha^{\ \beta} + \omega_\alpha^{\ \beta}\right) D_\beta f + h_\alpha^{\mu}(\dot{D_\mu f}) - \dot{f}\dot{u}_\alpha \tag{3.7}$$

$$D_{[\alpha} D_{\beta]} f = -\omega_{\alpha\beta}\dot{f}. \tag{3.8}$$

Given $\boldsymbol{u}$, the more general decomposition of the energy-momentum tensor field is given by

$$T_{\alpha\beta} = \rho u_\alpha u_\beta + p h_{\alpha\beta} + 2u_{(\alpha} q_{\beta)} + \pi_{\alpha\beta}\ , \tag{3.9}$$

where q_α is the energy-transfer function and $\pi_{\alpha\beta}$ is the anisotropic stress with $u^\alpha q_\alpha = 0$, $u^\alpha \pi_{\alpha\beta} = 0$, $\pi_{\alpha\beta} = \pi_{(\alpha\beta)}$, $\pi^\alpha_{\ \alpha} = 0$. In the multicomponent case, we assume that the total matter energy-momentum tensor is the sum of the individual energy-momentum tensors for the components plus an interaction term between these components:

$$T_{\alpha\beta} := \sum_{A=1}^{N} T^A_{\alpha\beta} + g_{\alpha\beta}\Pi \tag{3.10}$$

Moreover, given the preferred future directed time-like vector field $\boldsymbol{u}$, the EFEs (1.3) are expressed through the Ricci identities applied to $\boldsymbol{u}$ and the Bianchi identities in terms of the kinematic quantities, see e.g. [198].

3.1.1 Scalar fields

It was shown by Madsen [133] that if we require ϕ_A to be locally constant on a spacelike hypersurface, $D^\mu_A \phi_A = 0$ and $\nabla^\mu \phi_A \neq 0$, such that $\nabla^\mu \phi_A$ defines uniquely a time-like vector field orthogonal to the surfaces $\phi_A = const$ with

$$(\nabla_\lambda \phi_A)(\nabla^\lambda \phi_A) < 0 \quad \text{(time-like)} \tag{3.11}$$

From the local decomposition of the covariant derivative ($\nabla^\mu \phi_A = -u^\mu_A \psi_A$), we find that

$$u^\mu_A := -\frac{1}{\psi_A}\left(\nabla^\mu \phi_A\right) \tag{3.12}$$

is a unitary time-like vector field, with ψ_A the momentum-density defined by

$$\psi_A := \dot{\phi}_A = u^\lambda_A \nabla_\lambda \phi_A\ . \tag{3.13}$$

Because of the uniqueness of u^μ_A, we can use the $1+3$ covariant decomposition in this case using the local projector on the spacelike hypersurfaces of constant ϕ_A, which takes the form

$$h^A_{\alpha\beta} \equiv g_{\alpha\beta} + \frac{1}{\psi_A^2}\left(\nabla_\alpha \phi_A\right)\left(\nabla_\beta \phi_A\right). \tag{3.14}$$

Then, the energy-momentum tensor of each scalar field has the perfect fluid form

$$T^A_{\mu\nu} = \left[\frac{1}{2}\psi_A^2 + \mathcal{V}(\phi_A)\right] u^A_\mu u^A_\nu + \left[\frac{1}{2}\psi_A^2 - \mathcal{V}(\phi_A)\right] h^A_{\mu\nu} \tag{3.15}$$

with the identifications

$$\begin{aligned} \rho_A &= \frac{1}{2}\psi_A^2 + \mathcal{V}_A \\ p_A &= \frac{1}{2}\psi_A^2 - \mathcal{V}_A \end{aligned} \tag{3.16}$$

and, from the total energy-momentum tensor (3.10), we have $\Pi = -\mathcal{W}$. Finally, decomposing each $\boldsymbol{u}_A$ into components orthogonal and parallel to $\mathbf{u}$ it follows that [196]

$$u^\mu_A = \Gamma_A \left(u^\mu + v^\mu_A\right) \quad , \quad \Gamma_A = \frac{1}{\sqrt{1-v_A^2}} \quad , \quad u^\mu v_\mu = 0\ . \tag{3.17}$$

3.2 Universes close to the FL-scalar fields universe

In this section, we start by characterising the FL models in terms of the kinematic variables introduced above. Next, we define the *covariant and gauge-invariant* perturbation variables describing density inhomogeneities, and derive their evolution equations.

3.2.1 Characterization of Friedman-Lemaitre models

The particular case of FL models is caracterized by

$$\dot{u}_\mu = 0, \quad \sigma_{\mu\nu} = 0, \quad \omega_{\mu\nu} = 0. \tag{3.18}$$

and by the fact that spatial gradients of scalars are zero, in particular

$$D_\mu\phi = 0 \quad , \quad D_\mu\psi = 0, \quad , \quad D_\mu H = 0. \tag{3.19}$$

Furthermore, the symmetry of the spacetime forces the energy-momentum tensor (3.9) to have the algebraic form of a perfect fluid

$$q_\alpha = 0, \quad \pi_{\alpha\beta} = 0. \tag{3.20}$$

3.2.2 Covariant and gauge-invariant variables

We will use the following definitions for the covariant and gauge-invariant variables

$$\Delta_\alpha := a(t)\frac{D_\mu\psi}{\psi}, \quad \Delta^A_\alpha := a(t)\frac{D_\alpha\psi_A}{\psi_A}, \quad \mathcal{Z}_\alpha := a(t)D_\alpha\theta, \quad v^A_\alpha = -\frac{D_\alpha\phi_A}{\psi_A} \tag{3.21}$$

being, respectively, the total and each scalar field *comoving fractional momentum-density spatial gradients*, the *comoving spatial gradient of the expansion*, and the *velocity perturbations.* Above, we have defined the *effective scalar field momentum-density*

$$\psi^2 = \sum_{A=1}^{N} \psi_A^2 \tag{3.22}$$

which leads to the following relations between the perturbations variables

$$\Delta_\alpha = \frac{1}{\psi^2}\sum_{A=1}^{N}\psi_A^2\Delta^A_\alpha. \tag{3.23}$$

These variables contain information about three types of inhomogeneities and, similarly to the standard non-local decomposition, we follow [71] defining a local decomposition for the comoving gradient of the comoving fractional density gradient as

$$a(t)D_\alpha X_\beta := X_{\alpha\beta} = \frac{1}{3}X h_{\alpha\beta} + \Sigma_{\alpha\beta} + \Omega_{\alpha\beta} \;, \tag{3.24}$$

where $X = a(t)D^\alpha X_\alpha$ monitors "scalar" variations in the spatial distribution of matter (overdensities or underdensities). Generally, local scalar variables can be obtained by taking the divergence of any tensor and we define:

$$\Delta := a(t)D^\alpha \Delta_\alpha, \quad \mathcal{Z} := a(t)D^\alpha \mathcal{Z}_\alpha, \quad v_A := a(t)D^\alpha v^A_\alpha. \tag{3.25}$$

Also, we shall refer to the cosmological model with a self interacting scalar field of potential $\mathcal{V}(\phi)$ as *close to a Friedman-nonlinear scalar field universe* in some open set if, for some suitably small constant $\varepsilon \ll 1$ the following inequalities hold

$$\frac{|D_\alpha \psi|}{H\psi} < \varepsilon, \qquad \frac{|D_\alpha \phi|}{H^2\psi} < \varepsilon, \tag{3.26}$$

where $|D_\alpha\psi| = (D_\alpha\psi D^\alpha\psi)^{\frac{1}{2}}$.

3.2.3 Linearised equations

Let $\boldsymbol{u}$ be a time-like future-directed vector-field associated with the 4-velocity field of the total matter and $\boldsymbol{u}_A$ the orthogonal vectors to the surfaces $\phi_A = const.$, which are tilted from $\boldsymbol{u}$ by a small angle,

$$\Gamma_A \approx 1.$$

Then, to first-order, the relation between each $\boldsymbol{u}_A$ and $\boldsymbol{u}$ in the local rest frame defined by the latter vector field is given by [61]

$$\boldsymbol{u}_A \approx \boldsymbol{u} + \boldsymbol{v}_A. \tag{3.27}$$

By a small angle it is meant that $\boldsymbol{u}_A$ is time-like, which validates the space-like vector-field $\boldsymbol{v}_A$ as being a small deviation from the background solution. Thus, in a FL background, $\boldsymbol{v}_A = 0$, so that $\boldsymbol{v}_A$ will be a gauge-invariant perturbation variable. The total energy-momentum tensor in this frame is given by [61, 63]

$$T_{\alpha\beta} = \rho u_\alpha u_\beta + p h_{\alpha\beta} + 2u_{(\alpha}q_{\beta)} + \pi_{\alpha\beta} - g_{\alpha\beta}\mathcal{W}, \tag{3.28}$$

where, for N minimally coupled scalar-fields and, to first order

$$\rho = \sum_{A=1}^{N}\left[\frac{1}{2}\psi_A^2 + \mathcal{V}_A\right], \quad p = \sum_{A=1}^{N}\left[\frac{1}{2}\psi_A^2 - \mathcal{V}_A\right], \quad q_\alpha = \sum_{A=1}^{N}\psi_A^2 v^A_\alpha, \quad \pi_{\alpha\beta} = 0. \tag{3.29}$$

Also, to first order, we have that

$$\begin{aligned} \dot{q}^{<\mu>} &= \sum_{A=1}^{N}\left[2\frac{\dot{\psi}_A}{\psi_A}v_A^\mu + \dot{v}_A^{<\mu>}\right]\psi_A^2 \\ &= -6H\sum_{A=1}^{N}\psi_A^2 v_A^\mu - 2\sum_{A=1}^{N}\psi_A\left[\frac{d\mathcal{V}_A}{d\phi_A} + \frac{\partial\mathcal{W}}{\partial\phi_A}\right]v_A^\mu + \sum_{A=1}^{N}\psi_A^2\dot{v}_A^{<\mu>} \end{aligned} \tag{3.30}$$

and

$$\nabla^\mu\mathcal{W} = \sum_{A=1}^{N}\frac{\partial\mathcal{W}}{\partial\phi_A}\left[-u^\mu\psi_A + D_\mu\phi_A\right] = -\sum_{A=1}^{N}\psi_A\frac{\partial\mathcal{W}}{\partial\phi_A}\left[u^\mu + v_A^\mu\right]. \tag{3.31}$$

Then, the exact linearised evolution equations around a Friedman-Lemaitre-scalar field model in the frame defined by $\boldsymbol{u}$, are given by a wave-equation in the 1+3 covariant form for the effective momentum-density

$$\dot{\psi} = -3H\psi - \frac{1}{\psi}\sum_{A=1}^{N}\psi_A\left[\frac{d\mathcal{V}_A}{d\phi_A} + \frac{\partial\mathcal{W}}{\partial\phi_A}\right] - \frac{1}{\psi}\sum_{A=1}^{N}\psi_A^2 v_A \quad , \quad \psi \neq 0\ . \tag{3.32}$$

The momentum conservation equation is

$$\psi^2\dot{u}^\mu = -\psi(D^\mu\psi) - \sum_{A=1}^{N}\psi_A\left[\frac{d\mathcal{V}_A}{d\phi_A} + \frac{\partial\mathcal{W}}{\partial\phi_A}\right]v_A^\mu - 4H\sum_{A=1}^{N}\psi_A^2 v_A^\mu - \dot{q}^{<\mu>}$$

which, after multiplying by the scale factor $a(t)$ and using (3.30), simplifies to

$$a(t)\dot{u}^\mu = -\Delta^\mu + \frac{a(t)}{\psi^2}\sum_{A=1}^{N}\psi_A\left[\frac{d\mathcal{V}_A}{d\phi_A} + \frac{\partial\mathcal{W}}{\partial\phi_A}\right]v_A^\mu + 2H\frac{a(t)}{\psi^2}\sum_{A=1}^{N}\psi_A^2 v_A^\mu - \frac{a(t)}{\psi^2}\sum_{A=1}^{N}\psi_A^2\dot{v}_A^{<\mu>}\ . \tag{3.33}$$

In turn, the linearised Raychaudhuri equation for scalar fields is (see e.g. [29])

$$3\dot{H} = -3H^2 - \psi^2 + \sum_{A=1}^{N}\mathcal{V}_A + \mathcal{W} + D_\sigma\dot{u}^\sigma. \tag{3.34}$$

The set of linearised N wave-equations in the 1+3 covariant form, for each scalar field, are

$$\dot{\psi}_A = -3H\psi_A - \frac{\partial\mathcal{V}_A}{\partial\phi_A} - \frac{\partial\mathcal{W}}{\partial\phi_A} - \psi_A v_A \quad , \quad \psi_A \neq 0 \quad , \quad A = 1, ..., N \tag{3.35}$$

and the first order equation associated with the momentum conservation equation, for each scalar field, is

$$a(t)\dot{u}_\alpha + \Delta_\alpha^A + \left\{2\frac{\dot{\psi}_A}{\psi_A} + 4H + \frac{1}{\psi_A}\left[\frac{\partial\mathcal{W}}{\partial\phi_A} + \frac{d\mathcal{V}_A}{d\phi_A}\right]\right\}a(t)v_\alpha^A + a(t)\dot{v}^A_{<\alpha>} = 0\ . \tag{3.36}$$

3.2.4 Evolution equation for Δ

To obtain the evolution equation for Δ_μ, we take spatial gradients of equation (3.32) and, keeping the first order terms, we get

$$\begin{aligned} a(t)\frac{D_\mu\dot{\psi}}{\psi} =& -\mathcal{Z}_\mu - \left[3H - \frac{1}{\psi^2}\sum_{A=1}^{N}\psi_A\left(\frac{d\mathcal{V}_A}{d\phi_A} + \frac{\partial\mathcal{W}}{\partial\phi_A}\right)\right]\Delta_\mu - \frac{1}{\psi^2}\sum_{A=1}^{N}\psi_A\left(\frac{d\mathcal{V}_A}{d\phi_A} + \frac{\partial\mathcal{W}}{\partial\phi_A}\right)\Delta_\mu^A \\ &- \frac{a(t)}{\psi^2}D_\mu\sum_{A=1}\psi_A^2 v_A + \frac{a(t)}{\psi^2}\sum_{A=1}^{N}\psi_A^2\frac{d^2\mathcal{V}_A}{d\phi_A^2}v_\mu^A + \frac{a(t)}{\psi^2}\sum_{A,B=1}^{N}\psi_A\psi_B\frac{\partial^2\mathcal{W}}{\partial\phi_B\partial\phi_A}v_\mu^B. \end{aligned} \tag{3.37}$$

Then, using the relation (3.7) to first-order for the effective momentum-density variable ψ, we find

$$D_\mu\dot{\psi} = HD_\mu\psi + h_\mu^{\ \nu}(D_\nu\psi)^{\cdot} - \dot{\psi}\dot{u}_\mu$$

together with the relation

$$a(t)\frac{h_\mu^{\ \nu}(D_\nu\psi)^{\cdot}}{\psi} = \dot{\Delta}_{<\mu>} - \left[H - \frac{\dot{\psi}}{\psi}\right]\Delta_\mu$$

giving

$$a(t)\frac{D_\mu \dot{\psi}}{\psi} = \dot{\Delta}_{<\mu>} + \frac{\dot{\psi}}{\psi}\Delta_\mu - a(t)\frac{\dot{\psi}}{\psi}\dot{u}_\mu \ ,$$

which, upon inserting into (3.37), finally gives

$$\begin{aligned}\dot{\Delta}_{<\mu>} =& -\mathcal{Z}_\mu - \left[\frac{\dot{\psi}}{\psi} + 3H - \frac{1}{\psi^2}\sum_{A=1}^{N}\psi_A\left(\frac{d\mathcal{V}_A}{d\phi_A} + \frac{\partial\mathcal{W}}{\partial\phi_A}\right)\right]\Delta_\mu - \frac{1}{\psi^2}\sum_{A=1}^{N}\psi_A\left(\frac{d\mathcal{V}_A}{d\phi_A} + \frac{\partial\mathcal{W}}{\partial\phi_A}\right)\Delta_\mu^A \\ &- \frac{a(t)}{\psi^2}D_\mu\sum_{A=1}\psi_A^2 v_A + \frac{a(t)}{\psi^2}\sum_{A=1}^{N}\psi_A^2\frac{d^2\mathcal{V}_A}{d\phi_A^2}v_\mu^A + \frac{a(t)}{\psi^2}\sum_{A,B=1}^{N}\psi_A\psi_B\frac{\partial^2\mathcal{W}}{\partial\phi_B\partial\phi_A}v_\mu^B + a(t)\frac{\dot{\psi}}{\psi}\dot{u}_\mu.\end{aligned} \tag{3.38}$$

3.2.5 Evolution equation for $\mathcal{Z}$

The evolution equation for the perturbation variable $\mathcal{Z}_\mu$ is found by taking spatial gradients of the linearized Raychauduri equation (3.34), which after multiplication by the scale factor $a(t)$, reads

$$3a(t)D_\mu\dot{H} = -2H\mathcal{Z}_\mu - 2\psi^2\Delta_\mu - a(t)\sum_{A=1}^{N}\psi_A\left(\frac{d\mathcal{V}_A}{d\phi_A} + \frac{\partial\mathcal{W}}{\partial\phi_A}\right)v_\mu^A + a(t)D_\mu D_\sigma\dot{u}^\sigma. \tag{3.39}$$

Using relation (3.7) for the Hubble function gives

$$D_\mu\dot{H} = HD_\mu H + h_\mu^{\ \nu}(\dot{D_\nu H}) - \dot{H}\dot{u}_\mu \ ,$$

which together with

$$3a(t)h_\mu^{\ \nu}(\dot{D_\nu H}) = \dot{\mathcal{Z}}_{<\mu>} - H\mathcal{Z}_\mu \ ,$$

gives

$$3a(t)D_\mu\dot{H} = \dot{\mathcal{Z}}_{<\mu>} - 3\dot{H}a(t)\dot{u}_\mu \ .$$

Finally, inserting the last equation into (3.39)

$$\dot{\mathcal{Z}}_{<\mu>} = -2H\mathcal{Z}_\mu - 2\psi^2\Delta_\mu - a(t)\sum_{A=1}^{N}\psi_A\left(\frac{d\mathcal{V}_A}{d\phi_A} + \frac{\partial\mathcal{W}}{\partial\phi_A}\right)v_\mu^A + a(t)D_\mu D_\sigma\dot{u}^\sigma + 3\dot{H}a(t)\dot{u}_\mu \ . \tag{3.40}$$

3.2.6 Evolution equation for Δ^A and v^A

The evolution equation for each variable Δ_μ^A is obtained by taking spatial gradients of equation (3.35) and keeping first order terms as

$$a(t)\frac{D_\mu\dot{\psi}_A}{\psi_A} = -3H\Delta_\mu^A - \mathcal{Z}_\mu - a(t)D_\mu v_A + a(t)\frac{d^2\mathcal{V}_A}{d\phi_A^2}v_\mu^A + a(t)\sum_{B=1}^{N}\frac{\psi_B}{\psi_A}\frac{\partial^2\mathcal{W}}{\partial\phi_B\partial\phi_A}v_\mu^B$$

and

$$\begin{aligned}\dot{\Delta}^A_{<\mu>} =& -\left[\frac{\dot{\psi}_A}{\psi_A} + 3H\right]\Delta_\mu^A - \mathcal{Z}_\mu - a(t)D_\mu D_\sigma v_A^\sigma + a(t)\frac{d^2\mathcal{V}_A}{d\phi_A^2}v_\mu^A \\ &+ a(t)\sum_{B=1}^{N}\frac{\psi_B}{\psi_A}\frac{\partial^2\mathcal{W}}{\partial\phi_B\partial\phi_A}v_\mu^B + a(t)\frac{\dot{\psi}_A}{\psi_A}\dot{u}_\mu.\end{aligned} \tag{3.41}$$

To get the evolution equation for the velocity perturbation variables we can use the relation (3.7) for each of the scalar fields ϕ_A and get

$$a(t)\dot{v}^A_{<\mu>} = -a(t)\left[H + \frac{\dot{\psi}_A}{\psi_A}\right] v^A_\mu - \Delta^A_\mu - a(t)\dot{u}_\mu \quad , \quad \psi_A \neq 0. \tag{3.42}$$

This equation is identical to the momentum-conservation equation for the Ath component (3.36) after using the background nonlinear wave-equation in the $1+3$ form.

3.2.7 Energy frame

In order to close the system of evolution and constraint equations, we need to fix the frame for which we are constructing perturbation variables. Furthermore, this choice of frame must ensure that the perturbation variables are gauge-invariant. A suitable choice is the energy frame defined through

$$q^\mu = 0 \quad , \quad \dot{q}^{<\mu>} = 0. \tag{3.43}$$

Thus, if we choose $\boldsymbol{u} = \boldsymbol{u}_E$ to be the energy frame, then

$$\sum_{A=1}^{N} \psi_A^2 v_A^\mu = 0$$

and

$$\sum_{A=1}^{N} \psi_A^2 \dot{v}_A^{<\mu>} = 2\sum_{A=1}^{N} \psi_A \left[\frac{d\mathcal{V}_A}{d\phi_A} + \frac{\partial \mathcal{W}}{\partial \phi_A}\right] v_A^\mu \,.$$

In this frame, (3.33) reads

$$a(t)\dot{u}^\mu = -\Delta^\mu - \frac{a(t)}{\psi^2}\sum_{A=1}^{N} \psi_A \left[\frac{d\mathcal{V}_A}{d\phi_A} + \frac{\partial \mathcal{W}}{\partial \phi_A}\right] v_A^\mu \tag{3.44}$$

and the first order equation for the divergence of the acceleration is

$$a^2 D_\sigma \dot{u}^\sigma = -\Delta - \frac{a(t)}{\psi^2}\sum_{A=1}^{N} \psi_A \left[\frac{d\mathcal{V}_A}{d\phi_A} + \frac{\partial \mathcal{W}}{\partial \phi_A}\right] v_A \,. \tag{3.45}$$

By using (3.44) and (3.45) into (3.38), (3.40), (3.41) and (3.42), and after multiplying the resulting equation by the scale factor and taking the spatial divergence of these equations, we finally obtain

$$\begin{aligned}
\dot{\Delta} =& -\mathcal{Z} + \left[3H + \frac{3}{\psi^2}\sum_{A=1}^{N} \psi_A \left(\frac{d\mathcal{V}_A}{d\phi_A} + \frac{\partial \mathcal{W}}{\partial \phi_A}\right)\right]\Delta - \frac{1}{\psi^2}\sum_{A=1}^{N} \psi_A \left(\frac{d\mathcal{V}_A}{d\phi_A} + \frac{\partial \mathcal{W}}{\partial \phi_A}\right)\Delta_A \\
&+ a(t)\left[3H + \frac{1}{\psi^2}\sum_{C=1}^{N} \psi_C \left(\frac{d\mathcal{V}_C}{d\phi_C} + \frac{\partial \mathcal{W}}{\partial \phi_C}\right)\right]\frac{1}{\psi^2}\sum_{A=1}^{N} \psi_A \left(\frac{d\mathcal{V}_A}{d\phi_A} + \frac{\partial \mathcal{W}}{\partial \phi_A}\right) v_A \\
&+ \frac{a(t)}{\psi^2}\sum_{A=1}^{N} \psi_A^2 \frac{d^2\mathcal{V}_A}{d\phi_A^2} v_A + \frac{a(t)}{\psi^2}\sum_{A,B=1}^{N} \psi_A \psi_B \frac{\partial^2 \mathcal{W}}{\partial \phi_B \partial \phi_A} v_B
\end{aligned} \tag{3.46}$$

$$\dot{\mathcal{Z}} = -2H\mathcal{Z} - \left[3\dot{H} + 2\psi^2 + D^2\right]\Delta - \left[\psi^2 + 3\dot{H} + D^2\right]\frac{a(t)}{\psi^2}\sum_{A=1}^{N} \psi_A \left(\frac{d\mathcal{V}_A}{d\phi_A} + \frac{\partial \mathcal{W}}{\partial \phi_A}\right) v_A \tag{3.47}$$

$$
\begin{aligned}
\dot{\Delta}_A = &- \mathcal{Z} + \left[3H + \frac{1}{\psi_A}\left(\frac{d\mathcal{V}_A}{d\phi_A} + \frac{\partial \mathcal{W}}{\partial \phi_A}\right)\right]\Delta - \left[\frac{1}{\psi_A}\left(\frac{d\mathcal{V}_A}{d\phi_A} + \frac{\partial \mathcal{W}}{\partial \phi_A}\right)\right]\Delta_A \\
&- a(t)D^2 v_A + a(t)\frac{d^2\mathcal{V}_A}{d\phi_A^2}v_A + a(t)\sum_{B=1}^{N}\frac{\psi_B}{\psi_A}\frac{\partial^2 \mathcal{W}}{\partial\phi_B\partial\phi_A}v_B \\
&+ \left[3H + \frac{1}{\psi_A}\left(\frac{d\mathcal{V}_A}{d\phi_A} + \frac{\partial \mathcal{W}}{\partial \phi_A}\right)\right]\frac{a(t)}{\psi^2}\sum_{B=1}^{N}\psi_B\left[\frac{d\mathcal{V}_B}{d\phi_B} + \frac{\partial \mathcal{W}}{\partial \phi_B}\right]v_B
\end{aligned}
\tag{3.48}
$$

$$
a(t)\dot{v}_A = \Delta - \Delta_A + a(t)\left[2H + \frac{1}{\psi_A}\left(\frac{d\mathcal{V}_A}{d\phi_A} + \frac{\partial \mathcal{W}}{\partial \phi_A}\right)\right]v_A + \frac{a(t)}{\psi^2}\sum_{B=1}^{N}\psi_B\left[\frac{d\mathcal{V}_B}{d\phi_B} + \frac{\partial \mathcal{W}}{\partial \phi_B}\right]v_B. \tag{3.49}
$$

Equations (3.46)-(3.49), together with the background equations (2.6), form a closed system of equations for the variables Δ, $\mathcal{Z}$, Δ_A and v_A.

3.2.8 Relative variables

In order to simplify the notation, and the computation in the multiple scalar field case, let

$$
\alpha_A := \frac{1}{\psi_A}\left(\frac{d\mathcal{V}_A}{d\phi_A} + \frac{\partial \mathcal{W}}{\partial \phi_A}\right) \quad , \quad \beta_A^2 = \frac{\psi_A^2}{\psi^2} \tag{3.50}
$$

so that

$$
\dot{\alpha}_A = (3H + \alpha_A)\,\alpha_A + \frac{d^2\mathcal{V}_A}{d\phi_A^2} + \frac{\partial^2\mathcal{W}}{\partial\phi_A^2} + \sum_{B\neq A}^{N}\frac{\psi_B}{\psi_A}\frac{\partial^2\mathcal{W}}{\partial\phi_B\partial\phi_A}
$$

$$
\dot{\left(\beta_A^2\right)} = -2\sum_{B\neq A}^{N}(\alpha_A - \alpha_B)\,\beta_A^2\beta_B^2
$$

and, from the fact that β_A is an addictive quantity in the sense of [193], $\sum_{A=1}^{N}\beta_A^2 = 1$,

$$
\sum_{A=1}^{N}\dot{\left(\beta_A^2\right)} = 0\ . \tag{3.51}
$$

Defining the relative perturbation variables as

$$
\Delta_{[AB]} = \Delta_A - \Delta_B \quad , \quad v_{[AB]} = v_A - v_B \tag{3.52}
$$

then (3.23) reads

$$
\Delta_A = \Delta + \sum_{B\neq A}^{N}\beta_B^2\Delta_{[AB]}. \tag{3.53}
$$

For the relative velocity perturbation variables, using the energy-frame condition (3.43), it follows that

$$
v_A = \sum_{B\neq A}^{N}\beta_B^2 v_{[AB]}, \tag{3.54}
$$

and using the above variables we obtain from equations (3.46) and (3.47),

$$
\dot{\Delta} = A(t)\Delta - \mathcal{Z} - \sum_{A=1}^{N}\sum_{B>A}^{N}B_{AB}(t)\Delta_{[AB]} + a(t)\sum_{A=1}^{N}\sum_{B>A}^{N}C_{AB}(t)v_{[AB]} \tag{3.55}
$$

$$
\begin{aligned}
\dot{\mathcal{Z}} =& - 2H\mathcal{Z} - \left[3\dot{H} + 2\psi^2 + D^2\right]\Delta \\
& - a(t)\sum_{A=1}^{N}\sum_{B>A}^{N}\left(\alpha_A - \alpha_B\right)\beta_A^2\beta_B^2\left[\psi^2 + 3\dot{H} + D^2\right]v_{[AB]}\ .
\end{aligned}
\tag{3.56}
$$

To get the evolution equation for $\Delta_{[AB]}$ we take the difference $\dot{\Delta}_{[AB]} = \dot{\Delta}_A - \dot{\Delta}_B$, which upon using (3.53) and the following relation

$$
\begin{aligned}
(\alpha_A - \alpha_B)\Delta - \alpha_A\Delta_A + \alpha_B\Delta_B =& - \left(\alpha_A\beta_B^2 + \alpha_B\beta_A^2\right)\Delta_{[AB]} \\
& - \alpha_A\sum_{C\neq A,B}^{N}\beta_C^2\Delta_{[AC]} - \alpha_B\sum_{C\neq A,B}^{N}\beta_C^2\Delta_{[BC]}
\end{aligned}
$$

gives

$$
\begin{aligned}
\dot{\Delta}_{[AB]} =& - D_{AB}(t)\Delta_{[AB]} + \sum_{C\neq A,B}^{N}\left(\alpha_A\beta_C^2\Delta_{[AC]} - \alpha_B\beta_C^2\Delta_{[BC]}\right) \\
& + a(t)\sum_{C\neq A,B}^{N}\left[F_{AC}(t)v_{[AC]} - F_{BC}(t)v_{[BC]}\right] \\
& + a(t)\sum_{C\neq A,B}^{N}\sum_{D\neq A,B,C}^{N}\left(\frac{\psi_C}{\psi_A}\frac{\partial^2\mathcal{W}}{\partial\phi_C\partial\phi_A} - \frac{\psi_C}{\psi_B}\frac{\partial^2\mathcal{W}}{\partial\phi_C\partial\phi_B}\right)\beta_D^2 v_{[CD]}
\end{aligned}
\tag{3.57}
$$

as well as

$$
a\dot{v}_{[AB]} = -\Delta_{[AB]} + a(t)\left(2H + D_{AB}\right)v_{[AB]} + a(t)\sum_{C\neq A,B}^{N}\left(\alpha_A\beta_C^2 v_{[AC]} - \alpha_B\beta_C^2 v_{[BC]}\right), \tag{3.58}
$$

where

$$
\begin{aligned}
A(t) =& 3H + 2\left(\alpha_A\beta_A^2 + \alpha_B\beta_B^2\right) + 2\sum_{C\neq A,B}^{N}\alpha_C\beta_C^2 \\
B_{AB}(t) =& \left(\alpha_A - \alpha_B\right)\beta_A^2\beta_B^2 \\
C_{AB}(t) =& \left(3H + \left(\alpha_A\beta_A^2 + \alpha_B\beta_B^2\right)\right)B_{AB} \\
& + \left[\left(\frac{d^2\mathcal{V}}{d\phi_A^2} - \frac{d^2\mathcal{V}}{d\phi_B^2}\right) + \left(\frac{\partial^2\mathcal{W}}{\partial\phi_A^2} - \frac{\partial^2\mathcal{W}}{\partial\phi_B^2}\right) - \left(\frac{\psi_A}{\psi_B} - \frac{\psi_B}{\psi_A}\right)\frac{\partial^2\mathcal{W}}{\partial\phi_B\partial\phi_A}\right]\beta_A^2\beta_B^2 \\
& + \sum_{C\neq A,B}^{N}\left[\alpha_C\beta_C^2 B_{AB} + \left(\frac{\psi_C}{\psi_A}\frac{\partial^2\mathcal{W}}{\partial\phi_A\partial\phi_C} - \frac{\psi_C}{\psi_B}\frac{\partial^2\mathcal{W}}{\partial\phi_B\partial\phi_C}\right)\right] \\
D_{AB}(t) =& \alpha_A\beta_B^2 + \alpha_B\beta_A^2 \\
E_{AB}(t) =& \left(\frac{d^2\mathcal{V}}{d\phi_A^2} + \frac{\partial^2\mathcal{W}}{\partial\phi_A^2} - \frac{\psi_A}{\psi_B}\frac{\partial^2\mathcal{W}}{\partial\phi_A\partial\phi_B}\right)\beta_B^2 + \left(\frac{d^2\mathcal{V}}{d\phi_B^2} + \frac{\partial^2\mathcal{W}}{\partial\phi_B^2} - \frac{\psi_B}{\psi_A}\frac{\partial^2\mathcal{W}}{\partial\phi_B\partial\phi_A}\right)\beta_A^2 \\
& + \left(\alpha_A - \alpha_B\right)B_{AB} \\
F_{AB}(t) =& \alpha_A\left(\alpha_A + \alpha_B\right)\beta_A^2\beta_B^2
\end{aligned}
\tag{3.59}
$$

As usual, we can decouple the evolution equation for $\mathcal{Z}$ by differentiating (3.55) with respect to time t and using (3.56), obtaining

$$\begin{aligned}\ddot{\Delta} =& (A-2H)\,\dot{\Delta} + \left(\dot{A} + 2HA + 3\dot{H} + 2\psi^2 + D^2\right)\Delta \\ &- \sum_{A=1}^{N}\sum_{B>A}^{N}\left[\dot{B}_{AB} + (2H - D_{AB})\,B_{AB} + C_{AB}\right]\Delta_{[AB]} \\ &- \sum_{A=1}^{N}\sum_{B>A}^{N}\sum_{C\neq A,B}^{N} B_{AB}\left(\alpha_A\beta_C^2\Delta_{[AC]} - \alpha_B\beta_C^2\Delta_{[BC]}\right) \\ &+ a(t)\sum_{A=1}^{N}\sum_{B>A}^{N}\left[\dot{C}_{AB} + (5H + D_{AB})\,C_{AB} + B_{AB}\left(3\dot{H} + \psi^2 - E_{AB} + 2D^2\right)\right] v_{[AB]} \\ &- a(t)\sum_{A=1}^{N}\sum_{B>A}^{N}\sum_{C\neq A,B}^{N}\left[\left(B_{AB}F_{AC} - C_{AB}\alpha_A\beta_C^2\right)v_{[AC]} + \left(B_{AB}F_{BC} + C_{AB}\alpha_B\beta_C^2\right)v_{[BC]}\right] \\ &- a(t)\sum_{A=1}^{N}\sum_{B>A}^{N}\sum_{C\neq A,B}^{N}\sum_{D\neq A,B,C}^{N} B_{AB}\left(\frac{\psi_C}{\psi_A}\frac{\partial^2\mathcal{W}}{\partial\phi_C\partial\phi_A} - \frac{\psi_C}{\psi_B}\frac{\partial^2\mathcal{W}}{\partial\phi_C\partial\phi_B}\right)\beta_D^2 v_{[CD]}.\end{aligned} \tag{3.60}$$

Now

$$\begin{aligned}&\dot{\alpha}_A\beta_A^2 + \dot{\alpha}_B\beta_B^2 + \alpha_A(\dot{\beta_A^2}) + \alpha_B(\dot{\beta_B^2}) = \\ &\quad= 3H\left(\alpha_A\beta_A^2 + \alpha_B\beta_B^2\right) + \left(\alpha_A^2\beta_A^2 + \alpha_B^2\beta_B^2\right) - 2\left(\alpha_A - \alpha_B\right)B_{AB} \\ &\quad + \left(\frac{d^2\mathcal{V}}{d\phi_A^2} + \frac{\partial^2\mathcal{W}}{\partial\phi_A^2} + \frac{\psi_B}{\psi_A}\frac{\partial^2\mathcal{W}}{\partial\phi_B\partial\phi_A}\right)\beta_A^2 + \left(\frac{d^2\mathcal{V}}{d\phi_B^2} + \frac{\partial^2\mathcal{W}}{\partial\phi_B^2} + \frac{\psi_A}{\psi_B}\frac{\partial^2\mathcal{W}}{\partial\phi_A\partial\phi_B}\right)\beta_B^2 \\ &\quad + \sum_{C\neq A,B}^{N}\psi_C\left[\left(\frac{\beta_A^2}{\psi_A}\frac{\partial^2\mathcal{W}}{\partial\phi_C\partial\phi_A} + \frac{\beta_B^2}{\psi_B}\frac{\partial^2\mathcal{W}}{\partial\phi_C\partial\phi_B}\right) - 2\left(\alpha_A B_{AC} + \alpha_B B_{BC}\right)\right]\end{aligned} \tag{3.61}$$

so that

$$\begin{aligned}\dot{A} =& 3\dot{H} + 6H\left(\alpha_A\beta_A^2 + \alpha_B\beta_B^2\right) + 2\left(\alpha_A^2\beta_A^2 + \alpha_B^2\beta_B^2\right) - 4\left(\alpha_A - \alpha_B\right)B_{AB} \\ &+ 2\left(\frac{d^2\mathcal{V}}{d\phi_A^2} + \frac{\partial^2\mathcal{W}}{\partial\phi_A^2} + \frac{\psi_B}{\psi_A}\frac{\partial^2\mathcal{W}}{\partial\phi_B\partial\phi_A}\right)\beta_A^2 + 2\left(\frac{d^2\mathcal{V}}{d\phi_B^2} + \frac{\partial^2\mathcal{W}}{\partial\phi_B^2} + \frac{\psi_A}{\psi_B}\frac{\partial^2\mathcal{W}}{\partial\phi_A\partial\phi_B}\right)\beta_B^2 \\ &+ 2\sum_{C\neq A,B}^{N}\left[-2\left(\alpha_C - \alpha_A\right)B_{CA} - 2\left(\alpha_C - \alpha_B\right)B_{CB}\right] \\ &+ 2\sum_{C\neq A,B}^{N}\left[\frac{\psi_A}{\psi_C}\beta_C^2\frac{\partial^2\mathcal{W}}{\partial\phi_A\partial\phi_C} + \frac{\psi_B}{\psi_C}\beta_C^2\frac{\partial^2\mathcal{W}}{\partial\phi_B\partial\phi_C} + \psi_C\left(\frac{\beta_A^2}{\psi_A}\frac{\partial^2\mathcal{W}}{\partial\phi_C\partial\phi_A} + \frac{\beta_B^2}{\psi_B}\frac{\partial^2\mathcal{W}}{\partial\phi_C\partial\phi_B}\right)\right] \\ &+ 2\sum_{C\neq A,B}^{N}\sum_{D\neq A,B,C}^{N}\left[\left(3H + \alpha_C\right)\alpha_C + \frac{d^2\mathcal{V}}{d\phi_C^2} + \frac{\partial^2\mathcal{W}}{\partial\phi_C^2} + \frac{\psi_D}{\psi_C}\frac{\partial^2\mathcal{W}}{\partial\phi_D\partial\phi_C} - 2\alpha_C\left(\alpha_C - \alpha_D\right)\beta_D^2\right]\beta_C^2\end{aligned}$$

and

$$\begin{aligned}\dot{B}_{AB} =& \left[3H + \left(\alpha_A + \alpha_B\right) + 2\left(\alpha_A - \alpha_B\right)\left(\beta_A^2 - \beta_B^2\right)\right]B_{AB} \\ &+ \left[\left(\frac{d^2\mathcal{V}}{d\phi_A^2} - \frac{d^2\mathcal{V}}{d\phi_B^2}\right) + \left(\frac{\partial^2\mathcal{W}}{\partial\phi_A^2} - \frac{\partial^2\mathcal{W}}{\partial\phi_B^2}\right) + \left(\frac{\psi_B}{\psi_A}\frac{\partial^2\mathcal{W}}{\partial\phi_B\partial\phi_A} - \frac{\psi_A}{\psi_B}\frac{\partial^2\mathcal{W}}{\partial\phi_A\partial\phi_B}\right)\right]\beta_A^2\beta_B^2 \\ &+ \sum_{C\neq A,B}^{N}\left[-2B_{AB}\left(\alpha_A + \alpha_B - 2\alpha_C\right)\beta_C^2 + \left(\frac{\psi_C}{\psi_A}\frac{\partial^2\mathcal{W}}{\partial\phi_C\partial\phi_A} - \frac{\psi_C}{\psi_B}\frac{\partial^2\mathcal{W}}{\partial\phi_C\partial\phi_B}\right)\beta_A^2\beta_B^2\right],\end{aligned}$$

$$
\begin{aligned}
\dot{C}_{AB} = & \Big[3\dot{H} + 3H\left(\alpha_A\beta_A^2 + \alpha_B\beta_B^2\right) + \left(\alpha_A^2\beta_A^2 + \alpha_B^2\beta_B^2\right) - 2\left(\alpha_A - \alpha_B\right)B_{AB}\Big]B_{AB} \\
& + \left[\left(\frac{d^2\mathcal{V}}{d\phi_A^2} + \frac{\partial^2\mathcal{W}}{\partial\phi_A^2} + \frac{\psi_B}{\psi_A}\frac{\partial^2\mathcal{W}}{\partial\phi_B\partial\phi_A}\right)\beta_A^2 + \left(\frac{d^2\mathcal{V}}{d\phi_B^2} + \frac{\partial^2\mathcal{W}}{\partial\phi_B^2} + \frac{\psi_A}{\psi_B}\frac{\partial^2\mathcal{W}}{\partial\phi_A\partial\phi_B}\right)\beta_B^2\right]B_{AB} \\
& + \left(\frac{\psi_A}{\psi_B}\frac{\partial^2\mathcal{W}}{\partial\phi_B\partial\phi_A} - \frac{\psi_B}{\psi_A}\frac{\partial^2\mathcal{W}}{\partial\phi_A\partial\phi_B}\right)B_{AB} + \left(3H + \left(\alpha_A\beta_A^2 + \alpha_B\beta_B^2\right)\right)\dot{B}_{AB} \\
& + 2\left[\left(\frac{d^2\mathcal{V}}{d\phi_A^2} - \frac{d^2\mathcal{V}}{d\phi_B^2}\right) + \left(\frac{\partial^2\mathcal{W}}{\partial\phi_A^2} - \frac{\partial^2\mathcal{W}}{\partial\phi_B^2}\right) - \left(\frac{\psi_A}{\psi_B} - \frac{\psi_B}{\psi_A}\right)\frac{\partial^2\mathcal{W}}{\partial\phi_A\partial\phi_B}\right]\left(\beta_A^2 - \beta_B^2\right)B_{AB} \\
& + \left[\left(\frac{d^3\mathcal{V}}{d\phi_A^3} + \frac{\partial^3\mathcal{W}}{\partial\phi_A^3} - \frac{\partial^3\mathcal{W}}{\partial\phi_A\partial\phi_B^2} - \frac{\partial^3\mathcal{W}}{\partial\phi_B^2\partial\phi_A} - \frac{\psi_A}{\psi_B}\frac{\partial^3\mathcal{W}}{\partial\phi_A\partial\phi_B\partial\phi_A}\right)\psi_A\right]\beta_A^2\beta_B^2 \\
& - \left[\left(\frac{d^3\mathcal{V}}{d\phi_B^3} + \frac{\partial^3\mathcal{W}}{\partial\phi_B^3} - \frac{\partial^3\mathcal{W}}{\partial\phi_B\partial\phi_A^2} - \frac{\partial^3\mathcal{W}}{\partial\phi_A^2\partial\phi_B} - \frac{\psi_B}{\psi_A}\frac{\partial^3\mathcal{W}}{\partial\phi_B\partial\phi_A\partial\phi_B}\right)\psi_B\right]\beta_A^2\beta_B^2 \\
& + 2\sum_{C\neq A,B}^{N}\left[\left(\frac{d^2\mathcal{V}}{d\phi_A^2} - \frac{d^2\mathcal{V}}{d\phi_B^2}\right) + \left(\frac{\partial^2\mathcal{W}}{\partial\phi_A^2} - \frac{\partial^2\mathcal{W}}{\partial\phi_B^2}\right)\right]\left(B_{CA}\beta_B^2 + B_{CB}\beta_A^2\right) \\
& - 2\sum_{C\neq A,B}^{N}\left[\left(\frac{\psi_A}{\psi_B} - \frac{\psi_B}{\psi_A}\right)\frac{\partial^2\mathcal{W}}{\partial\phi_B\partial\phi_A}\right]\left(B_{CA}\beta_B^2 + B_{CB}\beta_A^2\right) \\
& + \sum_{C\neq A,B}^{N}\left[(3H + \alpha_C)\alpha_C + \frac{d^2\mathcal{V}}{d\phi_C^2} + \frac{\partial^2\mathcal{W}}{\partial\phi_C^2} + \frac{\psi_A}{\psi_C}\frac{\partial^2\mathcal{W}}{\partial\phi_A\partial\phi_C} + \frac{\psi_B}{\psi_C}\frac{\partial^2\mathcal{W}}{\partial\phi_B\partial\phi_C}\right]\beta_A^2\beta_B^2 \\
& + \sum_{C\neq A,B}^{N}\left[-2\left(\alpha_A B_{AC} + \alpha_B B_{BC}\right)B_{AB} - 2\alpha_C B_{AB}\left(B_{CA} + B_{CB}\right) + \alpha_C\beta_C^2\dot{B}_{AB}\right] \\
& + \sum_{C\neq A,B}^{N}\left[\psi_C\left(\frac{\beta_A^2}{\psi_A}\frac{\partial^2\mathcal{W}}{\partial\phi_C\partial\phi_A} + \frac{\beta_B^2}{\psi_B}\frac{\partial^2\mathcal{W}}{\partial\phi_C\partial\phi_B}\right)B_{AB}\right] \\
& + \sum_{C\neq A,B}^{N}\left[\left(\alpha_A - \alpha_C\right)\frac{\psi_C}{\psi_A}\frac{\partial^2\mathcal{W}}{\partial\phi_A\partial\phi_C} - \left(\alpha_B - \alpha_C\right)\frac{\psi_C}{\psi_B}\frac{\partial^2\mathcal{W}}{\partial\phi_B\partial\phi_C}\right] \\
& + \sum_{C\neq A,B}^{N}\psi_C\left[\frac{\partial^3\mathcal{W}}{\partial\phi_C\partial\phi_A^2} - \frac{\partial^3\mathcal{W}}{\partial\phi_C\partial\phi_B^2} - \left(\frac{\psi_A}{\psi_B}\frac{\partial^3\mathcal{W}}{\partial\phi_A\partial\phi_B\partial\phi_C} - \frac{\psi_B}{\psi_A}\frac{\partial^3\mathcal{W}}{\partial\phi_B\partial\phi_A\partial\phi_C}\right)\right]\beta_A^2\beta_B^2 \\
& + \sum_{C\neq A,B}^{N}\left[\frac{\psi_C}{\psi_A}\left(\psi_A\frac{\partial^3\mathcal{W}}{\partial\phi_C\partial\phi_A^2} + \psi_B\frac{\partial^3\mathcal{W}}{\partial\phi_B\partial\phi_A\partial\phi_C} + \psi_C\frac{\partial^3\mathcal{W}}{\partial\phi_C^2\partial\phi_A}\right)\right] \\
& - \sum_{C\neq A,B}^{N}\left[\frac{\psi_C}{\psi_B}\left(\psi_A\frac{\partial^3\mathcal{W}}{\partial\phi_A\partial\phi_B\partial\phi_C} + \psi_B\frac{\partial^3\mathcal{W}}{\partial\phi_C\partial\phi_B^2} + \psi_C\frac{\partial^3\mathcal{W}}{\partial\phi_C\partial\phi_B\partial\phi_C}\right)\right] \\
& + \sum_{C\neq A,B}^{N}\sum_{D\neq A,B,C}^{N}\left[-2\alpha_C B_{AB}B_{CD} + B_{AB}\beta_C^2\frac{\psi_D}{\psi_C}\frac{\partial^2\mathcal{W}}{\partial\phi_D\partial\phi_C}\right] \\
& + \sum_{C\neq A,B}^{N}\sum_{D\neq A,B,C}^{N}\left[\psi_D\left(\frac{\psi_C}{\psi_A}\frac{\partial^3\mathcal{W}}{\partial\phi_D\partial\phi_A\partial\phi_C} - \frac{\psi_C}{\psi_B}\frac{\partial^3\mathcal{W}}{\partial\phi_D\partial\phi_B\partial\phi_C}\right)\right].
\end{aligned}
$$

3.3 Decomposition into Scalar Harmonics and special solutions

A common procedure to analyse the PDE system of equations just derived above is to transform it into a system of ODEs by doing an harmonic decomposition. This is done by expanding the first-order gauge-invariant scalars in terms of the scalar harmonics [95],

$$\Delta = \sum_{\rm n} \Delta_{(\rm n)} Q_{(\rm n)} \tag{3.62}$$

which are comoving eigenfunctions of the operator

$$D^2 Q_{(n)} = -\frac{\rm n^2}{a^2} Q_{(\rm n)}, \quad \dot{Q}_{(\rm n)} = 0. \tag{3.63}$$

This is a usefull procedure that has been followed many times in the literature such as in models of structure formation in the universe [97, 28] and in the study of perturbations of spherically symmetric models, see e.g. [109] and references therein.

Using the time variable τ in (2.18) and the harmonic decomposition, the system of equations (3.57), (3.58) and (3.60) reads

$$\begin{aligned}
\Delta'' =& \left[q + 2 + 2\left(\frac{\alpha_A}{H}\beta_A^2 + \frac{\alpha_B}{H}\beta_B^2\right) + 2\sum_{C\neq A,B}^{N} \beta_C^2 \frac{\alpha_C}{H}\right]\Delta' + \left[\frac{2HA + 3\dot{H} + 2\psi^2 + \dot{A}}{H^2} - \frac{\rm n^2}{a^2H^2}\right]\Delta \\
& - \sum_{A=1}^{N}\sum_{B>A}^{N}\left[\frac{\dot{B}_{AB} + (2H - D_{AB})B_{AB} + C_{AB}}{H^2}\right]\Delta_{[AB]} \\
& + \sum_{A=1}^{N}\sum_{B>A}^{N}\sum_{C\neq A,B}^{N}\frac{B_{AB}}{H}\beta_C^2\left(\frac{\alpha_A}{H}\Delta_{[AC]} - \frac{\alpha_B}{H}\Delta_{[BC]}\right) \\
& + a(t)\sum_{A=1}^{N}\sum_{B>A}^{N}\left[\frac{\dot{C}_{AB} + (5H + D_{AB})C_{AB}}{H^2} + B_{AB}\left(\frac{3\dot{H} + \psi^2 - E_{AB}}{H^2} - 2\frac{\rm q^2}{a^2H^2}\right)\right]v_{[AB]} \\
& - a(t)\sum_{A=1}^{N}\sum_{B>A}^{N}\sum_{C\neq A,B}^{N}\left[\left(B_{AB}F_{AC} - C_{AB}\alpha_A\beta_C^2\right)v_{[AC]} + \left(B_{AB}F_{BC} + C_{AB}\alpha_B\beta_C^2\right)v_{[BC]}\right] \\
& - a(t)\sum_{A=1}^{N}\sum_{B>A}^{N}\sum_{C\neq A,B}^{N}\sum_{D\neq A,B,C}^{N} B_{AB}\left(\frac{\psi_C}{\psi_A}\frac{\partial^2\mathcal{W}}{\partial\phi_C\partial\phi_A} - \frac{\psi_C}{\psi_B}\frac{\partial^2\mathcal{W}}{\partial\phi_C\partial\phi_B}\right)\beta_D^2 v_{[CD]} \\
\Delta'_{[AB]} =& -\frac{D_{AB}}{H}\Delta_{[AB]} + \left(\frac{E_{AB}}{H^2} + \frac{\rm q^2}{a^2H^2}\right)v_{[AB]} + \sum_{C\neq A,B}^{N}\beta_C^2\left(\frac{\alpha_A}{H}\Delta_{[AC]} - \frac{\alpha_B}{H}\Delta_{[BC]}\right) \\
& + a(t)H\sum_{C\neq A,B}^{N}\left[\frac{F_{AC}}{H^2}v_{[AC]} - \frac{F_{BC}}{H^2}v_{[BC]}\right] \\
& + a(t)H\sum_{C\neq A,B}^{N}\left[\frac{1}{H^2}\sum_{D\neq A,B,C}^{N}\left(\frac{\psi_C}{\psi_A}\frac{\partial^2\mathcal{W}}{\partial\phi_C\partial\phi_A} - \frac{\psi_C}{\psi_B}\frac{\partial^2\mathcal{W}}{\partial\phi_C\partial\phi_B}\right)\beta_D^2 v_{[CD]}\right] \\
aHv'_{[AB]} =& -\Delta_{[AB]} + a\left(2H + D_{AB}\right)v_{[AB} + a(t)H\sum_{C\neq A,B}^{N}\left(\frac{\alpha_A}{H}\beta_C^2 v_{[AC]} - \frac{\alpha_B}{H}\beta_C^2 v_{[BC]}\right) .
\end{aligned} \tag{3.64}$$

In the case where the only matter present is a single nonlinear scalar field, then the relative density perturbations are identically zero $\Delta_{[AB]} = 0$ and the velocity perturbations vanish since, by construction, $D^2\phi = 0$. In this case, the system reduces to a *linear homogeneous second-order*

ODE

$$\Delta''_{(\mathrm{n})} - \left[1 - \frac{\dot{H}}{H^2} + \frac{2}{H\psi}\frac{d\mathcal{V}}{d\phi}\right]\Delta'_{(\mathrm{n})} - \left[6 + 6\frac{\dot{H}}{H^2} + 2\frac{\psi^2}{H^2} + \frac{2}{H\psi}\frac{d\mathcal{V}}{d\phi}\left(5 + \frac{1}{H\psi}\frac{d\mathcal{V}}{d\phi}\right) + \frac{2}{H^2}\frac{d^2\mathcal{V}}{d\phi^2} - \frac{\mathrm{n}^2}{a^2H^2}\right]\Delta_{(\mathrm{n})} = 0 \tag{3.65}$$

depending on the background quantities, H, ψ, and the first and second derivative of the potential with respect to ϕ which, in turn, depend only on time. In the case of a perfect fluid with linear equation of state, (1.9) the analogue differential equation can be found in chapter 14 of [198], equation 14.32. The case of dust in a flat background is then solved explicitly, with the particularity of the solution being independent of n while for general γ, the solution can be written in terms of Bessel functions, see Goode [91] for the corresponding Bardeen variable [12].

For scalar field density perturbations no solution is independent of n. However, in order to get solutions we can either solve equation (3.65) for each n or use the so-called *long-wavelength limit*, which amounts to consider solutions with wavelengths larger than the Hubble distance, or equivalently

$$\frac{\mathrm{n}}{aH} \ll 1. \tag{3.66}$$

For example, for the massless scalar field solution, after inserting (2.10) and (2.11) into (3.65), the general solution depending on the wave-number can be written in terms of special functions. In the long-wavelength limit the solution is

$$\Delta(\tau) = \frac{C_1}{4}e^{4\tau} + C_2 \tag{3.67}$$

or in terms of cosmic time t,

$$\Delta = C_2 + \frac{3}{4}C_1 t^{\frac{4}{3}}. \tag{3.68}$$

Even when it is possible to solve explicit the background equations for a given potential, the solutions to the perturbed system are, in principle, impossible to solve. In [211], Zimdahl used the slow-roll approximation in order to simplify the coefficients and study the behaviour of density inhomogeneities during slow-roll inflation. In what follows, we shall apply a dynamical system‘s approach to perform a qualitative analysis of the evolution of density inhomogeneities.

3.4 The Woszczyna-Bruni-Dunsby dynamical systems approach to density inhomogeneities

We have seen in Chapter 2 how a dynamical systems‘ approach, in particular the use of the HNV, is useful in characterizing asymptotic solutions as future attractors and their inflationary character in the background spacetime. We shall now use the system of equations derived in last section and employ a dynamical systems‘ approach initiated by Woszczyna [206, 207, 27, 208] to study the evolution of inhomogeneities. This method was also used to study stability problems in a universe with dust and radiation [31], magnetized cosmologies [101] and LRS Bianchi I models [62]. Following these works we introduce the dimensionless variables

$$\mathcal{U}_{(\mathrm{n})} := \frac{\Delta'_{(\mathrm{n})}}{\Delta_{(\mathrm{n})}} \quad , \quad \mathcal{X}_{[AB]} := \frac{\Delta_{[AB]}}{\Delta} \quad , \quad \mathcal{Y}_{[AB]} := \frac{aHv_{[AB]}}{\Delta} \ , \tag{3.69}$$

and we arrive at the following result:

Proposition 3.1. *The evolution for the first order scalar perturbations on a FL-scalar fields background, with arbitrary smooth potentials, is given by the following system of differential equations for the state vector* $\big((\Psi_A,\Phi_A,\Theta_A),\mathcal{U},\mathcal{X}_{[AB]},\mathcal{Y}_{[AB]}\big)$*:*

$$\begin{aligned}
\mathcal{U}' =& -\mathcal{U}^2-\xi\mathcal{U}-\zeta+\sum_{A=1}^{N}\sum_{B>A}^{N}\gamma_{AB}\mathcal{X}_{[AB]}+\sum_{A=1}^{N}\sum_{B>A}^{N}\eta_{AB}\mathcal{Y}_{[AB]}\\
&+\sum_{A=1}^{N}\sum_{B>A}^{N}\sum_{C\neq A,B}^{N}\frac{\Psi_C^2}{\Psi^2}\frac{B_{AB}}{H}\left(\frac{\alpha_A}{H}\mathcal{X}_{[AC]}-\frac{\alpha_B}{H}\mathcal{X}_{[BC]}\right)\\
&-\sum_{A=1}^{N}\sum_{B>A}^{N}\sum_{C\neq A,B}^{N}\left(\frac{(B_{AB}F_{AC}-C_{AB}\alpha_A\beta_C^2)}{H^3}\mathcal{Y}_{[AC]}+\frac{(B_{AB}F_{BC}-C_{AB}\alpha_B\beta_C^2)}{H^3}\mathcal{Y}_{[BC]}\right)\\
&-\sum_{A=1}^{N}\sum_{B>A}^{N}\sum_{C\neq A,B}^{N}\sum_{D\neq A,B,C}^{N}\frac{B_{AB}}{H^3}\left(\frac{\Psi_C}{\Psi_A}\frac{\partial^2\mathcal{W}}{\partial\phi_C\partial\phi_A}-\frac{\Psi_C}{\Psi_B}\frac{\partial^2\mathcal{W}}{\partial\phi_C\partial\phi_B}\right)\beta_D^2\mathcal{Y}_{[CD]}\\
\mathcal{X}'_{[AB]} =& -\mathcal{U}\mathcal{X}_{[AB]}+\varsigma_{AB}\mathcal{X}_{[AB]}+\varpi_{AB}\mathcal{Y}_{[AB]}+\sum_{C\neq A,B}\beta_C^2\left(\frac{\alpha_A}{H}\mathcal{X}_{[AC]}-\frac{\alpha_B}{H}\mathcal{X}_{[BC]}\right)\\
&+\sum_{C\neq A,B}^{N}\left[\frac{F_{AC}}{H^2}\mathcal{Y}_{[AC]}-\frac{F_{BC}}{H^2}\mathcal{Y}_{[BC]}+\frac{1}{H^2}\sum_{D\neq A,B,C}^{N}\left(\frac{\Psi_C}{\Psi_A}\frac{\partial^2\mathcal{W}}{\partial\phi_C\partial\phi_A}-\frac{\Psi_C}{\Psi_B}\frac{\partial^2\mathcal{W}}{\partial\phi_C\partial\phi_B}\right)\beta_D^2\mathcal{Y}_{[CD]}\right]\\
\mathcal{Y}'_{[AB]} =& -\mathcal{U}\mathcal{Y}_{[AB]}+\iota_{AB}\mathcal{Y}_{[AB]}-\mathcal{X}_{[AB]}\\
&+\sum_{C\neq A,B}^{N}\left(\frac{\alpha_A}{H}\beta_C^2\mathcal{Y}_{[AC]}-\frac{\alpha_B}{H}\beta_C^2\mathcal{Y}_{[BC]}\right)\\
\Psi'_A =&(q-2)\Psi_A-n\sqrt{6}\left[\Phi_A^{2n-1}\frac{d\Phi_A}{d\phi_A}+\Theta_A^{2n-1}\frac{d\Theta}{d\phi_A}\right]\\
\Phi'_A =&\frac{1}{n}(q+1)\Phi_A+\sqrt{6}\frac{d\Phi_A}{d\phi_A}\Psi_A\\
\Theta' =&\frac{1}{n}(q+1)\Theta+\sqrt{6}\sum_{A=1}^{N}\frac{\partial\Theta}{\partial\phi_A}\Psi_A
\end{aligned}\tag{3.70}$$

subject to the background constraint equation

$$\sum_{A=1}^{N}\Psi_A^2+\sum_{A=1}^{N}\Phi_A^{2n}+\Theta^{2n}=1-K, \tag{3.71}$$

where

$$q=2\sum_{A=1}^{N}\Psi_A^2-\sum_{A=1}^{N}\Phi_A^{2n}-\Theta^{2n}$$

and the coefficients are given by

$$\begin{aligned}
\xi =& -\left[q+2+2\left(\frac{\alpha_A}{H}\beta_A^2+\frac{\alpha_B}{H}\beta_B^2\right)+2\sum_{C\neq A,B}^{N}\beta_C^2\frac{\alpha_C}{H}\right]\\
=& -\left[2+2\sum_{A=1}^{N}\Psi_A^2-\sum_{A=1}^{N}\Phi_A^{2n}-\Theta^{2n}+\frac{12n}{\sqrt{6}}\sum_{A=1}^{N}\frac{\Psi_A^2}{\Psi^2}\left(\frac{\Phi_A^{2n-1}}{\Psi_A}\frac{d\Phi_A}{d\phi_A}+\frac{\Theta^{2n-1}}{\Psi_A}\frac{\partial\Theta}{\partial\phi_A}\right)\right]
\end{aligned}\tag{3.72}$$

$$
\begin{aligned}
\zeta =& -\left[\frac{2HA+3\dot{H}+2\psi^2+\dot{A}}{H^2}-\frac{\mathrm{n}^2}{a^2H^2}\right]\\
=& -\left[6\left(1+\frac{\dot{H}}{H^2}\right)+2\frac{\psi^2}{H^2}+2\left[\frac{\alpha_A}{H}\beta_A^2\left(5+\frac{\alpha_A}{H}\right)+\frac{\alpha_B}{H}\beta_B^2\left(5+\frac{\alpha_B}{H}\right)\right]-4\frac{(\alpha_A-\alpha_B)}{H}B_{AB}\right]\\
& -\frac{2}{H^2}\left[\left(\frac{d^2\mathcal{V}}{d\phi_A^2}+\frac{\partial^2\mathcal{W}}{\partial\phi_A^2}+\frac{\psi_B}{\psi_A}\frac{\partial^2\mathcal{W}}{\partial\phi_B\partial\phi_A}\right)\beta_A^2+\left(\frac{d^2\mathcal{V}}{d\phi_B^2}+\frac{\partial^2\mathcal{W}}{\partial\phi_B^2}+\frac{\psi_A}{\psi_B}\frac{\partial^2\mathcal{W}}{\partial\phi_A\partial\phi_B}\right)\beta_B^2\right]\\
& -2\sum_{C\neq A,B}^{N}\left[-2\left(\frac{\alpha_C}{H}-\frac{\alpha_A}{H}\right)\frac{B_{CA}}{H}-2\left(\frac{\alpha_C}{H}-\frac{\alpha_B}{H}\right)\frac{B_{CB}}{H}\right]\\
& -\frac{2}{H^2}\sum_{C\neq A,B}^{N}\left[\frac{\Psi_A}{\Psi_C}\beta_C^2\frac{\partial^2\mathcal{W}}{\partial\phi_A\partial\phi_C}+\frac{\Psi_B}{\Psi_C}\beta_C^2\frac{\partial^2\mathcal{W}}{\partial\phi_B\partial\phi_C}+\Psi_C\left(\frac{\beta_A^2}{\Psi_A}\frac{\partial^2\mathcal{W}}{\partial\phi_C\partial\phi_A}+\frac{\beta_B^2}{\Psi_B}\frac{\partial^2\mathcal{W}}{\partial\phi_C\partial\phi_B}\right)\right]\\
& -2\sum_{C\neq A,B}^{N}\sum_{D\neq A,B,C}^{N}\left[\left(3+\frac{\alpha_C}{H}\right)\frac{\alpha_C}{H}+\frac{1}{H^2}\frac{d^2\mathcal{V}}{d\phi_C^2}+\frac{1}{H^2}\frac{\partial^2\mathcal{W}}{\partial\phi_C^2}+\frac{1}{H^2}\frac{\Psi_D}{\Psi_C}\frac{\partial^2\mathcal{W}}{\partial\phi_D\partial\phi_C}\right]\beta_C^2
\end{aligned}
\tag{3.73}
$$

$$
\begin{aligned}
\gamma_{AB} =& -\left[\frac{\dot{B}_{AB}+(2H-D_{AB})B_{AB}+C_{AB}}{H^2}\right]\\
=& -\left[8+\frac{(\alpha_A+\alpha_B)}{H}+3\frac{(\alpha_A-\alpha_B)}{H}\left(\beta_A^2-\beta_B^2\right)\right]\frac{B_{AB}}{H}\\
& -\frac{1}{H^2}\left[2\left(\frac{d^2\mathcal{V}}{d\phi_A^2}-\frac{d^2\mathcal{V}}{d\phi_B^2}\right)+2\left(\frac{\partial^2\mathcal{W}}{\partial\phi_A^2}-\frac{\partial^2\mathcal{W}}{\partial\phi_B^2}\right)+\left(\frac{\partial^2\mathcal{W}}{\partial\phi_B\partial\phi_A}+\frac{\partial^2\mathcal{W}}{\partial\phi_A\partial\phi_B}\right)\left(\frac{\psi_B}{\psi_A}-\frac{\psi_A}{\psi_B}\right)\right]\beta_A^2\beta_B^2\\
& +\frac{B_{AB}}{H}\sum_{C\neq A,B}^{N}\left[\left(2\frac{\alpha_A}{H}+2\frac{\alpha_B}{H}-3\frac{\alpha_C}{H}\right)\beta_C^2\right]\\
& -\frac{1}{H^2}\sum_{C\neq A,B}^{N}\left[\frac{\psi_C}{\psi_A}\left(\frac{\partial^2\mathcal{W}}{\partial\phi_A\partial\phi_C}+\frac{\partial^2\mathcal{W}}{\partial\phi_C\partial\phi_A}\right)-\frac{\psi_C}{\psi_B}\left(\frac{\partial^2\mathcal{W}}{\partial\phi_B\partial\phi_C}+\frac{\partial^2\mathcal{W}}{\partial\phi_C\partial\phi_B}\right)\right]
\end{aligned}
$$

where in terms of the (2.17) we have

$$
\frac{d\mathcal{V}_A}{d\phi_A}=6nH^2\Phi_A^{2n-1}\frac{d\Phi_A}{d\phi_A}\quad,\quad\frac{d^2\mathcal{V}_A}{d\phi_A^2}=6nH^2\left[(2n-1)\Phi_A^{2(n-1)}\left(\frac{d\Phi_A}{d\phi_A}\right)^2+\Phi_A^{2n-1}\frac{d^2\Phi_A}{d\phi_A^2}\right]
$$

$$
\alpha_A=\frac{\sqrt{6}nH}{\Psi_A}\left[\Phi_A^{2n-1}\frac{d\Phi_A}{d\phi_A}+\Theta^{2n-1}\frac{\partial\Theta}{\partial\phi_A}\right]\quad,\quad\beta_A^2=\frac{\Psi_A^2}{\Psi^2}
\tag{3.74}
$$

where $\Psi^2=\sum_A^N\Psi_A^2$ and

$$
\frac{\partial^2\mathcal{W}}{\partial\phi_B\partial\phi_A}=6nH^2\left[(2n-1)\Theta^{2(n-1)}\left(\frac{\partial\Theta}{\partial\phi_B}\right)\left(\frac{\partial\Theta}{\partial\phi_A}\right)+\Theta^{2n-1}\frac{\partial^2\Theta}{\partial\phi_B\partial\phi_A}\right]
$$

$$
\frac{d^3\mathcal{V}_A}{d\phi_A^3}=6nH^2(2n-1)\left[3\Phi^{2(n-1)}\frac{d\Phi_A}{d\phi_A}\frac{d^2\Phi_A}{d\phi_A^2}+2(n-1)\Phi_A^{2\left(n-\frac{3}{2}\right)}\left(\frac{d\Phi_A}{d\phi_A}\right)^3\right]+6nH^2\Phi^{2n-1}\frac{d^3\Phi_A}{d\phi_A^3}
$$

$$
\begin{aligned}
\eta_{AB} =& \left[\frac{\dot{C}_{AB} + (5H + D_{AB})\, C_{AB}}{H^3} + \frac{B_{AB}}{H}\left(\frac{3\dot{H} + \psi^2 - E_{AB}}{H^2} - 2\frac{\mathrm{q}^2}{a^2 H^2}\right)\right] \\
=& \left\{6\frac{\dot{H}}{H^2} + 24 + \frac{\psi^2}{H^2} + \frac{(\alpha_A^2\beta_A^2 + \alpha_B^2\beta_B^2)}{H^2} + 3\frac{D_{AB}}{H} - \frac{E_{AB}}{H^2} - \frac{q^2}{a^2H^2}\right\}\frac{B_{AB}}{H} \\
&+ \left\{2\frac{(\alpha_A - \alpha_B)}{H}\left[\left[3 + \left(\frac{\alpha_A}{H}\beta_A^2 + \frac{\alpha_B}{H}\beta_B^2\right)\right](\beta_A^2 - \beta_B^2) - \frac{B_{AB}}{H}\right]\right\}\frac{B_{AB}}{H} \\
&+ \left\{\frac{(\alpha_A + \alpha_B)}{H}\left[3 + \left(\frac{\alpha_A}{H}\beta_A^2 + \frac{\alpha_B}{H}\beta_B^2\right)\right] + \left[11 + \frac{D_{AB}}{H}\right]\left(\frac{\alpha_A}{H}\beta_A^2 + \frac{\alpha_B}{H}\beta_B^2\right)\right\}\frac{B_{AB}}{H} \\
&+ \frac{1}{H^2}\left[\left(\frac{d^2\mathcal{V}}{d\phi_A^2} - \frac{d^2\mathcal{V}}{d\phi_B^2}\right) + \left(\frac{\partial^2\mathcal{W}}{\partial\phi_A^2} - \frac{\partial^2\mathcal{W}}{\partial\phi_B^2}\right) - \left(\frac{\Psi_A}{\Psi_B} - \frac{\Psi_B}{\Psi_A}\right)\frac{\partial^2\mathcal{W}}{\partial\phi_A\partial\phi_B}\right]\left[\left(8 + \left(\frac{\alpha_A}{H}\beta_A^2 + \frac{\alpha_B}{H}\beta_B^2\right)\frac{D_{AB}}{H}\right)\beta_A^2\beta_B^2\right] \\
&+ \frac{1}{H^2}\left[\left(\frac{d^2\mathcal{V}}{d\phi_A^2} - \frac{d^2\mathcal{V}}{d\phi_B^2}\right) + \left(\frac{\partial^2\mathcal{W}}{\partial\phi_A^2} - \frac{\partial^2\mathcal{W}}{\partial\phi_B^2}\right) - \left(\frac{\Psi_A}{\Psi_B} - \frac{\Psi_B}{\Psi_A}\right)\frac{\partial^2\mathcal{W}}{\partial\phi_A\partial\phi_B}\right]\left[2\left(\beta_A^2 - \beta_B^2\right)\frac{B_{AB}}{H}\right] \\
&\frac{1}{H^2}\left[\left(\frac{d^2\mathcal{V}}{d\phi_A^2} + \frac{\partial^2\mathcal{W}}{\partial\phi_A^2} + \frac{\Psi_B 1}{\Psi_A}\mathcal{W}''_{AB}\right)\beta_A^2 + \left(\frac{d^2\mathcal{V}}{d\phi_B^2} + \frac{\partial^2\mathcal{W}}{\partial\phi_B^2} + \frac{\Psi_A}{\Psi_B}\mathcal{W}''_{BA}\right)\beta_B^2 + \left(\frac{\Psi_A}{\Psi_B} - \frac{\Psi_B}{\Psi_A}\right)\frac{\partial^2\mathcal{W}}{\partial\phi_A\partial\phi_B}\right]\frac{B_{AB}}{H} \\
&+ \frac{1}{H^3}\left[\left(\frac{d^3\mathcal{V}}{d\phi_A^3} + \frac{\partial^3\mathcal{W}}{\partial\phi_A^3} - \frac{\partial^3\mathcal{W}}{\partial\phi_A\partial\phi_B^2} - \frac{\partial^3\mathcal{W}}{\partial\phi_B^2\partial\phi_A} - \frac{\psi_A}{\psi_B}\frac{\partial^3\mathcal{W}}{\partial\phi_A\partial\phi_B\partial\phi_A}\right)\psi_A\right]\beta_A^2\beta_B^2 \\
&- \frac{1}{H^3}\left[\left(\frac{d^3\mathcal{V}}{d\phi_B^3} + \frac{\partial^3\mathcal{W}}{\partial\phi_B^3} - \frac{\partial^3\mathcal{W}}{\partial\phi_B\partial\phi_A^2} - \frac{\partial^3\mathcal{W}}{\partial\phi_A^2\partial\phi_B} - \frac{\psi_B}{\psi_A}\frac{\partial^3\mathcal{W}}{\partial\phi_B\partial\phi_A\partial\phi_B}\right)\psi_B\right]\beta_A^2\beta_B^2 \\
&+ \left(\frac{5H + D_{AB}}{H}\right)\sum_{C\neq A,B}^{N}\left[\alpha_C\beta_C^2 B_{AB} + \left(\frac{\psi_C}{\psi_A}\frac{\partial^2\mathcal{W}}{\partial\phi_A\partial\phi_C} - \frac{\psi_C}{\psi_B}\frac{\partial^2\mathcal{W}}{\partial\phi_B\partial\phi_C}\right)\right] \\
&+ \left(\frac{3H + \alpha_A\beta_A^2 + \alpha_B\beta_B^2}{H}\right)\sum_{C\neq A,B}^{N}\left[-2B_{AB}\left(\alpha_A + \alpha_B - 2\alpha_C\right)\beta_C^2 + \left(\frac{\psi_C}{\psi_A}\frac{\partial^2\mathcal{W}}{\partial\phi_C\partial\phi_A} - \frac{\psi_C}{\psi_B}\frac{\partial^2\mathcal{W}}{\partial\phi_C\partial\phi_B}\right)\beta_A^2\beta_B^2\right] \\
&+ 2\sum_{C\neq A,B}^{N}\left[\left(\frac{d^2\mathcal{V}}{d\phi_A^2} - \frac{d^2\mathcal{V}}{d\phi_B^2}\right) + \left(\frac{\partial^2\mathcal{W}}{\partial\phi_A^2} - \frac{\partial^2\mathcal{W}}{\partial\phi_B^2}\right) - \left(\frac{\psi_A}{\psi_B}\frac{\partial^2\mathcal{W}}{\partial\phi_B\partial\phi_A} - \frac{\psi_B}{\psi_A}\frac{\partial^2\mathcal{W}}{\partial\phi_A\partial\phi_B}\right)\right]\left(B_{CA}\beta_B^2 + B_{CB}\beta_A^2\right) \\
&+ \sum_{C\neq A,B}^{N}\left[(3H + \alpha_C)\alpha_C + \frac{d^2\mathcal{V}}{d\phi_C^2} + \frac{\partial^2\mathcal{W}}{\partial\phi_C^2} + \frac{\psi_A}{\psi_C}\frac{\partial^2\mathcal{W}}{\partial\phi_A\partial\phi_C} + \frac{\psi_B}{\psi_C}\frac{\partial^2\mathcal{W}}{\partial\phi_B\partial\phi_C}\right]\beta_A^2\beta_B^2 \\
&+ \sum_{C\neq A,B}^{N}\left[-2\left(\alpha_A B_{AC} + \alpha_B B_{BC}\right)B_{AB} - 2\alpha_C B_{AB}\left(B_{CA} + B_{CB}\right) + \alpha_C\beta_C^2\dot{B}_{AB}\right] \\
&+ \sum_{C\neq A,B}^{N}\left[\psi_C\left(\frac{\beta_A^2}{\psi_A}\frac{\partial^2\mathcal{W}}{\partial\phi_C\partial\phi_A} + \frac{\beta_B^2}{\psi_B}\frac{\partial^2\mathcal{W}}{\partial\phi_C\partial\phi_B}\right)B_{AB} + (\alpha_A - \alpha_C)\frac{\psi_C}{\psi_A}\frac{\partial^2\mathcal{W}}{\partial\phi_A\partial\phi_C} - (\alpha_B - \alpha_C)\frac{\psi_C}{\psi_B}\frac{\partial^2\mathcal{W}}{\partial\phi_B\partial\phi_C}\right] \\
&+ \sum_{C\neq A,B}^{N}\psi_C\left[\frac{\partial^3\mathcal{W}}{\partial\phi_C\partial\phi_A^2} - \frac{\partial^3\mathcal{W}}{\partial\phi_C\partial\phi_B^2} - \left(\frac{\psi_A}{\psi_B}\frac{\partial^3\mathcal{W}}{\partial\phi_A\partial\phi_B\partial\phi_C} - \frac{\psi_B}{\psi_A}\frac{\partial^3\mathcal{W}}{\partial\phi_B\partial\phi_A\partial\phi_C}\right)\right]\beta_A^2\beta_B^2 \\
&+ \sum_{C\neq A,B}^{N}\left[\frac{\psi_C}{\psi_A}\left(\psi_A\frac{\partial^3\mathcal{W}}{\partial\phi_C\partial\phi_A^2} + \psi_B\frac{\partial^3\mathcal{W}}{\partial\phi_B\partial\phi_A\partial\phi_C} + \psi_C\frac{\partial^3\mathcal{W}}{\partial\phi_C\partial\phi_A\partial\phi_C}\right)\right] \\
&- \sum_{C\neq A,B}^{N}\left[\frac{\psi_C}{\psi_B}\left(\psi_A\frac{\partial^3\mathcal{W}}{\partial\phi_A\partial\phi_B\partial\phi_C} + \psi_B\frac{\partial^3\mathcal{W}}{\partial\phi_C\partial\phi_B^2} + \psi_C\frac{\partial^3\mathcal{W}}{\partial\phi_C\partial\phi_B\partial\phi_C}\right)\right] \\
&+ \sum_{C\neq A,B}^{N}\sum_{D\neq A,B,C}^{N}\left[-2\alpha_C B_{AB}B_{CD} + B_{AB}\beta_C^2\frac{\psi_D}{\psi_C}\frac{\partial^2\mathcal{W}}{\partial\phi_D\partial\phi_C} + \psi_D\left(\frac{\psi_C}{\psi_A}\frac{\partial^3\mathcal{W}}{\partial\phi_D\partial\phi_A\partial\phi_C} - \frac{\psi_C}{\psi_B}\frac{\partial^3\mathcal{W}}{\partial\phi_D\partial\phi_B\partial\phi_C}\right)\right].
\end{aligned}
$$

$$
\varsigma_{AB} = -\frac{D_{AB}}{H} = -\frac{1}{H}\left(\alpha_A\beta_B^2 + \alpha_B\beta_A^2\right)
$$

$$\varpi_{AB} = \frac{E_{AB}}{H^2} + \frac{\mathrm{q}^2}{a^2H^2}$$

$$\iota_{AB} = \left[1 - (1+q) + 2 + \frac{D_{AB}}{H}\right]$$

As explained by Dunsby in Chapter 14 of [198], the variable $\mathcal{U}_{\mathrm{n}}$ should be viewed as $\tan(\theta_{\mathrm{n}})$, where $0 \leq \theta_{\mathrm{n}} < 2\pi$, is the polar angle in the plane $(\Delta_{\mathrm{n}}, \Delta'_{\mathrm{n}})$. For scalar fields in a flat background, we consider the variables subset defined by

$$\sum_{A=1}^{N} \Psi_A^2 + \sum_{A=1}^{N} \Phi_A^2 = 1 \,, \quad -\infty < \mathcal{U}_{\mathrm{n}} < +\infty \,, \quad -\infty < \mathcal{X}_{[AB]} < +\infty \,, \quad -\infty < \mathcal{Y}_{[AB]} < +\infty \tag{3.75}$$

and regard the state space as $\mathbb{S}^N \times \mathbb{S}^1 \times \mathbb{R}^{N(N-1)}$. As we shall see ahead, in the case of assisted power-law inflation, the fixed points are restricted to the compact subset $\mathbb{S}^N \times \mathbb{S}^1$. When only a single scalar field is present, then we can use either the variable Ψ or Φ since they are related through the flat Friedman constraint (2.23), and the state space is regarded as the *cylinder* $[0,1] \times \mathbb{S}^1$.

The use of the variable $\mathcal{U}_{(\mathrm{n})}$ makes the analysis of the system's stability quite transparent: If an orbit is asymptotic to an equilibrium point, the perturbation approaches a stationary state either: decaying to zero if $(\mathcal{U} < 0)$, growing if $(\mathcal{U} > 0)$ or having a constant value $(\mathcal{U} = 0)$. If the orbit is asymptotic to a periodic orbit in the cylinder, the perturbation propagates as waves (see pag. 296 and 297 in Chapter 14 of [198]).

We shall now investigate separately the cases of one and two scalar fields.

3.4.1 Single scalar field

In the case of a single scalar field, we obtain from Proposition 3.1 the following result

The evolution for the phase of first order scalar perturbations of FL-nonlinear scalar field models, is given by the following system of differential equations for the state vector $\big((\Psi,\Phi),\mathcal{U}_{(n)}\big)$*:*

$$\begin{aligned}
\mathcal{U}'_{(n)} &= -\mathcal{U}^2_{(n)} - \xi(\Psi,\Phi)\mathcal{U}_{(n)} - \zeta(\Psi,\Phi) \\
\Psi' &= 2\Psi^3 - \left(2+\Phi^{2n}\right)\Psi - n\sqrt{6}\Phi^{2n-1}\frac{d\Phi}{d\phi} \\
\Phi' &= -\frac{\Phi^{2n+1}}{n} + \frac{\left(1+2\Psi^2\right)}{n}\Phi + \sqrt{6}\Psi\frac{d\Phi}{d\phi}
\end{aligned} \tag{3.76}$$

subject to the background constraint equation

$$\Psi^2 + \Phi^{2n} + K = 1 \tag{3.77}$$

and with the coefficients given by

$$\begin{aligned}
\xi(\Phi,\Psi) =& -\left[2 + 2\Psi^2 - \Phi^{2n} + \frac{12n}{\sqrt{6}}\frac{\Phi^{2n-1}}{\Psi}\frac{d\Phi}{d\phi}\right] \\
\zeta(\Phi,\Psi) =& -6\Phi^{2n} - \frac{6n}{\sqrt{6}}\frac{\Phi^{2n-1}}{\Psi}\frac{d\Phi}{d\phi}\left(10 + \frac{12n}{\sqrt{6}}\frac{\Phi^{2n-1}}{\Psi}\frac{d\Phi}{d\phi}\right) \\
& - 12n\left((2n-1)\Phi^{2n-2}\left(\frac{d\Phi}{d\phi}\right)^2 + \Phi^{2n-1}\frac{d^2\Phi}{d\phi^2}\right) + \frac{\mathrm{n}^2}{a^2H^2}.
\end{aligned} \tag{3.78}$$

As in the qualitative analysis in the background spacetime, we are interested on the dynamics in the invariant set of zero curvature models, corresponding to the background of spatially flat hypersurfaces. In particular, since the background evolution equations forms an autonomous subsystem, the background fixed points, given by Lemma 1, are also fixed points of (3.76) and the following result holds

Lemma 2. *For $K = 0$, the fixed points of system* (3.76) *are given by the conditions:*

$$\begin{aligned} \mathcal{P} \, : \quad \frac{d\Phi}{d\phi} &= -\frac{\sqrt{6}}{2n}\Phi\Psi \\ \mathcal{U}^{\pm}_{(n)}(\mathcal{P}) &= \frac{1}{2}\left(-\xi(\mathcal{P}) \pm \sqrt{\xi^2(\mathcal{P}) - 4\zeta(\mathcal{P})}\right) \end{aligned} \tag{3.79}$$

subject to $\Psi^2 + \Phi^{2n} = 1$, where

$$\begin{aligned} \xi(\mathcal{P}) &= -\left[4 - 9\Phi^{2n}_{\mathcal{P}}\right] \\ \zeta(\mathcal{P}) &= 18\left(\frac{1}{n} - \frac{2}{3}\right)\Phi^{2n}_{\mathcal{P}} - 18\left(\frac{1}{n} - 1\right)\Phi^{4n}_{\mathcal{P}} - 12n\Phi^{2n-1}_{\mathcal{P}}\left(\frac{d^2\Phi}{d\phi^2}\right)_{\mathcal{P}} + \frac{\mathrm{n}^2}{H^2a^2} \,. \end{aligned} \tag{3.80}$$

From the above considerations, and the fact that in a flat background we can reduce the linearised matrix at $\mathcal{P}$ given by (2.25), it also follows that the eigenvalues of the linearized system around the fixed points are the ones given by (2.26) together with

$$\omega_{\mathcal{U}^{\pm}(\mathcal{P})} = -2\mathcal{U}^{\pm}(\mathcal{P}) - \xi(\mathcal{P}) = \mp\sqrt{\xi^2(\mathcal{P}) - 4\zeta(\mathcal{P})} \,. \tag{3.81}$$

Thus, from (3.79), the fixed points exist if

$$\xi^2(\mathcal{P}) - 4\zeta(\mathcal{P}) \geq 0 \,, \tag{3.82}$$

and reduce to a single point when the equality is verified. In this case, the eigenvalues coincide and, from (3.81), are identically zero, resulting in a saddle point. If this is not the case, and if $\mathcal{P}_a$ is an atractor point of the background dynamical system, then it follows that $\mathcal{U}^+(\mathcal{P}_a)$ is the late time attractor of (3.76) having the following properties:

$$\begin{aligned} &\mathcal{U}^{+}_{(n)}(\mathcal{P}_a) < 0 \quad \text{if} \quad \zeta(\mathcal{P}_a) > 0 \quad \text{and} \quad \xi(\mathcal{P}_a) > 0, \\ &\mathcal{U}^{+}_{(n)}(\mathcal{P}_a) = 0 \quad \text{if} \quad \zeta(\mathcal{P}_a) = 0, \\ &\mathcal{U}^{+}_{(n)}(\mathcal{P}_a) > 0 \quad \text{if} \quad \xi(\mathcal{P}_a) < 0. \end{aligned} \tag{3.83}$$

3.4.2 Exponential potential: Power-law inflation

We have seen in Subsection 2.3.1 that, for an exponential potential, the background subsystem has two fixed points, $\mathcal{P}_0$ and $\mathcal{P}_1$ given by (2.30) and (2.31), respectively. Then, for the perturbed system (3.76) it follows from (3.79) that there are four fixed points

$$\left(\mathcal{P}_{0,1}, \mathcal{U}^{\pm}_{(\mathrm{n})}(\mathcal{P}_{0,1})\right), \tag{3.84}$$

where the coefficients (3.80) are

$$\xi(\mathcal{P}_0) = -4 \quad \text{and} \quad \xi(\mathcal{P}_1) \;\; = 5 - \frac{3}{2}\lambda^2 \tag{3.85}$$

$$\zeta(\mathcal{P}_0) = \frac{\mathrm{n}^2}{a^2H^2} \quad \text{and} \quad \zeta(\mathcal{P}_1) \quad = 6 - 4\lambda^2 + \frac{\lambda^4}{2} + \frac{\mathrm{n}^2}{a^2H^2}. \tag{3.86}$$

Also, the solutions are real if (3.82) is satisfied, which in turn implies that the wave number satisfies

$$\mathcal{P}_0 \; : \quad \mathrm{n}^2 \leq \mathrm{n}^2_{crit} = \frac{a^2H^2}{4}, \tag{3.87}$$

$$\mathcal{P}_1 \; : \quad \mathrm{n}^2 \leq \mathrm{n}^2_{crit} = \frac{a^2H^2}{4}\left(1 + \lambda^2 + \frac{\lambda^4}{4}\right). \tag{3.88}$$

When $\mathrm{n} = \mathrm{n}_{crit}$, the points $\mathcal{U}^{\pm}_{(\mathrm{n})}$ merge into a single saddle point. For $\mathrm{n} > \mathrm{n}_{crit}$, the fixed points cease to exist, the orbit is periodic and the perturbations behave as waves. We also saw that $\mathcal{P}_1$ was the late time attractor corresponding to the flat power-law solution of the background dynamical system (2.19). Thus, the attractor point of the dynamical system (3.76) is $\mathcal{U}^{+}_{\mathrm{n}}(\mathcal{P}_1)$ and we obtain from (3.83) that there exists

$$\mathrm{n}^2_{(-)}(\mathcal{P}_1) = a^2H^2\left(-6 + 4\lambda^2 - \frac{\lambda^4}{2}\right)$$

such that, if $\mathrm{n}^2_{(-)}(\lambda) < \mathrm{n}^2 < \mathrm{n}^2_{crit}(\lambda)$, then $\mathcal{U}^{+}_{(\mathrm{n})}(\mathcal{P}_2) < 0$ and the perturbations decay. When $\mathrm{n}^2 = \mathrm{n}^2_{(-)}(\lambda)$, then $\zeta(\mathcal{P}_1) = 0$ and the perturbations tend to a constant. Therefore, when the slope parameter satisfies $0 < \lambda < \sqrt{2}$, the density perturbation modes decay for all wavelenghts in the range for which there exists fixed points, i.e. as long as $0 \leq \mathrm{n}^2 < \mathrm{n}^2_{crit}(\lambda)$. In particular, for $\lambda = \sqrt{2}$ the modes decay, except in the long wavelenght limit, for which the perturbations tend to constant since $\mathrm{n}^2_{(-)}(\sqrt{2}) = 0$ (see Figs. 3.1, 3.2, 3.3, 3.4 and 3.5).

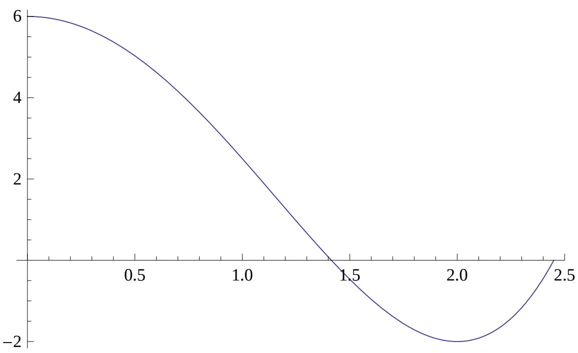

Figure 3.1: Plot of $\zeta(\lambda)$ using the long wavelenght limit. The positive region gives the values of λ for which the perturbations decay, for an exponential potential. The zeros are at $\lambda = \sqrt{2}$ and $\lambda = \sqrt{6}$. Notice that if $\mathrm{n}^2 > 0$ the graph is shifted along the ζ axis and, consequently, the interval for which it is positive gets bigger.

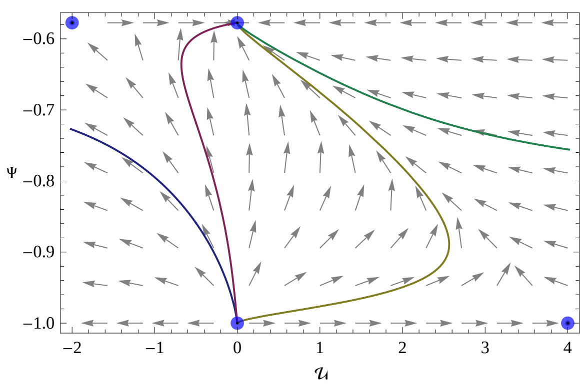

Figure 3.2: Density perturbations described by orbits in the phase plane $(\Psi, \mathcal{U}_{(n)})$ for an exponential potential with $\lambda = \sqrt{2}$ in the long wavelength limit. The figure shows the equilibrium points, one of which is the future attractor $(\mathcal{P}, \mathcal{U}^{+}_{(n)})$ having $\mathcal{U}_{(n)} = 0$.

Moreover, if $\sqrt{2} < \lambda < \sqrt{10/3}$ there are perturbation modes which can either decay, tend to constant or grow, depending on the value of n^2 (see Figs. 3.6, 3.7). For $\lambda = \sqrt{10/3}$, all perturbations grow, while n reaches its critical value. In this case, the saddle point is then at $\mathcal{U}_{\mathrm{n}_{crit}} = 0$ (see Figs. 3.8, 3.9, 3.10). Finally, for $\sqrt{\frac{10}{3}} \leq \lambda < \sqrt{6}$, all modes grow for all wavelenghts in the range for which there exist fixed points.

Notice that, by taking the long wavelength limit, $\mathcal{U}^{-}(\mathcal{P}_0) = 0$ (see Figs 3.2 and 3.7) which corresponds to the constant mode of (3.68), while $\mathcal{U}^{+}(\mathcal{P}_0) > 0$ corresponds to the growing mode.

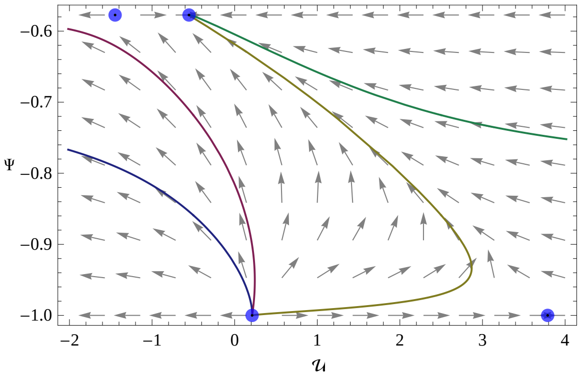

Figure 3.3: Density perturbations described by orbits in the phase plane $(\Psi, \mathcal{U}_{(\mathrm{n})})$ for an exponential potential with $\lambda = \sqrt{2}$ for $\frac{\mathrm{n}^2}{a^2H^2} = 0.8 > \frac{\mathrm{n}^2_{(-)}(\sqrt{2})}{a^2H^2} = 0$. The figure shows the equilibrium points, one of which is the future attractor $(\mathcal{P}, \mathcal{U}^{+}_{(n)})$ having $\mathcal{U}_{(n)} < 0$.

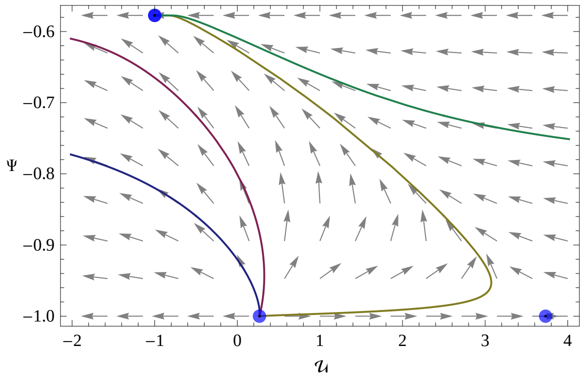

Figure 3.4: Density perturbations described by orbits in the phase plane $(\Psi, \mathcal{U}_{(n)})$ for an exponential potential with $\lambda = \sqrt{2}$ and $\mathrm{n}^2 = \mathrm{n}^2_{crit}(\sqrt{2}) = 1$. The figure shows the saddle point in the region $\mathcal{U}_{(n)} < 0$.

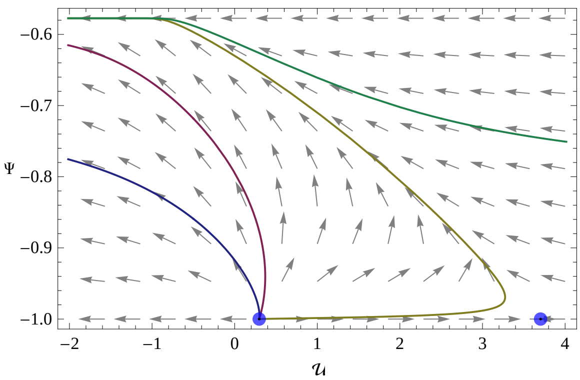

Figure 3.5: Density perturbations described by orbits in the phase plane $(\Psi, \mathcal{U}_{(n)})$ for an exponential potential with $\lambda = \sqrt{2}$ and $\mathrm{n}^2 > \mathrm{n}^2_{crit}(\sqrt{2}) = 1$. For this value of the slope parameter the orbits are periodic.

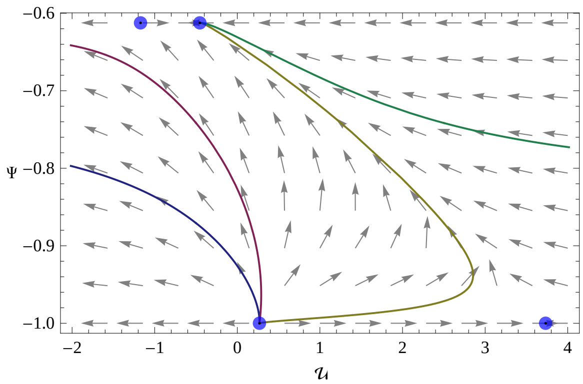

Figure 3.6: Density perturbations described by orbits in the phase plane $(\Psi, \mathcal{U}_{(n)})$ for an exponential potential with $\lambda = 1.5$ and $\frac{\mathrm{n}^2}{a^2H^2} = 1$. The figure shows the equilibrium points, one of which is the future attractor $(\mathcal{P}, \mathcal{U}^{+}_{(n)})$ having $\mathcal{U}_{(n)} < 0$.

3.4.3 Harmonic potentials: Chaotic inflation

Quadratic potential

We have seen, in Subsection 2.3.2, that for a quadratic potential the quasi-attractor method of [194] gives two fixed points $\mathcal{P}_0$ and $\mathcal{P}_1$ given by (2.50) and (2.51), respectively. Then, for the perturbed system (3.76), it follows from (3.79) that there are four fixed points

$$\left(\mathcal{P}_{0,1}, \mathcal{U}^{\pm}_{(\mathrm{n})}(\mathcal{P}_{0,1})\right), \tag{3.89}$$

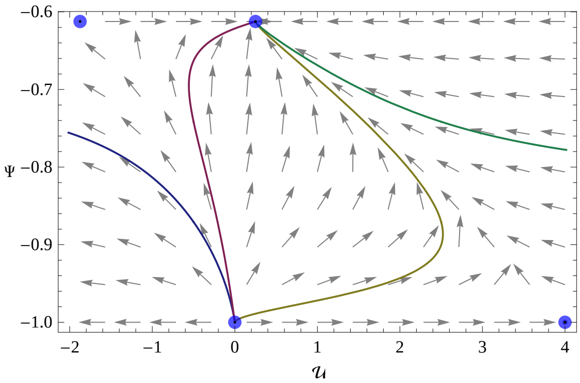

Figure 3.7: Density perturbations described by orbits in the phase plane $(\Psi, \mathcal{U}_{(n)})$ for an exponential potential with $\lambda = 1.5$ taking the long wavelength limit.

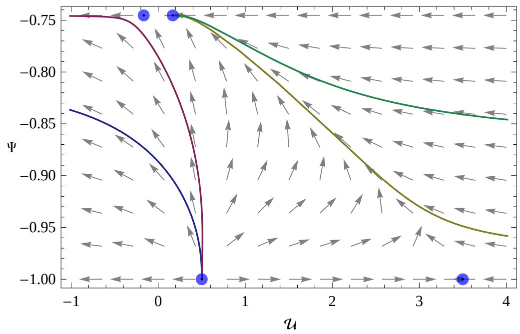

Figure 3.8: Density perturbations described by orbits in the phase plane $(\Psi, \mathcal{U}_{(n)})$ for an exponential potential with $\lambda = \sqrt{\frac{10}{3}}$ and $\frac{\mathrm{n}^2}{a^2H^2} = 1.75$.

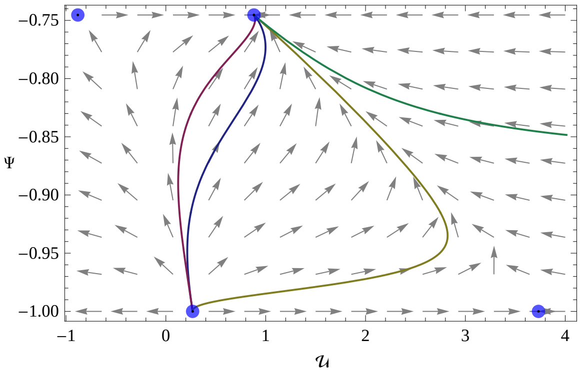

Figure 3.9: Density perturbations described by orbits in the phase plane $(\Psi, \mathcal{U}_{(n)})$ for an exponential potential with $\lambda = \sqrt{\frac{10}{3}}$ and $\frac{\mathrm{n}^2}{a^2H^2} = 1$. The figure shows the equilibrium points, one of which is the future attractor $(\mathcal{P}, \mathcal{U}^+_{(n)})$ having $\mathcal{U}_{(n)} > 0$.

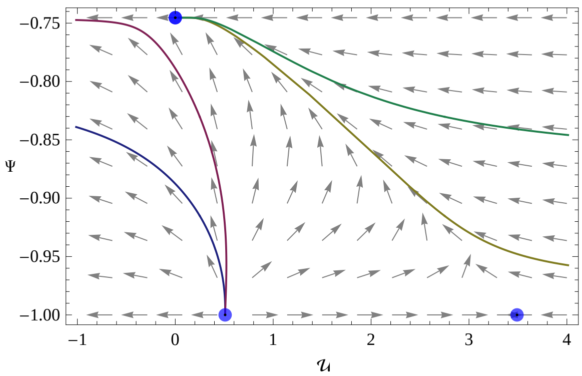

Figure 3.10: Density perturbations described by orbits in the phase plane $(\Psi, \mathcal{U}_{(n)})$ for an exponential potential with $\lambda = \sqrt{\frac{10}{3}}$ and $\frac{\mathrm{n}^2_{crit}}{a^2H^2} = \frac{16}{9}$. The figure shows the equilibrium points. The saddle point has $\mathcal{U}_{(n)} = 0$.

where the coefficients (3.80) are

$$\begin{aligned} \xi(\mathcal{P}_{0,1}) &= \frac{1}{2} \mp \frac{9}{2}\sqrt{1 - \frac{4}{9}\mathcal{M}^2} \\ \zeta(\mathcal{P}_{0,1}) &= 3 \mp 3\sqrt{1 - \frac{4}{9}\mathcal{M}^2} + \frac{n^2}{H^2a^2}\,, \end{aligned} \tag{3.90}$$

with the minus and plus signs standing for $\mathcal{P}_0$ and $\mathcal{P}_1$, respectively. We also saw that $\mathcal{P}_1$ is the late time attractor of the background subsystem. Therefore, the late time attractor of the perturbed dynamical system (3.70) is $\left(\mathcal{P}_1, \mathcal{U}^+_{(n)}(\mathcal{P}_1)\right)$. From (3.83) and (3.90), we easily see that this fixed point always lies in the region $\mathcal{U}_{(n)} < 0$ of the phase-space and it only exists if (3.82) is satisfied (see Fig. 3.11), i.e. for values of the wave number satisfying

$$n^2 \leq a^2H^2\left(\frac{17}{8} - \frac{15}{8}\sqrt{1 - \frac{4}{9}\mathcal{M}^2} - \frac{9}{4}\mathcal{M}^2\right), \tag{3.91}$$

which, in turn, implies

$$\mathcal{M} \leq \frac{2}{9}\sqrt{16 - 5\sqrt{7}}. \tag{3.92}$$

Thus, when $\mathcal{M} = \frac{2}{9}\sqrt{16 - 5\sqrt{7}}$, the fixed point only exists in the long wavelength limit and it is a saddle point (see Fig. 3.12), while for $\mathcal{M} > \frac{2}{9}\sqrt{16 - 5\sqrt{7}}$, the orbits are periodic. See Figs. 3.13, 3.14, 3.15 and 3.16 for the case $\mathcal{M} = 10^{-1} < \frac{2}{9}\sqrt{16 - 5\sqrt{7}}$.

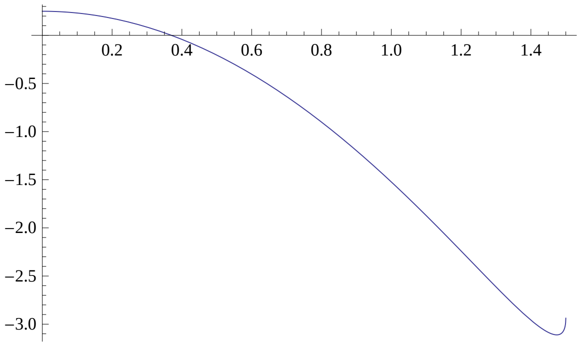

Figure 3.11: The non negative region shows the set of values of $\mathcal{M}$ for which there are fixed points.

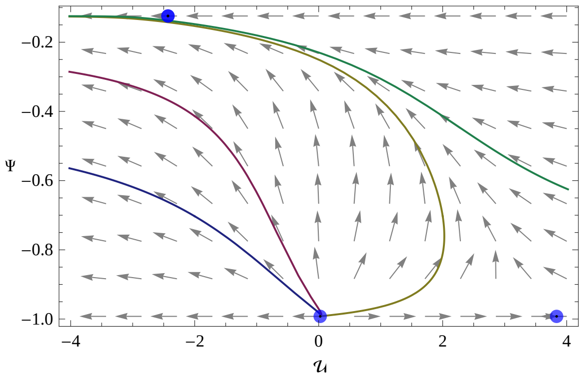

Figure 3.12: Density perturbations described by orbits in the phase plane $(\Psi, \mathcal{U}_{(n)})$ for an quadratic potential with $\mathcal{M} = \frac{2}{9}\sqrt{16 - 5\sqrt{7}}$ in the long wavelength limit. The figure shows the saddle point in the region $\mathcal{U}_{(n)} < 0$.

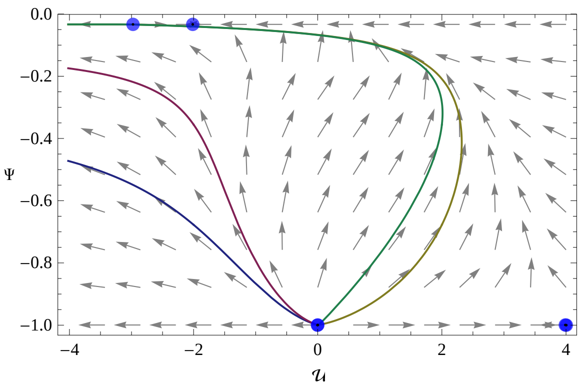

Figure 3.13: Density perturbations described by orbits in the phase plane $(\Psi, \mathcal{U}_{(n)})$ for an quadratic potential with $\mathcal{M} = 10^{-1}$ in the long wavelength limit. The figure shows the attractor point $\mathcal{U}^{+}(\mathcal{P}_1)$ in the region $\mathcal{U}_{(n)} < 0$.

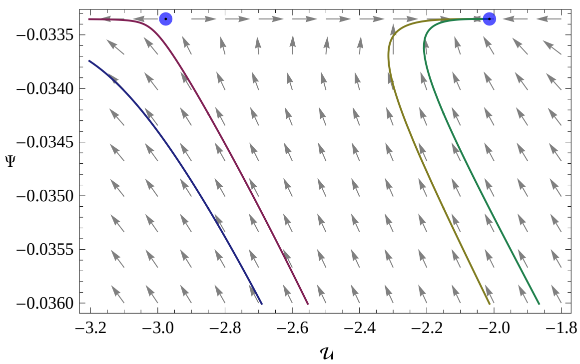

Figure 3.14: Density perturbations described by orbits in the phase plane $(\Psi, \mathcal{U}_{(n)})$ for an quadratic potential with $\mathcal{M} = 10^{-1}$ in the long wavelength limit. The figure shows in more detail the attractor point in the region $\mathcal{U}_{(n)} < 0$.

Quartic potential

We have seen, in Subsection 2.3.2, that for a quartic potential the quasi-attractor method of [194], used in [111], showed the existence of two fixed points $\mathcal{P}_0$ and $\mathcal{P}_1$ given by (2.55). Then, for the perturbed system (3.76) it follows, from (3.79) that there are four fixed points

$$\left(\mathcal{P}_{0,1}, \mathcal{U}^{\pm}_{(\mathrm{n})}(\mathcal{P}_{0,1})\right), \tag{3.93}$$

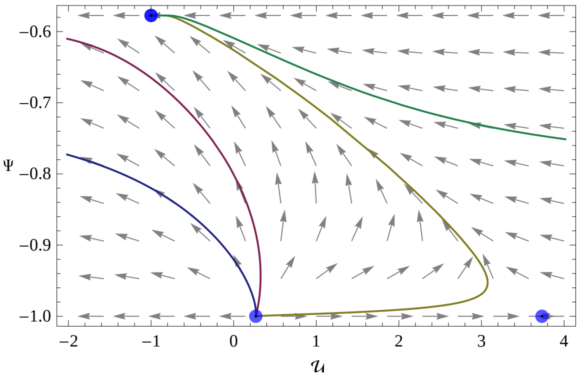

Figure 3.15: Density perturbations described by orbits in the phase plane $(\Psi, \mathcal{U}_{(n)})$ for a quadratic potential with $\mathcal{M} = 10^{-1}$ for $n^2 = n^2_{crit}$. The figure shows the saddle point, in the region $\mathcal{U}_{(n)} < 0$.

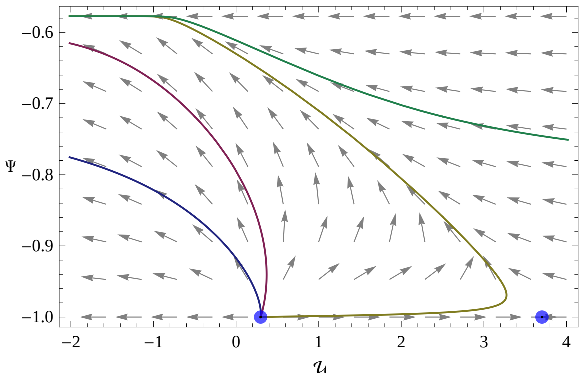

Figure 3.16: Density perturbations described by orbits in the phase plane $(\Psi, \mathcal{U}_{(n)})$ for an quadratic potential with $\mathcal{M} = 10^{-1}$ and $n^2 > n^2_{crit}$. The figure shows the existence of periodic orbits, and the perturbations behave as waves.

with the coefficients (3.80) given by

$$\xi(\mathcal{P}_0) = 2\left(1 - 3\cos\left(\frac{2}{3}\chi\right)\right) \quad \text{and} \quad \xi(\mathcal{P}_1) = 2\left(1 - 3\cos\left(\frac{2}{3}(\chi - \pi)\right)\right)$$
$$\zeta(\mathcal{P}_0) = \xi(\mathcal{P}_0) + 4\cos^2\left(\frac{2}{3}\chi\right) + \frac{\mathrm{n}^2}{H^2 a^2} \quad \text{and} \quad \zeta(\mathcal{P}_1) = \xi(\mathcal{P}_1) + 4\cos^2\left(\frac{2}{3}(\chi - \pi)\right) + \frac{\mathrm{n}^2}{H^2 a^2}. \tag{3.94}$$

From (3.82), the fixed points exist if

$$\mathrm{n}^2 \leq \mathrm{n}^2_{crit} = a^2 H^2 \left(5\cos^2\left(\frac{2}{3}(\chi - \pi)\right) - 1\right), \tag{3.95}$$

i.e. for χ satisfying

$$0 < \chi \leq \pi - \frac{3}{2}\arccos\left(-\frac{1}{\sqrt{5}}\right) \quad \text{or} \quad \pi - \frac{3}{2}\arccos\left(\frac{1}{\sqrt{5}}\right) \leq \chi \leq \frac{\pi}{2}. \tag{3.96}$$

see Fig. 3.19. Furthermore, we find that $\zeta(\mathcal{P}_1) > 0$ for all values of $0 < \chi < \pi/2$, and $\zeta(\mathcal{P}_1) = 0$ for $\chi = \pi/2$, see Fig. 3.18. We also find that (see also Fig. 3.17)

$$\begin{aligned} &\text{For} \quad 0 < \chi < \pi - \frac{3}{2}\arccos\left(\frac{1}{\sqrt{3}}\right) \quad \text{then} \quad \xi(\mathcal{P}_1) > 0 \\ &\text{For} \quad \chi = \pi - \frac{3}{2}\arccos\left(\frac{1}{\sqrt{3}}\right) \quad \text{then} \quad \xi(\mathcal{P}_1) = 0 \\ &\text{For} \quad \pi - \frac{3}{2}\arccos\left(\frac{1}{\sqrt{3}}\right) < \chi \leq \frac{\pi}{2} \quad \text{then} \quad \xi(\mathcal{P}_1) < 0\,. \end{aligned} \tag{3.97}$$

Therefore, there are perturbation modes which decay for $0 < \chi < \pi - \frac{3}{2}\arccos\left(-\frac{1}{\sqrt{5}}\right)$ (see Figs. 3.20, 3.21 and 3.22) and which grow for $\pi - \frac{3}{2}\arccos\left(\frac{1}{\sqrt{5}}\right) < \chi < \pi/2$ (see Figs. 3.23, 3.24, 3.25 and 3.26). Since the solution is inflationary for $\chi < \pi/4$ (see (2.57)), we notice that the instabilities (in the sense of growing modes) occur when the solution is non-inflationary, while when inflation occurs there are only decaying modes present.

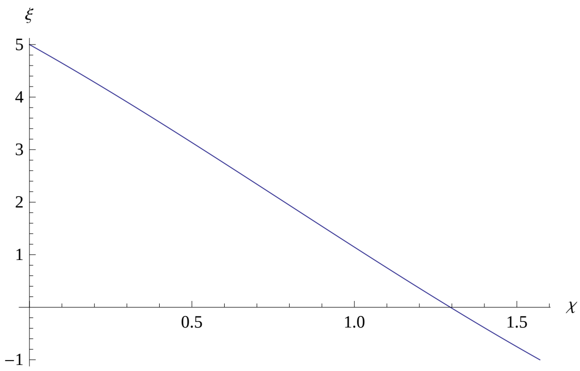

Figure 3.17: Plot of $\xi(\mathcal{P}_1)$ in the interval $0 \leq \chi \leq \frac{\pi}{2}$. The figure shows a zero at $\chi = \pi - \frac{3}{2}\arccos\left(\frac{1}{\sqrt{3}}\right)$. The positive region shows the admitted values for decay as long as there exists fixed points.

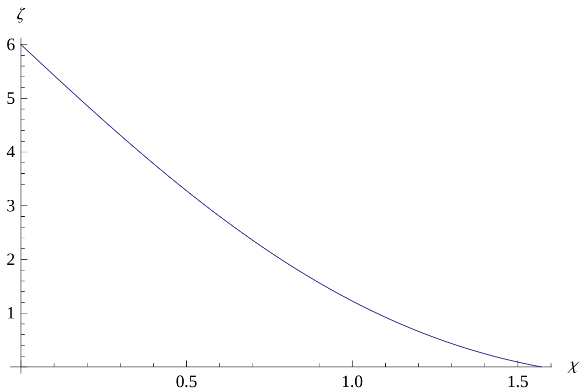

Figure 3.18: Plot of $\zeta(\mathcal{P}_1)$ in the interval $0 \leq \chi \leq \frac{\pi}{2}$ taking the long-wavelength limit. The figure shows that ζ is a positive monotone function of χ achieving the value zero at $\chi = \frac{\pi}{2}$.

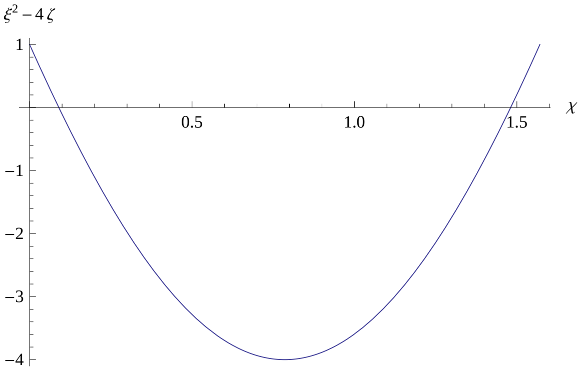

Figure 3.19: Plot of $\xi^2(\mathcal{P}_1) - 4\zeta(\mathcal{P}_1)$ in the interval $0 \leq \chi \leq \frac{\pi}{2}$. The figure shows two zeros at $\chi = \pi - \frac{3}{2}\arccos\left(\pm\frac{1}{\sqrt{5}}\right)$. The nonnegative region shows the admitted values of χ for which there exists fixed points.

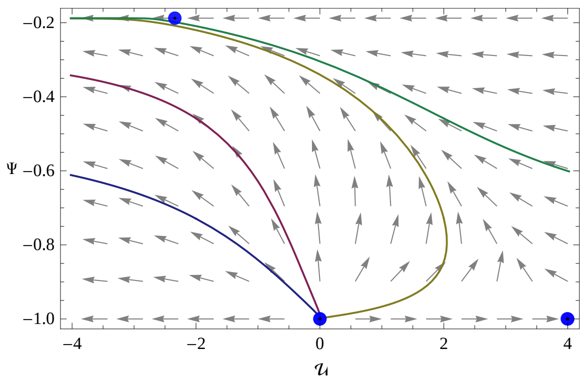

Figure 3.20: Phase portrait for $\chi = \pi - \frac{3}{2}\arccos\left(-\frac{1}{\sqrt{5}}\right)$ taking the long wavelength limit. The figure shows the saddle point in the region $\mathcal{U} < 0$.

New inflation

In the linearly perturbed case, we find that the fixed points of the system (3.70) are

$$\left(\mathcal{P}, \mathcal{U}^{\pm}_{(n)}(\mathcal{P})\right), \tag{3.98}$$

where $\mathcal{U}^{\pm}_{(n)}(\mathcal{P})$ are given by Eqs. (3.79) with

$$\xi(\mathcal{P}) = -\left[4 - 9\Phi^4_{\mathcal{P}}\right]$$
$$\zeta(\mathcal{P}) = -3\Phi^4_{\mathcal{P}}\left(1 - 3\Phi^4_{\mathcal{P}}\right) + \mathcal{M}^4\mathcal{N}^2 + \frac{\mathrm{n}^2}{H^2 a^2},$$

which will be investigated numerically. See Figs. 3.27 and 3.28.

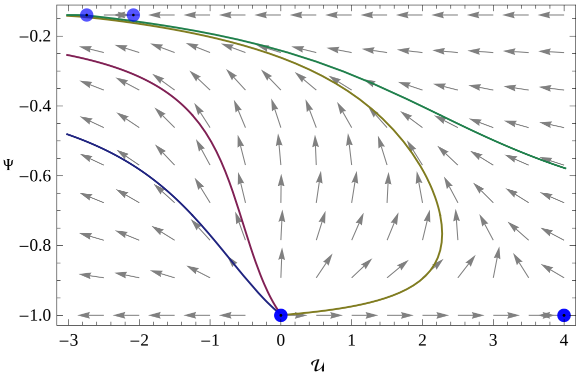

Figure 3.21: Density perturbations described by orbits in the phase plane $(\Psi, \mathcal{U}_{(n)})$ for an quartic potential with $\chi = 20^{-1}$ taking the long wavelength limit. The figure shows the late time attractor in the region $\mathcal{U} < 0$.

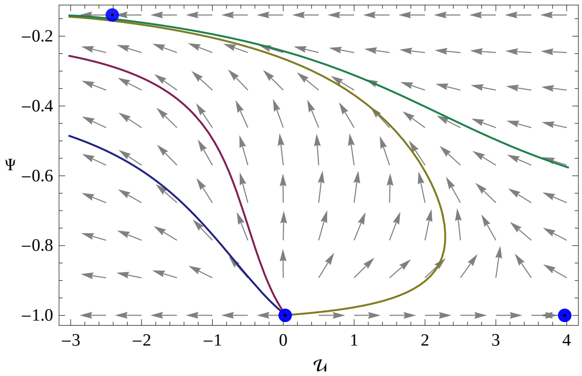

Figure 3.22: Density perturbations described by orbits in the phase plane $(\Psi, \mathcal{U}_{(n)})$ for an quartic potential with $\chi = 20^{-1}$ for $n^2 = n^2_{crit}(20^{-1})$. The figure shows the saddle point in the region $\mathcal{U} < 0$.

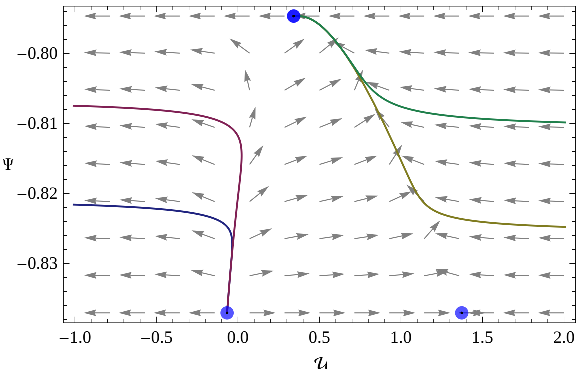

Figure 3.23: Density perturbations described by orbits in the phase plane $(\Psi, \mathcal{U}_{(n)})$ for an quartic potential with $\chi = \pi - \frac{3}{2} \arccos\left(\frac{1}{\sqrt{5}}\right)$ taking the long wavelength limit. The figure shows the saddle point in the region $\mathcal{U} > 0$.

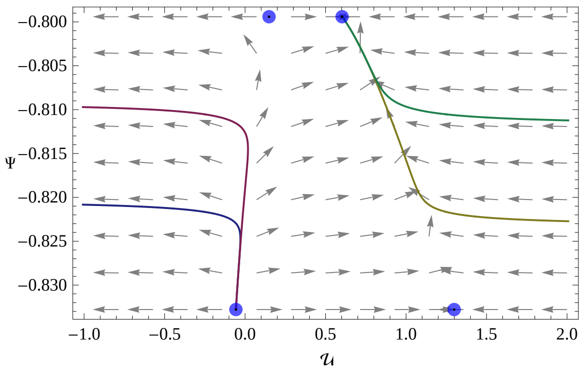

Figure 3.24: Density perturbations described by orbits in the phase plane $(\Psi, \mathcal{U}_{(n)})$ for an quartic potential with $\chi = 3/2$ taking the long wavelength limit. The figure shows the late time attractor in the region $\mathcal{U} > 0$.

3.4.4 Two scalar fields

In this section, we consider two scalar fields which do not interact with each other, i.e. with $\mathcal{W} = 0$. In this case, we find from Proposition 3.1 the following result:

The evolution for the phase of first order scalar perturbations on a FL background with two nonlinear smooth scalar fields , is given by the following system of differential equations for the

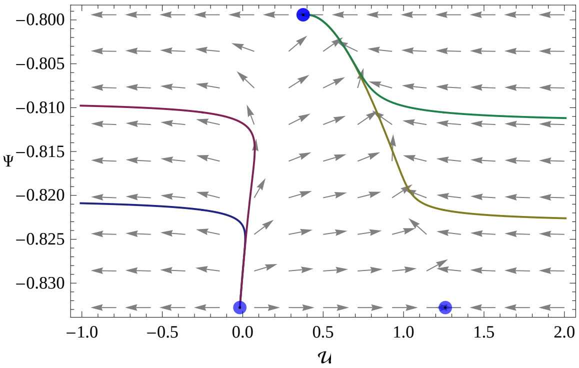

Figure 3.25: Density perturbations described by orbits in the phase plane $(\Psi, \mathcal{U}_{(n)})$ for a quartic potential with $\chi = 3/2$ for $n^2 = n^2_{crit}(3/2)$. The figure shows the saddle point in the region $\mathcal{U} > 0$.

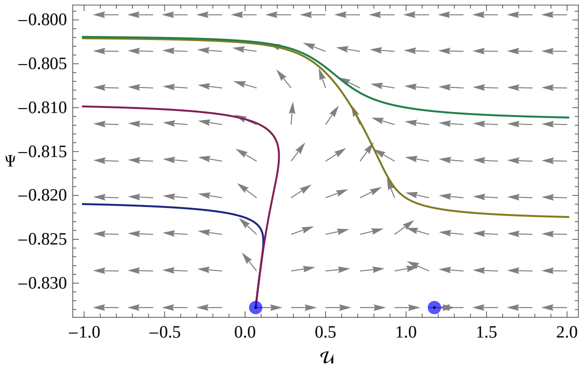

Figure 3.26: Density perturbations described by orbits in the phase plane $(\Psi, \mathcal{U}_{(n)})$ for a quartic potential with $\chi = 3/2$, for $n^2 > n^2_{crit}(3/2)$. The figure shows periodic orbits and the perturbations behave as waves.

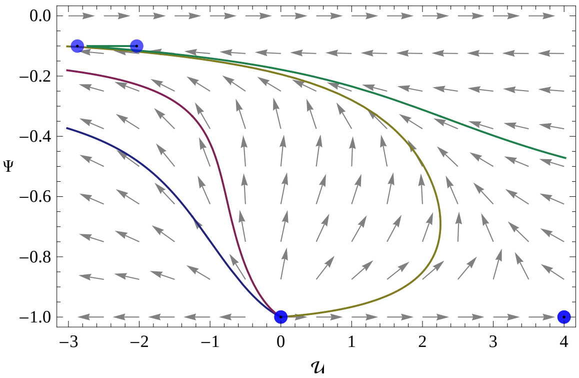

Figure 3.27: Density perturbations described by orbits in the phase plane $(\Psi, \mathcal{U}_{(n)})$ for new inflation with $\mathcal{M} = 0.3$ and $\mathcal{N} = 0.1$ in the long wavelength limit.

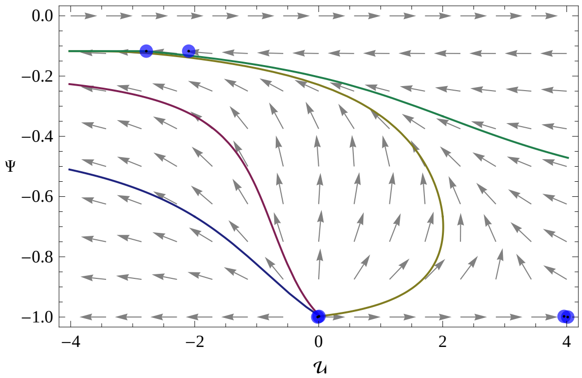

Figure 3.28: Density perturbations described by orbits in the phase plane $(\Psi, \mathcal{U}_{(n)})$ for new inflation with $\mathcal{M} = 0.3$ and $\mathcal{N} = 2$ in the long wavelength limit.

state vector $\left((\Psi_1, \Psi_2, \Phi_1, \Phi_1), \mathcal{U}, \mathcal{X}_{[12]}, \mathcal{Y}_{[12]}\right)$*:*

$$\begin{aligned}
\mathcal{U}' &= -\mathcal{U}^2 - \xi\mathcal{U} - \zeta + \gamma_{12}\mathcal{X}_{[12]} + \eta_{12}\mathcal{Y}_{[12]} \\
\mathcal{X}'_{[12]} &= \left(-\mathcal{U} + \varsigma_{12}\right)\mathcal{X}_{[12]} + \varpi_{12}\mathcal{Y}_{[12]} \\
\mathcal{Y}'_{[12]} &= \left(-\mathcal{U} + \iota_{12}\right)\mathcal{Y}_{[12]} - \mathcal{X}_{[12]} \\
\Psi'_1 &= (q-2)\Psi_1 - n\sqrt{6}\Phi_1^{2n-1}\frac{d\Phi_1}{d\phi_1} \\
\Psi'_2 &= (q-2)\Psi_2 - n\sqrt{6}\Phi_2^{2n-1}\frac{d\Phi_2}{d\phi_2} \\
\Phi'_1 &= \frac{1}{n}(q+1)\Phi_1 + \sqrt{6}\Psi_1\frac{d\Phi_1}{d\phi_1} \\
\Phi'_2 &= \frac{1}{n}(q+1)\Phi_2 + \sqrt{6}\Psi_2\frac{d\Phi_2}{d\phi_2} \ ,
\end{aligned} \tag{3.99}$$

subject to the background constraint equation

$$\Psi_1^2 + \Psi_2^2 + \Phi_1^{2n} + \Phi_2^{2n} = 1 - K \tag{3.100}$$

where

$$q = 2\left(\Psi_1^2 + \Psi_2^2\right) - \left(\Phi_1^{2n} + \Phi_2^{2n}\right) \tag{3.101}$$

and the coefficients are given by

$$\xi = -\left[2 + 2\left(\Psi_1^2 + \Psi_2^2\right) - \left(\Phi_1^{2n} + \Phi_2^{2n}\right) + 2\sqrt{6}n\left[\frac{\Phi_1^{2n-1}}{\Psi_1}\left(\frac{\Psi_1^2}{\Psi^2}\frac{d\Phi_1}{d\phi_1}\right) + \frac{\Phi_2^{2n-1}}{\Psi_2}\left(\frac{\Psi_2^2}{\Psi^2}\frac{d\Phi_2}{d\phi_2}\right)\right]\right] \tag{3.102}$$

$$\begin{aligned}\zeta = &-\left[6\Phi^{2n} + 2\sqrt{6}n\left[\frac{\Phi_1^{2n-1}}{\Psi_1}\frac{d\Phi_1}{d\phi_1}\frac{\Psi_1^2}{\Psi^2}\left(5 + \sqrt{6}n\frac{\Phi_1^{2n-1}}{\Psi_1}\frac{d\Phi_1}{d\phi_1}\right) + \frac{\Phi_2^{2n-1}}{\Psi_2}\frac{d\Phi_2}{d\phi_2}\frac{\Psi_2^2}{\Psi^2}\left(5 + \sqrt{6}n\frac{\Phi_2^{2n-1}}{\Psi_2}\frac{d\Phi_2}{d\phi_2}\right)\right]\right]\\
&+ 24n^2\left[\frac{\Phi_1^{2n-1}}{\Psi_1}\frac{d\Phi_1}{d\phi_1} - \frac{\Phi_2^{2n-1}}{\Psi_2}\frac{d\Phi_2}{d\phi_2}\right]^2\frac{\Psi_1^2\Psi_2^2}{\Psi^4}\\
&- 12n(2n-1)\left[\Phi_1^{2(n-1)}\left(\frac{d\Phi_1}{d\phi_1}\right)^2\frac{\Psi_1^2}{\Psi^2} + \Phi_2^{2(n-1)}\left(\frac{d\Phi_2}{d\phi_2}\right)^2\frac{\Psi_2^2}{\Psi^2}\right]\\
&- 12n\left[\Phi_1^{2n-1}\frac{d^2\Phi_1}{d\phi_1^2}\frac{\Psi_1^2}{\Psi^2} + \Phi_2^{2n-1}\frac{d^2\Phi_2}{d\phi_2^2}\frac{\Psi_2^2}{\Psi^2}\right] + \frac{\mathrm{n}^2}{a^2H^2}\end{aligned} \tag{3.103}$$

$$\begin{aligned}\gamma_{12} = &-8\sqrt{6}n\left(\frac{\Phi_1^{2n-1}}{\Psi_1}\frac{d\Phi_1}{d\phi_1} - \frac{\Phi_2^{2n-1}}{\Psi_2}\frac{d\Phi_2}{d\phi_2}\right)\frac{\Psi_1^2}{\Psi^2}\frac{\Psi_2^2}{\Psi^2}\\
&- 18n^2\left(\frac{\Phi_1^{2n-1}}{\Psi_1}\frac{d\Phi_1}{d\phi_1} - \frac{\Phi_2^{2n-1}}{\Psi_2}\frac{d\Phi_2}{d\phi_2}\right)^2\frac{\left(\Psi_1^2 - \Psi_2^2\right)}{\Psi^2}\frac{\Psi_1^2}{\Psi^2}\frac{\Psi_2^2}{\Psi^2}\\
&- 6n^2\left[\left(\frac{\Phi_1^{2n-1}}{\Psi_1}\frac{d\Phi_1}{d\phi_1}\right)^2 - \left(\frac{\Phi_2^{2n-1}}{\Psi_2}\frac{d\Phi_2}{d\phi_2}\right)^2\right]\frac{\Psi_1^2}{\Psi^2}\frac{\Psi_2^2}{\Psi^2}\\
&- 12n(2n-1)\left(\Phi_1^{2(n-1)}\left(\frac{d\Phi_1}{d\phi_1}\right)^2 - \Phi_2^{2(n-1)}\left(\frac{d\Phi_2}{d\phi_2}\right)^2\right)\frac{\Psi_1^2}{\Psi^2}\frac{\Psi_2^2}{\Psi^2}\\
&- 12n\left(\Phi_1^{2n-1}\frac{d^2\Phi_1}{d\phi_1^2} - \Phi_2^{2n-1}\frac{d^2\Phi_2}{d\phi_2^2}\right)\frac{\Psi_1^2}{\Psi^2}\frac{\Psi_2^2}{\Psi^2}\end{aligned}$$

$$\begin{aligned}\eta_{12} = &-\left[18\sqrt{6}n - 6\sqrt{6}n\left(\Psi^2 - \Phi^{2n}\right) - 2\sqrt{6}n\frac{\mathrm{q}^2}{a^2H^2}\right]\left(\frac{\Phi_1^{2n-1}}{\Psi_1}\frac{d\Phi_1}{d\phi_1} - \frac{\Phi_2^{2n-1}}{\Psi_2}\frac{d\Phi_2}{d\phi_2}\right)\frac{\Psi_1^2\Psi_2^2}{\Psi^4}\\
&- 48n^2\left[\frac{\Phi_1^{2n-1}}{\Psi_1}\frac{\Psi_1^2}{\Psi^2}\frac{d\Phi_1}{d\phi_1} + \frac{\Phi_2^{2n-1}}{\Psi_2}\frac{\Psi_2^2}{\Psi^2}\frac{d\Phi_2}{d\phi_2}\right]\left(\frac{\Phi_1^{2n-1}}{\Psi_1}\frac{d\Phi_1}{d\phi_1} - \frac{\Phi_2^{2n-1}}{\Psi_2}\frac{d\Phi_2}{d\phi_2}\right)\frac{\Psi_1^2\Psi_2^2}{\Psi^4}\\
&- 18\sqrt{6}n^3\left[\frac{\Phi_1^{2n-1}}{\Psi_1}\frac{\Psi_1^2}{\Psi^2}\frac{d\Phi_1}{d\phi_1} + \frac{\Phi_2^{2n-1}}{\Psi_2}\frac{\Psi_2^2}{\Psi^2}\frac{d\Phi_2}{d\phi_2}\right]^2\left(\frac{\Phi_1^{2n-1}}{\Psi_1}\frac{d\Phi_1}{d\phi_1} - \frac{\Phi_2^{2n-1}}{\Psi_2}\frac{d\Phi_2}{d\phi_2}\right)\frac{\Psi_1^2\Psi_2^2}{\Psi^4}\\
&- 6\sqrt{6}n^3\left[\left(\frac{\Phi_1^{2n-1}}{\Psi_1}\frac{d\Phi_1}{d\phi_1}\right)^2\frac{\Psi_1^2}{\Psi^2} + \left(\frac{\Phi_2^{2n-1}}{\Psi_2}\frac{d\Phi_2}{d\phi_2}\right)^2\frac{\Psi_2^2}{\Psi^2}\right]\left(\frac{\Phi_1^{2n-1}}{\Psi_1}\frac{d\Phi_1}{d\phi_1} - \frac{\Phi_2^{2n-1}}{\Psi_2}\frac{d\Phi_2}{d\phi_2}\right)\frac{\Psi_1^2\Psi_2^2}{\Psi^4}\\
&- 36n^2\left(\frac{\Phi_1^{2n-1}}{\Psi_1}\frac{d\Phi_1}{d\phi_1} - \frac{\Phi_2^{2n-1}}{\Psi_2}\frac{d\Phi_2}{d\phi_2}\right)^2\frac{\Psi_1^2 - \Psi_2^2}{\Psi^2}\frac{\Psi_1^2\Psi_2^2}{\Psi^4}\\
&- 24n^2\left[\left(\frac{\Phi_1^{2n-1}}{\Psi_1}\frac{d\Phi_1}{d\phi_1}\right)^2 - \left(\frac{\Phi_2^{2n-1}}{\Psi_2}\frac{d\Phi_2}{d\phi_2}\right)^2\right]\frac{\Psi_1^2\Psi_2^2}{\Psi^4}\\
&+ 18\sqrt{6}n^3\left(\frac{\Phi_1^{2n-1}}{\Psi_1}\frac{d\Phi_1}{d\phi_1} - \frac{\Phi_2^{2n-1}}{\Psi_2}\frac{d\Phi_2}{d\phi_2}\right)^3\frac{\Psi_1^4\Psi_2^4}{\Psi^8}\\
&- 6n(2n-1)\left[8 + \sqrt{6}n\left(\frac{\Phi_1^{2n-1}}{\Psi_1}\frac{d\Phi_1}{d\phi_1} + \frac{\Phi_2^{2n-1}}{\Psi_2}\frac{d\Phi_2}{d\phi_2}\right)\right]\left(\Phi_1^{2(n-1)}\left(\frac{d\Phi_1}{d\phi_1}\right)^2 - \Phi_2^{2(n-1)}\left(\frac{d\Phi_2}{d\phi_2}\right)^2\right)\frac{\Psi_1^2\Psi_2^2}{\Psi^4}\\
&- 6n\left[8 + \sqrt{6}n\left(\frac{\Phi_1^{2n-1}}{\Psi_1}\frac{d\Phi_1}{d\phi_1} + \frac{\Phi_2^{2n-1}}{\Psi_2}\frac{d\Phi_2}{d\phi_2}\right)\right]\left(\Phi_1^{2n-1}\frac{d^2\Phi_1}{d\phi_1^2} - \Phi_2^{2n-1}\frac{d^2\Phi_2}{d\phi_2^2}\right)\frac{\Psi_1^2\Psi_2^2}{\Psi^4}\end{aligned}$$

$$\varsigma_{12} = -\sqrt{6}n\left[\frac{\Phi_1^{2n-1}}{\Psi_1}\frac{d\Phi_1}{d\phi_1}\frac{\Psi_2^2}{\Psi^2} + \frac{\Phi_2^{2n-1}}{\Psi_2}\frac{d\Phi_2}{d\phi_2}\frac{\Psi_1^2}{\Psi^2}\right]$$

$$\begin{aligned}\varpi_{12} =& 6n^2\left[\frac{\Phi_1^{2n-1}}{\Psi_1}\frac{d\Phi_1}{d\phi_1} - \frac{\Phi_2^{2n-1}}{\Psi_2}\frac{d\Phi_2}{d\phi_2}\right]^2\frac{\Psi_1^2\Psi_2^2}{\Psi^4}\\ &+ 6n\left[(2n-1)\Phi_1^{2(n-1)}\left(\frac{d\Phi_1}{d\phi_1}\right)^2 + \Phi_1^{2n-1}\frac{d^2\Phi_1}{d\phi_1^2}\right]\frac{\Psi_2^2}{\Psi^2}\\ &+ 6n\left[(2n-1)\Phi_2^{2(n-1)}\left(\frac{d\Phi_2}{d\phi_2}\right)^2 + \Phi_2^{2n-1}\frac{d^2\Phi_2}{d\phi_2^2}\right]\frac{\Psi_1^2}{\Psi^2} + \frac{\mathrm{q}^2}{a^2H^2}\end{aligned}$$

$$\iota_{12} = 2 - 2\left(\Psi_1^2 + \Psi_2^2\right) + \left(\Phi_1^{2n} + \Phi_2^{2n}\right) - \varsigma_{12}\,.$$

We also prove from (3.99):

Lemma 3. *For $K = 0$, the fixed points of system (3.70) are given by Lemma 1 together with:*

$$-\mathcal{U}^2 - \xi(\mathcal{P})\mathcal{U} - \zeta(\mathcal{P}) = 0 \quad \wedge \quad \gamma_{12}(\mathcal{P})\mathcal{X}_{[12]} + \eta_{12}(\mathcal{P})\mathcal{Y}_{[12]} = 0$$

or

$$-\mathcal{U}^2 - \left(\varsigma_{12}(\mathcal{P}) + \iota_{12}(\mathcal{P})\right)\mathcal{U} + \left(\varsigma_{12}(\mathcal{P})\iota_{12}(\mathcal{P}) + \varpi_{12}(\mathcal{P})\right) = 0 \quad \wedge \quad \gamma_{12}(\mathcal{P})\mathcal{X}_{[12]} + \eta_{12}(\mathcal{P})\mathcal{Y}_{[12]} \neq 0.$$

where

$$\xi(\mathcal{P}) = -\left[4 - 9\Phi_1^{2n}(\mathcal{P}) - 9\Phi_2^{2n}(\mathcal{P})\right]$$

$$\begin{aligned}\zeta(\mathcal{P}) =& \left[24 - 9\left(4 - \frac{2}{n}\right)\Psi^2\right]\left(\Phi_1^{2n} + \Phi_2^{2n}\right) - 18\left[\frac{\Phi_1^{4n}}{\Psi_1^2} + \frac{\Phi_2^{4n}}{\Psi_2^2}\right]\Psi^2\\ &+ 36\left[\Phi_1^{2n}\frac{\Psi_2}{\Psi_1} - \Phi_2^{2n}\frac{\Psi_1}{\Psi_2}\right]^2 - 12n\left[\Phi_1^{2n-1}\frac{\Psi_1^2}{\Psi^2}\frac{d^2\Phi_1}{d\phi_1^2} + \Phi_2^{2n-1}\frac{\Psi_2^2}{\Psi^2}\frac{d^2\Phi_2}{d\phi_2^2}\right] + \frac{n^2}{a^2H^2}\end{aligned}$$

$$\begin{aligned}\gamma_{12}(\mathcal{P}) =& 24\left[\Phi_1^{2n}\frac{\Psi_2^2}{\Psi_1^2 + \Psi_2^2} - \Phi_2^{2n}\frac{\Psi_1^2}{\Psi_1^2 + \Psi_2^2}\right] - 9\left[\Phi_1^{4n}\frac{\Psi_2^2}{\Psi_1^2} - \Phi_2^{4n}\frac{\Psi_1^2}{\Psi_2^2}\right]\\ &- 24\left[\Phi_1^{2n}\frac{\Psi_2}{\Psi_1} - \Phi_2^{2n}\frac{\Psi_1}{\Psi_2}\right]^2\frac{\Psi_1^2 - \Psi_2^2}{\Psi^2} - 9\left(4 - \frac{2}{n}\right)\left[\Phi_1^{2n}\Psi_2^2 - \Phi_2^{2n}\Psi_1^2\right]\\ &- 12n\left[\Phi_1^{2n-1}\frac{d^2\Phi_1}{d\phi_1^2} - \Phi_2^{2n-1}\frac{d^2\Phi_2}{d\phi_2^2}\right]\frac{\Psi_1^2\Psi_2^2}{\left(\Psi_1^2 + \Psi_2^2\right)^2}\end{aligned}$$

$$\begin{aligned}\eta_{12}(\mathcal{P}) =& \left[36\Psi^2 + 81\Phi^{4n} + 24\left(\frac{\Phi_1^{4n}}{\Psi_1^2} + \frac{\Phi_2^{4n}}{\Psi_2^2}\right)\Psi^2 - 6\frac{q^2}{a^2H^2}\right]\left[\Phi_1^{2n}\frac{\Psi_2^2}{\Psi^2} - \Phi_2^{2n}\frac{\Psi_1^2}{\Psi^2}\right]\\ &- 54\left[\Phi_1^{2n}\frac{\Psi_2}{\Psi_1} - \Phi_2^{2n}\frac{\Psi_1}{\Psi_2}\right]^2\frac{\Psi_1^2 - \Psi_2^2}{\Psi^2} - 36\left[\Phi_1^{4n}\frac{\Psi_2^2}{\Psi_1^2} - \Phi_2^{4n}\frac{\Psi_1^2}{\Psi_2^2}\right]\\ &- 81\left[\Phi_1^{2n}\frac{\Psi_2}{\Psi_1} - \Phi_2^{2n}\frac{\Psi_1}{\Psi_2}\right]^3\frac{\Psi_1\Psi_2}{\Psi^4}\\ &- 9\left(2 - \frac{1}{n}\right)\left[8 - 3\left(\frac{\Phi_1^{2n}}{\Psi_1^2} + \frac{\Phi_2^{2n}}{\Psi_2^2}\right)\Psi^2\right]\left(\Phi_1^{2n}\Psi_2^2 - \Phi_2^{2n}\Psi_1^2\right)\\ &- 6n\left[8 - 3\left(\frac{\Phi_1^{2n}}{\Psi_1^2} + \frac{\Phi_2^{2n}}{\Psi_2^2}\right)\Psi^2\right]\left[\Phi_1^{2n-1}\frac{d^2\Phi_1}{d\phi_1^2} - \Phi_2^{2n-1}\frac{d^2\Phi_2}{d\phi_2^2}\right]\frac{\Psi_1^2\Psi_2^2}{\left(\Psi_1^2 + \Psi_2^2\right)^2}\end{aligned}$$

$$
\begin{aligned}
\varsigma_{12}(\mathcal{P}) =& \frac{3}{\Psi_1^2\Psi_2^2}\left[\Phi_1^{2n}\Psi_2^4 + \Phi_2^{2n}\Psi_1^4\right] \\
\varpi_{12}(\mathcal{P}) =& 9\frac{\left(\Phi_1^{2n}\Psi_2^2 - \Phi_2^{2n}\Psi_1^2\right)^2}{\Psi_1^2\Psi_2^2} + \frac{9}{n}(2n-1)\Psi^2\left(\Phi_1^{2n}\left(\frac{\Psi_2}{\Psi_1}\right)^2 + \Phi_2^{2n}\left(\frac{\Psi_1}{\Psi_2}\right)^2\right) \\
&+ \frac{6n}{\Psi^2}\left(\Phi_1^{2n-1}\Psi_2^2\frac{d^2\Phi_1}{d\phi_1^2} + \Phi_2^{2n-1}\Psi_1^2\frac{d^2\Phi_2}{d\phi_2^2}\right) + \frac{q^2}{a^2H^2} \\
\iota_{12}(\mathcal{P}) =& 3\left(\Phi_1^2 + \Phi_2^2\right) - \varsigma_{12}(\mathcal{P})
\end{aligned}
$$

Corolary 1. *In particular, if*

$$\gamma_{12}(\mathcal{P}) = \eta_{12}(\mathcal{P}) = 0$$

then

$$\mathcal{Y}_{[12]}(\mathcal{P}) = \mathcal{X}_{[12]}(\mathcal{P}) = 0, \quad \textit{and} \quad \mathcal{U}_n^{\pm}(\mathcal{P}) = \frac{1}{2}\left(-\xi(\mathcal{P}) \pm \sqrt{\xi(\mathcal{P})^2 - 4\zeta(\mathcal{P})}\right). \tag{3.104}$$

In this particular case, the linearised matrix of the system at the fixed points has eigenvalues given by the background eigenvalues together with

$$-2\mathcal{U}^{\pm}(\mathcal{P}) - \xi(\mathcal{P}) = \mp\sqrt{\xi^2(\mathcal{P}) - 4\zeta(\mathcal{P})} \tag{3.105}$$

and

$$\frac{1}{2}\left[-2\mathcal{U}^{\pm} + \iota_{12} + \varsigma_{12} \pm \sqrt{\left(2\mathcal{U}^{\pm} - \iota_{12} - \varsigma_{12}\right)^2 - 4\left((\mathcal{U}^{\pm})^2 - (\iota_{12} + \varsigma_{12})\mathcal{U}^{\pm} + \varpi_{12} + \iota_{12}\varsigma_{12}\right)}\right]. \tag{3.106}$$

In the next section, we shall apply Lemma 3 to the case of two scalar fields with independent exponential potentials.

3.4.5 Flat assisted power-law inflation

In this case, we have

$$\mathcal{P}: \quad \frac{d^2\Phi_A}{d\phi_A^2} = -\frac{\sqrt{6}}{4}\lambda_A\frac{\Phi_A}{\Psi_A}\Psi^2 \quad , \quad \frac{d^3\Phi_A}{d\phi_A^3} = -\frac{\sqrt{6}}{8}\lambda_A^2\frac{\Phi_A}{\Psi_A}\Psi^2 \tag{3.107}$$

so that, at $\mathcal{P}$, and recalling (2.33)-(2.36), the coefficients read

$$\xi(\mathcal{P}_0) = -4 \quad , \quad \xi(\mathcal{P}_{1,2}) = 5 - \frac{3}{2}\lambda_{1,2}^2 \quad , \quad \xi(\mathcal{P}_3) = 5 - \frac{3}{2}\lambda^2$$

$$\zeta(\mathcal{P}_0) = \frac{\mathrm{n}^2}{a^2H^2} \quad , \quad \zeta(\mathcal{P}_{1,2}) = 6 - 4\lambda_{1,2}^2 + \frac{\lambda_{1,2}^4}{2} + \frac{\mathrm{n}^2}{a^2H^2} \quad , \quad \zeta(\mathcal{P}_3) = 6 - 4\lambda^2 + \frac{\lambda^4}{2} + \frac{\mathrm{n}^2}{a^2H^2}$$

$$\gamma_{12}(\mathcal{P}_0) = 0 \quad , \quad \gamma_{12}(\mathcal{P}_{1,2}) = 0 \quad , \quad \gamma_{12}(\mathcal{P}_3) = 0$$

$$\eta_{12}(\mathcal{P}_0) = 0 \quad , \quad \eta_{12}(\mathcal{P}_{1,2}) = 0 \quad , \quad \eta_{12}(\mathcal{P}_3) = 0$$

$$\varsigma_{12}(\mathcal{P}_0) = 0 \quad , \quad \varsigma_{12}(\mathcal{P}_{1,2}) = 0 \quad , \quad \varsigma_{12}(\mathcal{P}_3) = \frac{6 - \lambda^2}{2}$$

$$\iota_{12}(\mathcal{P}_0) = 0 \quad , \quad \iota_{12}(\mathcal{P}_{1,2}) = \frac{6 - \lambda_{1,2}^2}{2} \quad , \quad \iota_{12}(\mathcal{P}_3) = 0$$

$$\varpi_{12}(\mathcal{P}_0) = \frac{\mathrm{q}^2}{a^2H^2} \quad , \quad \varpi_{12}(\mathcal{P}_{1,2}) = \frac{\mathrm{q}^2}{a^2H^2} \quad , \quad \varpi_{12}(\mathcal{P}_3) = \frac{\lambda^2}{2}\left(6-\lambda^2\right) + \frac{\mathrm{q}^2}{a^2H^2}$$

and by Corolary 1, the fixed points of the system reduce to

$$\left(\left(\Psi_A, \Phi_A\right), \mathcal{U}, \mathcal{X}_{[12]}, \mathcal{Y}_{[12]}\right) = \left(\mathcal{P}_i, \mathcal{U}^{\pm}_{\mathrm{n}}(\mathcal{P}_i), 0, 0\right) \tag{3.108}$$

with eigenvalues given by the background solution (2.37)-(2.39) together with

$$\omega^{\pm}(\mathcal{P}_0) = \mp 2\sqrt{4 - \frac{\mathrm{n}^2}{a^2H^2}} \; , \; -2 \mp \sqrt{4 - \frac{\mathrm{n}^2}{a^2H^2}} \pm \sqrt{-\frac{\mathrm{q}^2}{a^2H^2}} \; , \tag{3.109}$$

$$\begin{aligned} \omega^{\pm}(\mathcal{P}_{1,2}) = \mp\, 2 &\sqrt{\frac{1}{4}\left(1 + \lambda_{1,2}^2 + \frac{\lambda_{1,2}^4}{4}\right) - \frac{\mathrm{n}^2}{a^2H^2}} \; , \\ &\left(4 - \lambda_{1,2}^2\right) \mp \sqrt{\frac{1}{4}\left(1 + \lambda_{1,2}^2 + \frac{\lambda_{1,2}^4}{4}\right) - \frac{\mathrm{n}^2}{a^2H^2}} \pm \sqrt{\frac{1}{4}\left(\frac{6 - \lambda_{1,2}^2}{2}\right)^2 - \frac{\mathrm{q}^2}{a^2H^2}} \; , \end{aligned} \tag{3.110}$$

$$\begin{aligned} \omega^{\pm}(\mathcal{P}_3) = \mp\, 2 &\sqrt{\frac{1}{4}\left(1 + \lambda^2 + \frac{\lambda^4}{4}\right) - \frac{\mathrm{n}^2}{a^2H^2}} \; , \\ &\left(4 - \lambda^2\right) \mp \sqrt{\frac{1}{4}\left(1 + \lambda^2 + \frac{\lambda^4}{4}\right) - \frac{\mathrm{n}^2}{a^2H^2}} \pm \sqrt{\frac{1}{4}\left(\frac{6 - \lambda^2}{2}\right)^2 - \frac{\lambda^2}{2}(6 - \lambda^2) - \frac{\mathrm{q}^2}{a^2H^2}} \; . \end{aligned} \tag{3.111}$$

Thus, the behaviour of $\mathcal{U}^{+}$, being the future attractor, is very similar to the single scalar field solution with more restrictions due to the new eigenvalue (3.106). For instance, the fixed points for the perturbations of the massless scalar field solution only exists if $\mathrm{q}/(aH) \ll 1$.

3.5 Conclusions and future work

We have considered the problem of the linear stability homogenization of scalar field cosmologies using a dynamical systems‘ approach in covariant and gauge-invariant linear perturbation theory. In particular, we have established conditions under which the linearly perturbed inhomogeneous cosmologies homogenize having inflationary solutions as global attractors, for quadratic and exponential potentials. We have shown that the homogeneization occurs for open sets in the respective control parameter spaces and, therefore, that this process is stable in these settings. Our results are summarised in Table 3.1.

It would be interesting to apply these methods to more generic backgrounds such as the Bianchi type I, see [62], or to Kantowski-Sachs models as in [25].

$\mathcal{V}(\phi)$	**Fut. Att.:** $(\mathcal{P}_1, \mathcal{U}^+_{\mathbf{n}}(\mathcal{P}_1))$	**Parameter Space**	**Phys. Mean.**	**Inf. Sol.**
$\Lambda e^{\lambda\phi}$	Pt $\Psi = -\frac{\lambda}{\sqrt{6}}, \mathcal{U}^+_{(n)} < 0$	$0 < \lambda < \sqrt{2}$, $0 \leq n^2 < n^2_{crit}(\lambda)$	decays	$0 < \lambda < \sqrt{2}$
	Pt $\Psi = -\frac{\lambda}{\sqrt{6}}, \mathcal{U}^+_{(n)} < 0$	$\lambda = \sqrt{2}$, $0 < n^2 < n^2_{crit}(\lambda)$	decays	
	Pt $\Psi = -\frac{\lambda}{\sqrt{6}}, \mathcal{U}^+_{(n)} = 0$	$\lambda = \sqrt{2}$, $n^2 = 0$	tend to a const.	
	Pt $\Psi = -\frac{\lambda}{\sqrt{6}}, \mathcal{U}^+_{(n)} < 0$	$\sqrt{2} < \lambda < \sqrt{\frac{10}{3}}$, $0 < n^2_{-}(\lambda) < n^2 < n^2_{crit}(\lambda)$	decays	
	Pt $\Psi = -\frac{\lambda}{\sqrt{6}}, \mathcal{U}^+_{(n)} = 0$	$\sqrt{2} < \lambda < \sqrt{\frac{10}{3}}$, $n^2 = n^2_{-}(\lambda)$	tend to a const.	
	Pt $\Psi = -\frac{\lambda}{\sqrt{6}}, \mathcal{U}^+_{(n)} > 0$	$\sqrt{2} < \lambda < \sqrt{\frac{10}{3}}$, $0 \leq n^2 < n^2_{-}(\lambda)$	grows	
	Pt $\Psi = -\frac{\lambda}{\sqrt{6}}, \mathcal{U}^+_{(n)} > 0$	$\frac{10}{3} \leq \lambda < \sqrt{6}$, $0 \leq n^2 < n^2_{crit}(\lambda)$	grows	
	Per. orb. $\Psi = -\frac{\lambda}{\sqrt{6}}$	for all λ, $n^2 > n^2_{crit}(\lambda)$	Pert. is a wave	
$\frac{m^2}{2}\phi^2$	Point $\Psi = \Psi_{\mathcal{P}_1}, \mathcal{U}^+_{(n)} < 0$	$0 < \mathcal{M} < \frac{2}{9}\sqrt{16 - 5\sqrt{7}}$, $n^2 < n^2_{crit}$	Pert. decays	$0 < \mathcal{M} < \sqrt{2}$
	Periodic orbit $\Psi = \Psi_{\mathcal{P}_1}$	$\mathcal{M} > \frac{2}{9}\sqrt{16 - 5\sqrt{7}}$	Pert. is a wave	
$\frac{\lambda^4}{4}\phi^4$	Point $\Psi = \Psi_{\mathcal{P}_1}, \mathcal{U}^+_{(n)} < 0$	See (3.97)	Pert. decays	$0 < \mathcal{M} < 6^{\frac{1}{4}}$
	Periodic orbit $\Psi = \Psi_{\mathcal{P}_1}$		Pert. is a wave	
	Point $\Psi = \Psi_{\mathcal{P}_1}, \mathcal{U}^+_{(n)} > 0$		Pert. grows	

Table 3.1: Summary of the results about the behaviour of density perturbations in flat FL scalar field backgrounds with exponential, quadratic and quartic potentials.

Chapter 4

Progresses Towards the Exponential Decay of Non-Linear Perturbations

A natural way to analyse the stability of spacetimes is to ask whether small perturbations of a given solution to the EFEs asymptotically decay to the background solution. Like in the last chapter, most approaches to this question have been limited to the use of linear or higher-order truncated perturbation theory, and thus, they never take fully into account the nonlinearity of the EFEs —see e.g. [29, 30, 138] and also [8]. This type of analysis has been hampered by the lack of a suitable formulation of the EFEs for which the theory of systems of first order hyperbolic partial differential equations can be applied.

In [84], Friedrich has introduced a frame representation of the vacuum EFEs. The evolution equations implied by this alternative representation of the equations of General Relativity constitute a *first-order quasi-linear symmetric hyperbolic system (FOSH)*. In general, these systems are of the form

$$\mathbf{A}^0(\mathbf{u})\partial_t\mathbf{u} - \mathbf{A}^j(\mathbf{u})\partial_j\mathbf{u} = \mathbf{B}(\mathbf{u})\mathbf{u} \tag{4.1}$$

where $\mathbf{u} = \mathbf{u}(\boldsymbol{x}, t)$ is a smooth vector-valued function of dimension s with domain in $\Sigma \times [0, T]$ where Σ is a spacelike 3-dimensional manifold. Moreover, $\mathbf{A}^0$, $\mathbf{A}^j$, and $\mathbf{B}$ denote smooth $s \times s$ matrix valued-functions. The matrices $\mathbf{A}^0$ and $\mathbf{A}^j$ are symmetric. In addition, the matrix $\mathbf{A}^0$ is positive definite. The operators ∂_t and ∂_j stand, respectively, for the partial derivatives with respect to the coordinates $t \in [0, T]$ and $(x^j) \in \Sigma$, with the specific latin indices $i, j = 1, 2, 3$.

The construction for vacuum spacetimes given in [84] has been extended in [85, 86] to the case of a *perfect fluid* using a Lagrangian description of the fluid flow —see also [37, 38]. In both the vacuum and the perfect fluid cases the introduction of a frame formalism gives rise to extra gauge freedom. This freedom is associated to the evolution of the spatial frame coefficients along the flow of the time-like frame. If one fixes conveniently this gauge (using, for example, the Fermi gauge) one obtains a hyperbolic reduction for the evolution equations. As a consequence, given smooth initial data satisfying the constraints, local in time existence and uniqueness of a solution to the EFEs can be established —see e.g. [86, 165] and also [37, 38] for details. A natural way of performing a stability analysis is to consider a sequence of smooth initial data sets $\mathbf{u}_0^\epsilon$ for the EFEs satisfying the constraints equations on a Cauchy hypersurface Σ. The sequence is assumed to depend continuously on the parameter ϵ in such a way that the limit $\epsilon \to 0$ renders the data of the reference solution $\mathring{\mathbf{u}}_0$. In particular, one can write the full solution to the EFEs as the Ansatz

$$\mathbf{u}^\epsilon = \mathring{\mathbf{u}} + \epsilon\breve{\mathbf{u}}, \tag{4.2}$$

where $\breve{\mathbf{u}}$ is a (nonlinear) perturbation whose size is controlled by the parameter ϵ. Using the

Ansatz in equation (4.1), and writing

$$\begin{aligned} \mathbf{B}(\mathring{\mathbf{u}}+\epsilon\breve{\mathbf{u}}) &\equiv \mathbf{B}(\mathring{\mathbf{u}})+\epsilon\mathbf{B}(\mathring{\mathbf{u}},\breve{\mathbf{u}},\epsilon), \\ \mathbf{A}^{\mu}(\mathring{\mathbf{u}}+\epsilon\breve{\mathbf{u}}) &\equiv \mathbf{A}^{\mu}(\mathring{\mathbf{u}})+\epsilon\mathbf{A}^{\mu}(\mathring{\mathbf{u}},\breve{\mathbf{u}},\epsilon), \quad \mu=0,1,2,3 \end{aligned} \tag{4.3}$$

we are led to consider an initial value problem for the nonlinear perturbations of the form:

$$\begin{aligned} &\left(\mathring{\mathbf{A}}^0+\epsilon\breve{\mathbf{A}}^0\right)\partial_t\breve{\mathbf{u}}-\left(\mathring{\mathbf{A}}^j+\epsilon\breve{\mathbf{A}}^j\right)\partial_j\breve{\mathbf{u}}=\left(\mathring{\mathbf{B}}+\epsilon\breve{\mathbf{B}}\right)\breve{\mathbf{u}}, \\ &\breve{\mathbf{u}}(\boldsymbol{x},0)=\breve{\mathbf{u}}_0(\boldsymbol{x}). \end{aligned} \tag{4.4}$$

Here, the coefficients $\mathring{\mathbf{B}}\equiv\mathbf{B}(\mathring{\mathbf{u}})$ and $\mathring{\mathbf{A}}^{\mu}=\mathbf{A}^{\mu}(\mathring{\mathbf{u}})$ in the splitting (4.3) are defined uniquely by the condition $\epsilon=0$. Also

$$\breve{\mathbf{A}}^{\mu}\equiv\mathbf{A}^{\mu}(\mathring{\mathbf{u}},\breve{\mathbf{u}},\epsilon)$$

and

$$\breve{\mathbf{B}}\breve{\mathbf{u}}\equiv\mathbf{B}(\mathring{\mathbf{u}},\breve{\mathbf{u}},\epsilon)\mathring{\mathbf{u}}+\mathbf{B}(\mathring{\mathbf{u}},\breve{\mathbf{u}},\epsilon)\breve{\mathbf{u}}+\breve{\mathbf{A}}^j\partial_j\mathring{\mathbf{u}}-\breve{\mathbf{A}}^0\partial_t\mathring{\mathbf{u}}.$$

where it has been assumed that

$$\mathring{\mathbf{A}}^0\partial_t\mathring{\mathbf{u}}-\mathring{\mathbf{A}}^j\partial_j\mathring{\mathbf{u}}=\mathring{\mathbf{B}}\mathring{\mathbf{u}}.$$

A particular approach to the existence and the asymptotic exponential decay to zero of solutions to the Cauchy problem (4.4) for the case where the coefficients of the linearised system ($\epsilon=0$) are constant matrices has been discussed in [120, 122, 147]. In this approach, the stability of solutions follows from the existence of eigenvalues for the non-principal part of the linearised system having a negative real part (*strictly dissipative systems*). In the case where the system is only strongly hyperbolic, the inner product in L^2 has to be replaced by the so-called $\mathcal{H}$-inner product —see [122]. A procedure to analyse stability in the case of systems where $\mathring{\mathbf{B}}$ has vanishing eigenvalues (*dissipative systems*) has been given in [122] —see also [121, 147]. In principle, these methods can be applied to systems of the type considered here where the matrices $\mathring{\mathbf{B}}$, $\mathring{\mathbf{A}}^{\mu}$ are not constant but depend smoothly on time —see also [166].

The approach described in the previous paragraph has been applied by Reula [166] to the Einstein-perfect fluid system of [85] with a positive cosmological constant $\Lambda>0$, to prove the exponential decay of nonlinear perturbations for a wide class of homentropic fluids in flat Friedman-Lemaitre backgrounds[1]. An advantage of this approach is that it avoids the problem of gauge-dependence in perturbation theory and gauge-invariant conclusions, such as geodesic completeness, can be inferred.

Here, we pursue a similar approach to the one used by Reula in [166] to analyse the nonlinear stability of FL-nonlinear scalar field models. To this end, we first construct a first order symmetric hyperbolic system for the EFEs with a scalar field as the matter source. This construction is performed by splitting the wave equation for the scalar field into two first order equations. In our analysis, the scalar field is used to construct an adapted orthogonal frame. Written in terms of this adapted frame, the energy-momentum tensor is diagonal, independently of further gauge choices. A similar construction has been considered in the analysis of linear perturbations in last chapter, see also [29, 133]. Afterwards, we discuss the invariant characterization of FL-nonlinear scalar field models and derive the corresponding linearised system of evolution equations.

Some final remarks are then made about the analysis of the eigenvalues of the non-principal part of the system and, in particular, about the application of the so called *stability eigenvalue condition*, for coefficients depending on time.

[1]The presence of a cosmological constant is crucial for global existence and exponential decay, since the minimum of the Hubble function must be strictly positive, namely $H_{min}=\sqrt{\Lambda/3}$.

4.1 Friedrich's frame formulation of the Einstein Field Equations

In this section, we provide a brief introduction to Friedrich's frame formulation of the Einstein field equations. The basic equation of Friedrich's construction is the contracted Bianchi identity. From the latter, it is possible to deduce hyperbolic propagation equations for the conformal Weyl tensor for a wide class of gauge choices.

4.1.1 Basic definitions and notation

In order to implement the frame formulation of the Einstein field equations, one defines locally an orthonormal moving frame or *tetrad* with respect to the metric $\mathbf{g}$ in an open neighbourhood $\mathcal{U} \subset \mathcal{M}$. The frame is a set $\{\boldsymbol{e}_a\}$ of linearly independent vector fields in the tangent space $T_p(\mathcal{M})$ at each point $p \in \mathcal{U}$ such that

$$\mathbf{g}\left(\boldsymbol{e}_a, \boldsymbol{e}_b\right) = \eta_{ab}, \quad a,\, b = 0, \ldots 3, \tag{4.5}$$

where $\eta_{ab} = \operatorname{diag}(-1,1,1,1)$ and latin letters (except for the i, j) are used for frame indices . The *norm* of a vector field, $\boldsymbol{v} \in T_p(\mathcal{M})$, in an orthonormal frame is defined as

$$|\boldsymbol{v}|^2 \equiv \mathbf{g}(\boldsymbol{v}, \boldsymbol{v}) = v^a v^b \eta_{ab}$$

and in terms of a coordinate basis set $\{\partial_\alpha\}$ we have $\boldsymbol{e}_a = e_a{}^\mu \partial_\mu$. Condition (4.5) gives

$$\eta_{ab} = e_a{}^\mu e_b{}^\nu g_{\mu\nu}.$$

where as usual $\mu, \nu = 0, 1, 2, 3$. The *frame commutator* is written as

$$[\boldsymbol{e}_a, \boldsymbol{e}_b] = c^c{}_{ab} \boldsymbol{e}_c, \tag{4.6}$$

where $c^c{}_{ab}$ are the *structure coefficients*. The *dual basis* or *coframe* is the set of linear forms $\{\boldsymbol{\theta}^b\}$ belonging to the dual space $T^*_p(\mathcal{M})$ at each point $p \in \mathcal{U}$ defined by the pairing $\langle \boldsymbol{\theta}^b, \boldsymbol{e}_a \rangle = \delta^b_a$. In terms of the dual basis we can write condition (4.5) as

$$\mathbf{g} = -\left(\boldsymbol{\theta}^0\right)^2 + \sum_{a=1}^{3} \left(\boldsymbol{\theta}^a\right)^2.$$

The *spacetime (Levi-Civita) connection* in an orthonormal basis is defined by

$$\boldsymbol{\nabla}_a \boldsymbol{e}_b \equiv \gamma^c{}_{ba} \boldsymbol{e}_c,$$

where $\gamma^c{}_{ba}$ are the *connections coefficients*. The covariant derivative of a tensor in $\mathcal{M}$ can be written as

$$\begin{aligned} \boldsymbol{\nabla}_a v_{q_1 \ldots q_s}{}^{p_1 \ldots p_r} = \boldsymbol{e}_a \left(v_{q_1 \cdots q_s}{}^{p_1 \cdots p_r}\right) + \gamma^{p_1}{}_{fa} v_{q_1 \cdots q_s}{}^{f \cdots p_r} + \cdots \\ \cdots + \gamma^{p_r}{}_{fa} v_{q_1 \cdots q_s}{}^{p_1 \cdots f} - \gamma^f{}_{q_1 a} v_{f \cdots q_s}{}^{p_1 \cdots p_r} - \cdots - \gamma^f{}_{q_s a} v_{q_1 \cdots f}{}^{p_1 \cdots p_r}. \end{aligned}$$

The torsion free and metric compatibility conditions imply, respectively, that

$$c^c{}_{ab} = \gamma^c{}_{ba} - \gamma^c{}_{ab}, \quad \gamma^e{}_{ba} \eta_{ec} + \gamma^e{}_{ca} \eta_{eb} = 0.$$

The equations for the frame coefficients $\{e_a{}^\mu\}$ are given by equation (4.6) in terms of the connection coefficients. In turn, equations for the connection coefficients are obtained from the Ricci identity

$$R^a{}_{bcd} = \boldsymbol{e}_c\left(\gamma^a{}_{bd}\right) - \boldsymbol{e}_d\left(\gamma^a{}_{bc}\right) + \gamma^a{}_{fc}\gamma^f{}_{bd} - \gamma^a{}_{fd}\gamma^f{}_{bc} - \gamma^a{}_{bf}\left(\gamma^f{}_{dc} - \gamma^f{}_{cd}\right). \tag{4.7}$$

The Riemann tensor can be decomposed in terms of the conformal Weyl tensor $\boldsymbol{C}$ and the Schouten tensor $\boldsymbol{S}$ as

$$R^a{}_{bcd} = C^a{}_{bcd} + \delta^a{}_{[c}S_{d]b} - \eta_{b[c}S_{d]}{}^a. \tag{4.8}$$

For future use, we introduce the *Friedrich tensor* $\boldsymbol{F}$ via

$$F_{abcd} \equiv C_{abcd} - \eta_{a[c}S_{d]b}, \tag{4.9}$$

and its dual with respect to the last pair of indices

$${}^\star F_{abcd} = {}^\star C_{abcd} + \frac{1}{2}S_{pb}\epsilon^p{}_{acd}, \tag{4.10}$$

where ϵ_{abcd} is the usual Levi-Civita totally antisymmetric symbol with $\epsilon_{0123} = 1$. In terms of the Friedrich tensor, one finds that the contracted Bianchi identities read

$$\nabla_a F^a_{bcd} = 0, \quad \nabla_a {}^\star F^a_{bcd} = 0. \tag{4.11}$$

4.1.2 Orthonormal decomposition of the field equations

The equations of Friedrich's frame formulation of the Einstein field equations are given by (4.6) (4.7) and (4.11), together with the decomposition (4.8). The independent variables of the system are

$$\left(e_a{}^\mu, \gamma^a{}_{bc}, C^a{}_{bcd}, S_{bc}\right).$$

In what follows, we will shall decompose the equations and relevant tensors in terms of their parallel and orthogonal components with respect to the time-like frame. We write $\boldsymbol{N} \equiv \boldsymbol{e}_0$ and set

$$\boldsymbol{N} = N^a\boldsymbol{e}_a, \quad N^a = \delta_0{}^a,$$

where $N_a = -\delta_a{}^0$ in our signature. In terms of these objects, tensor fields which are orthogonal to the timelike frame-vector are defined by

$$T_{a_1\dots a_p\dots a_q}N^{a_p} = 0, \quad p = 1, 2, \dots, q.$$

Next, one defines

$$h_{ab} \equiv \eta_{ab} + N_aN_b,$$

where the projector onto the orthogonal 3-subspaces satisfies $h_a{}^c = \eta^{bc}h_{ab}$. The *spatial covariant derivative* is then given by

$$D_aT_{q_1\cdots q_r} = h_a{}^bh_{q_1}{}^{p_1}\cdots h_{q_r}{}^{p_r}\nabla_bT_{p_1\cdots p_r}.$$

In particular, one has that

$$D_ah_{bd} = 0, \quad D_a\epsilon_{bcd} = 0,$$

where ϵ_{bcd} is the spatial Levi-Civita symbol and the indices run from 1 to 3. In order to further proceed with the geometric decomposition one defines the *acceleration vector* by

$$\boldsymbol{a} \equiv \boldsymbol{\nabla}_0\boldsymbol{e}_0 = \gamma^p{}_{00}\boldsymbol{e}_p, \qquad p = 1, 2, 3.$$

It follows then that $a^p = \gamma^p{}_{00}$ or equivalently, $a_p = \gamma^0{}_{p0}$. We will also consider the so-called *Weingarten map* given by

$$\boldsymbol{\chi}(\boldsymbol{e}_a) \equiv \boldsymbol{\nabla}_a\boldsymbol{e}_0 = \gamma^p{}_{0a}\boldsymbol{e}_p, \quad a, p = 1, 2, 3,$$

so that $\chi_a{}^p = \gamma^p{}_{0a}$. The tensor χ_{ab} can be written in terms of its irreducible parts as

$$\chi_{ab} = \gamma^0{}_{ba} = \left(\chi^{ST}\right)_{ab} + \frac{1}{3}\chi h_{ab} + \left(\chi^A\right)_{ab},$$

where $(\chi^{ST})_{ab}$, χ, $(\chi^A)_{ab}$ denote, respectively, its symmetric trace-free, trace and antisymmetric parts. If the flow of $\boldsymbol{e}_0$ is hypersurface orthogonal then one has that $(\chi^A)_{ab} = 0$ and that

$$\frac{1}{2}\pounds_N h_{ab} = \chi_{(ab)} = (\chi^{ST})_{ab} + \frac{1}{3}\chi h_{ab}, \tag{4.12}$$

where $\pounds_N$ denotes the Lie derivative along $\boldsymbol{N}$ and $\nabla_a N^p = -N_a a^p + \chi_a{}^p$. Finally, the 4-dimensional Levi-Civita symbol is also decomposed using

$$\epsilon_{abcd} = 2\epsilon_{ab[c}N_{d]} - 2N_{[a}\epsilon_{b]cd}.$$

Now, defining $\tilde{F}_{bcd} \equiv \nabla_a F^a{}_{bcd}$, it follows that the first contracted Bianchi identity can be written as

$$\tilde{F}_{bcd} = N_b\left[\bar{\tilde{F}}_{0c0}N_d - \bar{\tilde{F}}_{0d0}N_c\right] + 2\bar{\tilde{F}}_{b0[c}N_{d]} - N_b\bar{\tilde{F}}_{0cd} + \bar{\tilde{F}}_{bcd} = 0, \tag{4.13}$$

where contractions with $\boldsymbol{N}$ are denoted by the index 0 and the bar $^-$ indicates that the remaining indices are spatial. For example, $\bar{\tilde{F}}_{b0d} \equiv h_b{}^q N^r h_d{}^s \tilde{F}_{qrs}$. Given the vector $\boldsymbol{N}$, the Weyl tensor is uniquely determined through its *electric* and *magnetic* parts defined, respectively, by

$$E_{ab} \equiv h_a{}^q h_b{}^d N^p N^c C_{pqcd}, \quad B_{bd} \equiv h_b{}^p h_d{}^q N^a N^c C^\star_{apcq}.$$

In terms of the latter, the Weyl tensor and its dual can be written as

$$C_{abcd} = 2\left[l_{a[c}E_{d]b} - l_{b[c}E_{d]a}\right] - 2\left[N_{[c}B_{d]p}\epsilon^p{}_{ab} + N_{[a}B_{b]p}\epsilon^p{}_{cd}\right] \tag{4.14}$$

$${}^\star C_{abcd} = 2N_{[a}E_{b]p}\epsilon^p{}_{cd} - 4E_{p[a}\epsilon_{b]}{}^p{}_{[c}N_{d]} - 4N_{[a}B_{b][c}N_{d]} - B_{pq}\epsilon^p{}_{ab}\epsilon^q{}_{cd} \tag{4.15}$$

where $l_{ab} \equiv h_{ab} + N_a N_b$.

4.2 Nonlinear scalar fields in the frame formalism

In this section, we introduce a description of nonlinear scalar fields which is particularly well adapted to the analysis of the present chapter.

4.2.1 Basic equations

In general the energy-momentum tensor for a smooth nonlinear scalar field (1.11) has the form

$$\boldsymbol{T} = \boldsymbol{\psi}\otimes\boldsymbol{\psi} - \left(\frac{1}{2}|\boldsymbol{\psi}|^2 + \mathcal{V}(\phi)\right)\mathbf{g},$$

where we have defined the 1-form

$$\boldsymbol{\psi} \equiv \boldsymbol{\nabla}\phi$$

with $\boldsymbol{\nabla}$ the spacetime connection. Accordingly, we define

$$\psi_a \equiv \boldsymbol{\psi}\left(\boldsymbol{e}_a\right) = (\psi, \bar{\psi}_a), \tag{4.16}$$

where we have written

$$\psi \equiv \psi_0 = \pounds_N\phi \tag{4.17}$$

and

$$\bar{\psi}_a \equiv h_a{}^b\psi_b = D_a\phi. \tag{4.18}$$

The components of the energy-momentum tensor $\boldsymbol{T}$ with respect to the tetrad $\{\boldsymbol{e}_a\}$ are then given by

$$T_{ab} = \psi_a\psi_b - \left(\frac{1}{2}|\boldsymbol{\psi}|^2 + \mathcal{V}(\phi)\right)\eta_{ab}, \tag{4.19}$$

while its trace is

$$T = -|\boldsymbol{\psi}|^2 - 4\mathcal{V}(\phi).$$

The Einstein field equations (1.3), imply for the components of the Ricci tensor that

$$R_{ab} = \psi_a\psi_b + \mathcal{V}(\phi)\eta_{ab},$$

while the Ricci scalar is given by

$$R = -T = |\boldsymbol{\psi}|^2 + 4\mathcal{V}(\phi).$$

From these expressions, it follows that the components of the Schouten tensor with respect to the frame $\{\boldsymbol{e}_a\}$ are given by

$$S_{ab} = \psi_a\psi_b - \frac{1}{3}\left(\frac{1}{2}|\boldsymbol{\psi}|^2 - \mathcal{V}(\phi)\right)\eta_{ab}.$$

4.2.2 Gauge considerations

In order to construct an adapted frame to our particular problem, we let $\boldsymbol{\psi} \equiv \alpha\boldsymbol{e}_0$. It follows that

$$\psi^a = \alpha\delta_0^a, \tag{4.20}$$

so that

$$\alpha = -\psi \quad \text{and} \quad D^a\phi = 0.$$

Accordingly,

$$|\boldsymbol{\psi}|^2 = \mathbf{g}(\boldsymbol{\psi}, \boldsymbol{\psi}) = \alpha^2\eta_{00} = -\alpha^2, \quad \alpha = \pm\sqrt{-|\boldsymbol{\psi}|^2}. \tag{4.21}$$

If the vector $\boldsymbol{\psi}$ is taken to be future oriented, then one must choose α to be positive. In terms of a coordinate basis the latter implies

$$\psi^\mu = \alpha e_0{}^\mu = -\psi e_0{}^\mu, \quad e_0{}^\mu = \frac{\nabla^\mu\phi}{\sqrt{-|\boldsymbol{\psi}|^2}} \tag{4.22}$$

and

$$D_a\phi = 0, \quad \bar{e}_a{}^\mu\nabla_\mu\phi = 0. \tag{4.23}$$

With this choice, we have that

$$\psi_a = -\psi N_a,$$

and therefore

$$T_{ab} = \left(\frac{1}{2}\psi^2 + \mathcal{V}(\phi)\right)N_aN_b + \left(\frac{1}{2}\psi^2 - \mathcal{V}(\phi)\right)h_{ab}, \tag{4.24}$$

$$S_{ab} = \frac{1}{3}\left(\frac{5}{2}\psi^2 - \mathcal{V}(\phi)\right)N_aN_b + \frac{1}{3}\left(\frac{1}{2}\psi^2 + \mathcal{V}(\phi)\right)h_{ab}. \tag{4.25}$$

Remark 1. *In general, $\boldsymbol{\psi}$ might not be timelike for all times. However, when fixing $\boldsymbol{\psi} = \alpha \boldsymbol{e}_0$, we assume it to be always timelike. If this is not the case, then our gauge breaks and the evolution stops. Thus, we are only considering a subset of solutions to the EFEs for which our gauge is valid. It is important to notice that this choice is not empty as there are examples of FL-scalar field solutions for which $\boldsymbol{\psi}$ is always timelike —see section 5.1. Moreover, one expects that this property will also hold for small enough perturbations of these reference FL solutions.*

Using equations (4.16) and (4.21), the expression for the conservation of the energy-momentum tensor takes the form:

$$\begin{aligned}
\nabla^a T_{ab} =& \nabla^a \left(\psi^2 N_a N_b + \left(\frac{1}{2}\psi^2 - \mathcal{V}(\phi) \right) \eta_{ab} \right) \\
=& 2\psi N_b N^a \left(\nabla_a \psi \right) + \psi^2 \left(N_b \left(\nabla_a N^a \right) + N^a \left(\nabla_a N_b \right) \right) + \nabla_b \left(\frac{1}{2}\psi^2 - \mathcal{V}(\phi) \right) \qquad (4.26) \\
=& \left(2\psi \pounds_{\boldsymbol{N}} \psi + \psi^2 \chi + \psi \frac{d\mathcal{V}}{d\phi} \right) N_b + \psi^2 a_b + \psi \nabla_b \psi = 0.
\end{aligned}$$

From the latter, projecting with respect to the timelike frame one obtains:

$$N^b \left(\nabla^a T_{ab} \right) = 0, \quad \pounds_{\boldsymbol{N}} \psi + \chi \psi + \frac{d\mathcal{V}}{d\phi} = 0, \qquad (4.27)$$

$$h_c{}^b \left(\nabla^a T_{ab} \right) = 0, \qquad D_c \psi + \psi a_c = 0. \qquad (4.28)$$

Moreover, using the fact that $D_a \phi = 0$ in the orthogonal subspaces to $\boldsymbol{e}_0$, one obtains from equation (4.6)

$$\left[\bar{\boldsymbol{e}}_a, \bar{\boldsymbol{e}}_b \right] \phi = 2 \left(\chi^A \right)_{ab} \psi = 0,$$

which implies

$$\left(\chi^A \right)_{ab} = 0. \qquad (4.29)$$

Remark 2. *Following Friedrich in [86], one could as well have defined*

$$\nabla^a T_{ab} = q_b + q N_b, \quad J_{ab} = \nabla_{[a} q_{b]}. \qquad (4.30)$$

Then, instead of using the condition on the vanishing of the divergence of the energy-momentum tensor, one could include the equations $q = 0$ and $q_b = 0$ as a part of the equations determining the Einstein-scalar field system in the frame representation. Once the gauge is fixed, the first equation in (4.30) *appears in the reduced system of evolution equations while the second part is regarded as a zero quantity —see equation (4.44) in [86]. It can be shown that the zero quantities satisfy a system of* subsidiary evolution equations*. For this, it can be shown that the zero quantities vanish if they are zero on the initial hypersurface. For the quantity q_b the relevant subsidiary equation is given in equation (4.70) of [86]. We also notice that the evolution for the acceleration can be computed from the tensor J_{ab}.*

4.3 The Einstein-Friedrich-nonlinear scalar field system

In this section, we derive a first order symmetric hyperbolic system for the EFEs coupled to a nonlinear scalar field. Making use of the Bianchi identity and the energy-momentum tensor given by equation (4.19), we derive the propagation equations for the *electric* and *magnetic* parts of the conformal Weyl tensor. After fixing the gauge, we complete the reduced system of evolution equations by deriving equations for the frame and the connection coefficients. In the last part of this section, we make some remarks concerning the hyperbolicity of the system.

4.3.1 Basic expressions

We start by computing the various components for the Friedrich tensor $\boldsymbol{F}$. Using equations (4.14) and (4.25), one finds

$$\begin{aligned}
\bar{F}_{00c0} &= 0 = -\bar{F}_{000c}, \quad \bar{F}_{00cd} = 0 = -\bar{F}_{00dc}, \\
\bar{F}_{a00d} &= -E_{ad} + \frac{1}{6}\left(\frac{5}{2}\psi^2 - \mathcal{V}(\phi)\right) h_{ad} = -\bar{F}_{a0d0}, \\
\bar{F}_{ab0d} &= B_{dp}\epsilon^p{}_{ab} = -\bar{F}_{abd0} = -\bar{F}_{ba0d}, \\
\bar{F}_{0bcd} &= B_{bp}\epsilon^p{}_{cd} = -\bar{F}_{0bdc} = -\bar{F}_{b0cd}, \\
\bar{F}_{0b0d} &= E_{bd} + \frac{1}{6}\left(\frac{1}{2}\psi^2 + \mathcal{V}(\phi)\right) h_{bd} = -\bar{F}_{0bd0}, \\
\bar{F}_{abcd} &= -2\left(h_{b[c}E_{d]a} - h_{a[c}E_{d]b}\right) - \frac{1}{6}\left(\frac{1}{2}\psi^2 + \mathcal{V}(\phi)\right)\left(h_{ac}h_{db} - h_{ad}h_{cb}\right),
\end{aligned} \tag{4.31}$$

with the non-vanishing traces

$$\begin{aligned}
h^{ac}\bar{F}_{a0c0} &= \frac{1}{2}\left(\frac{5}{2}\psi^2 - \mathcal{V}(\phi)\right), \\
h^{bd}\bar{F}_{0b0d} &= \frac{1}{2}\left(\frac{1}{2}\psi^2 + \mathcal{V}(\phi)\right) = -h^{bd}\bar{F}_{0bd0}, \\
h^{bd}\bar{F}_{abcd} &= E_{ac} - \frac{1}{3}\left(\frac{1}{2}\psi^2 + \mathcal{V}(\phi)\right) h_{ac} = F'^b{}_{abc}, \\
h^{ac}h^{bd}\bar{F}_{abcd} &= -\frac{1}{2}\psi^2 - \mathcal{V}(\phi).
\end{aligned} \tag{4.32}$$

Using the expression (4.10) with equations (4.15) and (4.25), we get the following components of the dual ${}^\star\boldsymbol{F}$:

$$\begin{aligned}
{}^\star\bar{F}_{00c0} &= -{}^\star\bar{F}_{000c} = 0, \quad {}^\star\bar{F}_{00cd} = -{}^\star\bar{F}_{00dc} = 0, \\
{}^\star\bar{F}_{a0c0} &= -{}^\star\bar{F}_{a00c} = B_{ac}, \\
{}^\star\bar{F}_{abc0} &= -2E_{p[b}\epsilon_{a]}{}^p{}_c - \frac{1}{6}\left(\frac{1}{2}\psi^2 + \mathcal{V}(\phi)\right)\epsilon_{bac} = -{}^\star\bar{F}_{ab0c}, \\
{}^\star\bar{F}_{a0cd} &= E_{ap}\epsilon^p{}_{cd} - \frac{1}{6}\left(\frac{5}{2}\psi^2 - \mathcal{V}(\phi)\right)\epsilon_{acd}, \\
{}^\star\bar{F}_{0bcd} &= -E_{bp}\epsilon^p{}_{cd} - \frac{1}{6}\left(\frac{1}{2}\psi^2 + \mathcal{V}(\phi)\right)\epsilon_{bcd}, \\
{}^\star\bar{F}_{0b0d} &= B_{bd}, \quad {}^\star\bar{F}_{abcd} = -B_{pq}\epsilon^p{}_{ab}\epsilon^q{}_{cd}.
\end{aligned} \tag{4.33}$$

4.3.2 The Bianchi equations

If one substitutes the expressions for the Friedrich tensor derived in the previous section into the first Bianchi identities (4.13), one obtains the following relations for the components of the zero quantity $\tilde{F}_{abc}$:

$$\begin{aligned}
\tilde{\bar{F}}_{0c0} &= -\pounds_{\boldsymbol{N}}\bar{F}_{00c0} + D^q\bar{F}_{q0c0} + \chi_c{}^s\bar{F}_{00s0} - \chi\bar{F}_{00c0} - \chi^{qb}\left(\bar{F}_{qbc0} + \bar{F}_{q0cb}\right) + a^b\left(\bar{F}_{0bc0} + \bar{F}_{00cb} + \bar{F}_{b0c0}\right), \\
\tilde{\bar{F}}_{0cd} &= -\pounds_{\boldsymbol{N}}\bar{F}_{00cd} + D^q\bar{F}_{q0cd} + a^b\bar{F}_{0bcd} + a^q\bar{F}_{q0cd} - \chi^{qb}\bar{F}_{qbcd} - \chi\bar{F}_{00cd} - \chi^q{}_c\bar{F}_{q00d} - \chi^q{}_d\bar{F}_{q0c0} \\
&\quad + \chi_c{}^s\bar{F}_{00sd} + \chi_d{}^s\bar{F}_{00cs} + a_c\bar{F}_{000d} + a_d\bar{F}_{00c0}, \\
\tilde{\bar{F}}_{b0d} &= -\pounds_{\boldsymbol{N}}\bar{F}_{0b0d} + D^a\bar{F}_{ab0d} - \chi\bar{F}_{0b0d} - \chi^a{}_b\bar{F}_{a00d} - \chi^{ac}\bar{F}_{abcd} + \chi_b{}^s\bar{F}_{0s0d} + a_b\bar{F}_{000d} + \chi_d{}^s\bar{F}_{0b0s} \\
&\quad + a^q\bar{F}_{qb0d} + a^c\bar{F}_{0bcd}, \\
\bar{F}_{bcd} &= -\pounds_{\boldsymbol{N}}\bar{F}_{0bcd} + D^a\bar{F}_{abcd} + a^q\bar{F}_{qbcd} + \left(a_b\bar{F}_{00cd} + a_c\bar{F}_{0b0d} + a_d\bar{F}_{0bc0}\right) - \chi\bar{F}_{0bcd} \\
&\quad + \chi_b{}^q\bar{F}_{0qcd} + \chi_c{}^q\bar{F}_{0bqd} + \chi_d{}^q\bar{F}_{0bcq} - \chi^q{}_b\bar{F}_{q0cd} - \chi^q{}_c\bar{F}_{qb0d} - \chi^q{}_d\bar{F}_{qbc0},
\end{aligned} \tag{4.34}$$

where we have used the fact that $\boldsymbol{F}$ is anti-symmetric in the last two indices —see e.g. [85]. Similar relations hold for the dual ${}^\star\tilde{\boldsymbol{F}}$.

Remark 3. *In [86] —cfr. equation (4.47)— suitable zero quantities are defined by using the decomposition in terms of irreducible components of* $\tilde{\boldsymbol{F}}$.

The evolution equation for the electric part of the Weyl tensor

An evolution equation for the electric part of the Weyl tensor can be obtained using the third equation of (4.34) together with the expressions (4.31)-(4.33), and then symmetrising with respect to the indices (bd). One obtains the equation

$$\begin{aligned}\bar{\tilde{F}}_{(b|0|d)} =& -\pounds_{\boldsymbol{N}} E_{bd} - \frac{1}{6}\left(\frac{1}{2}\psi^2 + \mathcal{V}(\phi)\right)\pounds_{\boldsymbol{N}} h_{bd} - \frac{1}{6} h_{bd}\pounds_{\boldsymbol{N}}\left(\frac{1}{2}\psi^2 + \mathcal{V}(\phi)\right) + D_a B_{p(d}\epsilon_{b)}{}^{pa} \\ & + 2a_a B_{p(b}\epsilon_{d)}{}^{pa} - 2\chi E_{bd} + 2\chi^a{}_{(b}E_{d)q} + 3\chi_{(b}{}^q E_{d)q} - h_{db}\chi^{ac}E_{ac} - \frac{1}{3}\left(\psi^2 - \mathcal{V}(\phi)\right)\chi_{(bd)}.\end{aligned}$$

Similarly, using equation (4.12), we get

$$\begin{aligned}\bar{\tilde{F}}_{(b|0|d)} =& -\pounds_{\boldsymbol{N}} E_{bd} + D_a B_{p(d}\epsilon_{b)}{}^{pa} + 2a_a B_{p(b}\epsilon_{d)}{}^{pa} - 2\chi E_{bd} + 2\chi^a{}_{(b}E_{d)q} + 3\chi_{(b}{}^q E_{d)q} - h_{db}\chi^{ac}E_{ac} \\ & - \frac{1}{2}\psi^2\chi_{(bd)} - \frac{1}{6}h_{bd}\pounds_{\boldsymbol{N}}\left(\frac{1}{2}\psi^2 + \mathcal{V}(\phi)\right).\end{aligned}$$

The trace of the previous expression is given by

$$h^{rs}\bar{\tilde{F}}_{(r|0|s)} = -\frac{1}{2}\psi^2\chi - \frac{1}{2}\pounds_{\boldsymbol{N}}\left(\frac{1}{2}\psi^2 + \mathcal{V}(\phi)\right),$$

which is the evolution equation for the scalar field —i.e. the equation expressing the conservation of energy. From this, it follows that E_{ab} remains trace free during the evolution if the data is given accordingly. Thus, taking the difference of the last two equations, and taking into account (4.29), the evolution equation for the components of the tensor E_{ab} can be written as

$$\begin{aligned}2\pounds_{\boldsymbol{N}} E_{bd} - 2D_a B_{p(d}\epsilon_{b)}{}^{pa} \;=&\; 4a_a B_{p(b}\epsilon_{d)}{}^{pa} - 4\chi E_{bd} + 10\chi^q{}_{(b}E_{d)q} - 2h_{db}\chi^{ac}E_{ac} \\ & -\psi^2\left(\chi_{(bd)} - \frac{1}{3}\chi h_{bd}\right).\end{aligned} \tag{4.35}$$

The evolution equation for the magnetic part of the Weyl tensor

An evolution equation for the magnetic part of the Weyl tensor can also be derived from the third equation of (4.34) using the expressions (4.31)-(4.33). A computation yields

$$\begin{aligned}\tilde{F}_{b0d} =& -\pounds_{\boldsymbol{N}} B_{bd} + D^a\left(2E_{p[b}\epsilon_{a]}{}^p{}_d + \frac{1}{6}\left(\frac{1}{2}\psi^2 + \mathcal{V}(\phi)\right)\epsilon_{bad}\right) - \chi B_{bd} + \chi^a{}_b B_{ad} + 2\chi_{(b}{}^a B_{d)a} \\ & + 2a_q B_{pb}\epsilon^{qp}{}_d + a^q E_{pq}\epsilon^p{}_{bd} + \chi^{qb}B_{pq}\epsilon^p{}_{ab}\epsilon^q{}_{cd}.\end{aligned}$$

Now, since B_{bd} is a symmetric tensor, all the information about its evolution is contained in the symmetrised expression $\tilde{F}_{(b|0|d)}$. Consequently, symmetrising the previous equation with respect to the spatial indices (bd), and using (4.29), we get

$$2\pounds_{\boldsymbol{N}} B_{bd} - 2D_a E_{p(b}\epsilon_{d)}{}^{ap} = -4a_a E_{p(b}\epsilon_{d)}{}^{pa} + 6\chi_{(b}{}^a B_{d)a} - 2\chi B_{bd} + 2\chi_{ac}B_{pq}\epsilon^{pa}{}_{(b}\epsilon_{d)}{}^{qc}. \tag{4.36}$$

Ignoring the information about the trace, the principal part of the equations (4.35) and (4.36) is a symmetric matrix for the variables E_{cd}, B_{cd}, $c \leq d$. Explicitly,

$$
\begin{pmatrix}
2\boldsymbol{e}_0 & 0 & 0 & 0 & 0 & 0 & 0 & -D_1 & D_2 & D_3 & -D_3 & 0 \\
0 & 2\boldsymbol{e}_0 & 0 & 0 & 0 & 0 & D_1 & 0 & -D_3 & -D_2 & 0 & D_2 \\
0 & 0 & 2\boldsymbol{e}_0 & 0 & 0 & 0 & -D_2 & D_3 & 0 & 0 & D_1 & -D_1 \\
0 & 0 & 0 & \boldsymbol{e}_0 & 0 & 0 & -D_3 & D_2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \boldsymbol{e}_0 & 0 & D_3 & 0 & -D_1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \boldsymbol{e}_0 & 0 & -D_2 & D_1 & 0 & 0 & 0 \\
0 & D_1 & -D_2 & -D_3 & D_3 & 0 & 2\boldsymbol{e}_0 & 0 & 0 & 0 & 0 & 0 \\
-D_1 & 0 & D_3 & D_2 & 0 & -D_2 & 0 & 2\boldsymbol{e}_0 & 0 & 0 & 0 & 0 \\
D_2 & -D_3 & 0 & 0 & -D_1 & D_1 & 0 & 0 & 2\boldsymbol{e}_0 & 0 & 0 & 0 \\
D_3 & -D_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boldsymbol{e}_0 & 0 & 0 \\
-D_3 & 0 & D_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boldsymbol{e}_0 & 0 \\
0 & D_2 & -D_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boldsymbol{e}_0
\end{pmatrix}
\begin{pmatrix}
E_{12} \\ E_{13} \\ E_{23} \\ E_{11} \\ E_{22} \\ E_{33} \\ B_{12} \\ B_{13} \\ B_{23} \\ B_{11} \\ B_{22} \\ B_{33}
\end{pmatrix}.
$$

Remark 4. *The trace-freeness of the tensors E_{ab} and B_{ab} can be recovered by assuming it initially. Then, using the evolution equations, it can be showed that E_{ab} and B_{ab} are trace-free at later times —see e.g. the discussion in [85] for the perfect fluid case.*

4.3.3 The Lagrangian description and Fermi transport

In order to deduce the remaining evolution equations, we will adopt a Lagrangian description. This point of view amounts to requiring the timelike vector of the orthonormal frame to follow the matter flow lines. Accordingly, we introduce coordinates $(t, \boldsymbol{x})$ such that

$$\boldsymbol{e}_0 = \partial_t, \quad e_0{}^\mu = \delta_0^\mu. \tag{4.37}$$

This particular choice is equivalent to setting $\boldsymbol{\theta}^b = \theta^b{}_j \mathrm{d}x^j$ while at the same time fixing the lapse function to one[2]. With this choice (since $\pounds_N = \partial_t$), we have from equations (4.17), (4.27), and (4.23) that

$$\partial_t \phi = \psi \equiv N^a \psi_a = -\alpha < 0, \tag{4.38}$$

$$\partial_t \psi = -\psi\chi - \frac{d\mathcal{V}}{d\phi}, \tag{4.39}$$

$$\bar{e}_a{}^0 \psi = -\bar{e}_a{}^j \nabla_j \phi. \tag{4.40}$$

Now, the timelike coframe is given in terms of the natural cobasis through the relation

$$\boldsymbol{\theta}^0 = \mathrm{d}t + \beta_j \mathrm{d}x^j,$$

while the spatial frame vectors are found to be

$$\bar{\boldsymbol{e}}_a = (\theta_a{}^j)^{-1}(\partial_j - \beta_j \partial_t), \quad \bar{e}_a{}^0 = (\theta_a{}^j)^{-1}\beta_j, \quad \bar{e}_a{}^j = (\theta_a{}^j)^{-1}. \tag{4.41}$$

It then follows from equations (4.40) and (4.41) that

$$\beta_j = -\frac{1}{\psi}\partial_j \phi. \tag{4.42}$$

Thus, since β_j is nonzero, the surfaces of constant time are not necessarily spacelike for the characteristic cone and this could be a problem for the hyperbolicity of the system —see [37, 38].

[2]See also [37], where a symmetric hyperbolic system was obtained for the Einstein-Euler system. This construction holds for an arbitrary Eulerian picture.

Finally, the remaining frame components are chosen to be Fermi propagated along $\boldsymbol{e}_0$. That is, we require

$$\nabla_0 \boldsymbol{e}_a - \left(\mathbf{g}\left(\boldsymbol{e}_a, \nabla_0 \boldsymbol{e}_0\right) \boldsymbol{e}_0 - \mathbf{g}\left(\boldsymbol{e}_a, \boldsymbol{e}_0\right) \nabla_0 \boldsymbol{e}_0\right) = 0,$$

which implies

$$\bar{\gamma}^a{}_{b0} = 0.$$

4.3.4 Evolution equation for the frame coefficients

As already mentioned, the evolution equations are obtained from the relation (4.6) which yields

$$[\boldsymbol{e}_0, \bar{\boldsymbol{e}}_b] = a_b \boldsymbol{e}_0 - \bar{\gamma}^c{}_{0b} \bar{\boldsymbol{e}}_c,$$

where $\bar{\gamma}^c{}_{b0} = 0$ (Fermi gauge) has been used. Therefore, the evolution equations for the remaining frame coefficients read

$$\begin{aligned} \partial_t \bar{e}_b{}^i &= -\chi_b{}^c \bar{e}_c{}^i, \\ \partial_t \bar{e}_b{}^0 &= a_b - \chi_b{}^c \bar{e}_c{}^0, \end{aligned} \tag{4.43}$$

which, together with (4.41), imply propagation equations for the components of the metric in the local coordinate system. In particular, one has

$$\partial_t \beta_j = \bar{\theta}^b{}_j a_b$$

with β_j given by equation (4.42) —see also equation (6.2) in [37] for an arbitrary lapse U.

4.3.5 Evolution equations for the connection coefficients

The equations for the connection coefficients are obtained from the splitting of the Riemann tensor with respect to the frame $\{\boldsymbol{e}_a\}$. In general, we have that

$$\begin{aligned} \bar{R}^a{}_{b0d} &= \boldsymbol{e}_0\left(\gamma^a{}_{bd}\right) - D_d \gamma^a{}_{b0} - a_d \gamma^a{}_{b0} - \left(\gamma^p{}_{d0} - \chi_d{}^p\right) \bar{\gamma}^a{}_{bp} - a_b \chi_d{}^a + a^a \chi_{db}, \\ \bar{R}^0{}_{b0d} &= \boldsymbol{e}_0\left(\chi_{db}\right) - D_d a_b - a_b a_d + \chi_{pb} \chi_d{}^p - \chi_{dp} \gamma^p{}_{b0} - \chi_{pb} \gamma^p{}_{d0}, \\ \bar{R}^a{}_{0cd} &= D_c \chi_d{}^a - D_d \chi_c{}^a - a^a\left(\chi_{cd} - \chi_{dc}\right), \\ \bar{R}^a{}_{bcd} &= \tilde{R}^a{}_{bcd} + \chi_c{}^a \chi_{db} - \chi_d{}^a \chi_{cb} - \gamma^a{}_{b0}\left(\chi_{cd} - \chi_{dc}\right), \end{aligned} \tag{4.44}$$

where $\tilde{R}^a{}_{bcd}$ denotes the Riemann tensor constructed only with the spatial connection coefficients $\bar{\gamma}^c{}_{ab}$. The first two identities give evolution equations once the Lagrangian gauge is introduced. The remaining two equations are the *quasi-constraints* for the connection coefficients —see [37, 38]. No equations for the connection coefficient associated to the acceleration can be deduced from these identities. In the sequel, it will be shown how evolution equations for the acceleration can be obtained for our particular problem.

From equations (4.44), we can also deduce two important equations relating the Ricci tensor to the connection:

$$\begin{aligned} R_{00} &= -\boldsymbol{e}_0\left(\chi\right) + D_p a^p - \chi_p{}^b \chi_b{}^p + a_p a^p \\ \bar{R}_{0d} &= D_c \chi_d{}^c - D_d \chi - 2\, a^c \left(\chi^A\right)_{cd}. \end{aligned} \tag{4.45}$$

The first identity in (4.44), together with the conditions for the Lagrangian and Fermi gauge provide the equation

$$\partial_t \gamma^a{}_{bd} = -\gamma^a{}_{bp} \chi_d{}^p + 2 h^{ap} \chi_{d[p} a_{b]} + B_{dp} \epsilon^{pa}{}_b, \tag{4.46}$$

describing the evolution of the spatial connection coefficients $\bar{\gamma}^c{}_{ab}$. To obtain the last equation we have used $\bar{R}^a{}_{b0d} = \bar{C}^a{}_{b0d} = B_{dp} \epsilon^{pa}{}_b$.

The evolution equation for the part of the connection described by χ_{bd} is obtained from the second identity in (4.44). In order to do so, first, we will derive the evolution and the quasi-constraint equations for the acceleration. The evolution equation for the acceleration can be obtained from

$$\begin{aligned}[\boldsymbol{e}_0, \bar{\boldsymbol{e}}_c]\,\psi &= c^0{}_{0c}\boldsymbol{e}_0(\psi) + c^p{}_{0c}\bar{\boldsymbol{e}}_p(\psi) \\ &= \gamma^0{}_{c0}(\partial_t\psi) + (\gamma^p{}_{c0} - \gamma^p{}_{0c})\,(D_p\psi) \\ &= a_c\,(\partial_t\psi) - \chi_c{}^p\,(D_p\psi)\,,\end{aligned}$$

where the properties of the Lagrangian and Fermi gauge have been employed. Now, expanding the left hand side and making use of the evolution and the quasi-constraint equation for the energy-momentum tensor of the scalar field one has

$$\partial_t a_c - D_c\chi = -\chi_c{}^p a_p + \left(\chi + \frac{2}{\psi}\frac{d\mathcal{V}}{d\phi}\right) a_c$$

so that using the second equation in (4.45) we arrive at

$$\partial_t a_c - D_p\chi_c{}^p = -2\left(\chi^A\right)_{pc} a^p - \chi_c{}^p a_p + \left(\chi + \frac{2}{\psi}\frac{d\mathcal{V}}{d\phi}\right) a_c. \tag{4.47}$$

In the case of the quasi-constraint, a computation yields

$$D_c a_b - D_b a_c = 2\left(\chi + \frac{1}{\psi}\frac{d\mathcal{V}}{d\phi}\right)\left(\chi^A\right)_{cb}.$$

Thus, making use of this equation in the second identity of (4.44) and recalling the properties of the Fermi gauge one finds that

$$\partial_t\chi_{db} - D_b a_d = E_{db} + \frac{1}{3}\left(\mathcal{V}(\phi) - \psi^2\right) h_{db} - \chi_d{}^p\chi_{pb} + 2\left(\chi + \frac{1}{\psi}\frac{d\mathcal{V}}{d\phi}\right)\left(\chi^A\right)_{db} + a_d a_b. \tag{4.48}$$

The principal part of the combined system of equations (4.47) and (4.48) is given by

$$\begin{pmatrix} \boldsymbol{e}_0 & -D_1 & -D_2 & -D_3 \\ -D_1 & \boldsymbol{e}_0 & 0 & 0 \\ -D_2 & 0 & \boldsymbol{e}_0 & 0 \\ -D_3 & 0 & 0 & \boldsymbol{e}_0 \end{pmatrix}\begin{pmatrix} a_d \\ \chi_d{}^1 \\ \chi_d{}^2 \\ \chi_d{}^3 \end{pmatrix}.$$

which is clearly symmetric. Finally, since for our particular problem one has $(\chi^A)_{ab} = 0$, equation (4.47) takes the form

$$\partial_t a_c - D_p\chi_c{}^p = \left(\frac{2}{\psi}\frac{d\mathcal{V}}{d\phi} + \chi\right) a_c - \chi_c{}^p a_p, \tag{4.49}$$

and after symmetrising equation (4.48) we obtain

$$2\partial_t\chi_{(bd)} - 2D_{(b}a_{d)} = \frac{2}{3}\left(\mathcal{V}(\phi) - \psi^2\right) h_{bd} - 2\chi^p{}_{(d}\chi_{b)p} + 2a_b a_d + 2E_{bd}. \tag{4.50}$$

Also, from the first equation in (4.45) it follows that

$$\partial_t\chi - D_p a^p = \mathcal{V}(\phi) - \psi^2 - \chi^{cd}\chi_{cd} + a^2,$$

where $a^2 = a_c a^c$.

Then the principal part of the sytem reads

$$\begin{pmatrix} \boldsymbol{e}_0 & 0 & 0 & -D_2 & -D_3 & 0 & -D_1 & 0 & 0 \\ 0 & \boldsymbol{e}_0 & 0 & -D_1 & 0 & -D_3 & 0 & -D_2 & 0 \\ 0 & 0 & \boldsymbol{e}_0 & 0 & -D_1 & -D_2 & 0 & 0 & -D_3 \\ -D_2 & -D_1 & 0 & 2\boldsymbol{e}_0 & 0 & 0 & 0 & 0 & 0 \\ -D_3 & 0 & -D_1 & 0 & 2\boldsymbol{e}_0 & 0 & 0 & 0 & 0 \\ 0 & -D_3 & -D_2 & 0 & 0 & 2\boldsymbol{e}_0 & 0 & 0 & 0 \\ -D_1 & 0 & 0 & 0 & 0 & 0 & \boldsymbol{e}_0 & 0 & 0 \\ 0 & -D_2 & 0 & 0 & 0 & 0 & 0 & \boldsymbol{e}_0 & 0 \\ 0 & 0 & -D_3 & 0 & 0 & 0 & 0 & 0 & \boldsymbol{e}_0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ \chi_{12} \\ \chi_{13} \\ \chi_{23} \\ \chi_{11} \\ \chi_{22} \\ \chi_{33} \end{pmatrix}.$$

which is clearly symmetric.

4.3.6 Hyperbolicity considerations

The system consisting of equations (4.35), (4.36), (4.38), (4.39), (4.43), (4.46), (4.49) and (4.50) can be written matricially as

$$\mathbf{A}^0 \partial_t \mathbf{u} - \mathbf{A}^p \bar{\boldsymbol{e}}_p(\mathbf{u}) = \mathbf{B}(\mathbf{u})\mathbf{u}. \tag{4.51}$$

As discussed in [37], these systems are not hyperbolic in the usual sense as, in general, the time lines are not hypersurface orthogonal and the "spatial" frame vectors $\bar{\boldsymbol{e}}_a$ have components in the time direction —cfr. equation (4.41). Since the surfaces of constant time t are not necessarily spacelike, this type of system is referred to as a *quasi-FOSH* system —see [37]. In terms of the partial derivatives, equation (4.51) reads

$$\tilde{\mathbf{A}}^0(\mathbf{u}) \partial_t \mathbf{u} - \mathbf{A}^j(\mathbf{u}) \partial_j \mathbf{u} = \mathbf{B}(\mathbf{u})\mathbf{u} \tag{4.52}$$

with

$$\tilde{\mathbf{A}}^0(\mathbf{u}) \equiv \mathbf{A}^0 - \mathbf{A}^p \bar{e}_p{}^0, \quad \mathbf{A}^j(\mathbf{u}) \equiv \mathbf{A}^p \bar{e}_p{}^j. \tag{4.53}$$

In order to have a well posed initial value problem, the matrix $\tilde{\mathbf{A}}^0(\mathbf{u})$ must be positive definite. This is the case as long the quadratic form

$$\sum_{b=1,2,3} \theta^b{}_i \theta^b{}_j - \beta_i \beta_j \tag{4.54}$$

is positive definite —see Proposition 9 in [37]. In the next section, we will consider a reference solution admitting a foliation by homogeneous spacelike hypersurfaces. As a consequence, the linearisation of the system (4.4) is well posed without the need to control the smallness of β_i. The smallness of these terms is taken care by the perturbation fields —see also [166].

Written in terms of partial derivatives, our system of evolution equations reads

$$
\begin{aligned}
\partial_t\phi &= \psi, \\
\partial_t\psi &= -\psi\chi - \frac{d\mathcal{V}}{d\phi}, \\
2\partial_t\chi_{(bd)} - 2\,\bar{e}_{(b}{}^{0}\partial_t a_{d)} - 2\,\bar{e}_{(b}{}^{j}\partial_j a_{d)} &= \frac{2}{3}\left(\mathcal{V}(\phi)-\psi^2\right)h_{bd} - 2\chi^{p}{}_{(d}\chi_{b)p} + 2a_b a_d + 2E_{bd} \\
&\quad - \left(\gamma^{p}{}_{bd}+\gamma^{p}{}_{db}\right)a_p, \\
\partial_t a_c - \bar{e}_p{}^{0}\partial_t\chi_c{}^{p} - \bar{e}_p{}^{j}\partial_j\chi_c{}^{p} &= \left(\frac{2}{\psi}\frac{d\mathcal{V}}{d\phi}+\chi\right)a_c - \chi_c{}^{p}a_p - \gamma^{q}{}_{cp}\chi_q{}^{p} + \gamma^{p}{}_{qp}\chi_c{}^{q}, \\
2\partial_t E_{bd} - 2\epsilon^{pa}{}_{(b|}\bar{e}_a{}^{0}\partial_t B_{p|d)} - 2\epsilon^{pa}{}_{(b|}\bar{e}_a{}^{j}\partial_j B_{p|d)} &= -\psi^2\left(\chi_{(bd)}-\frac{1}{3}\chi h_{bd}\right) - 4\chi E_{bd} + 10\chi^{q}{}_{(b}E_{d)q} \\
&\quad - 2h_{bd}\chi^{qp}E_{qp} + 4a_a B_{p(b}\epsilon_{d)}{}^{pa} - 2\gamma^{q}{}_{pa}B_{q(d}\epsilon_{b)}{}^{pa} \\
&\quad - 2\epsilon^{pa}{}_{(d}\gamma^{q}{}_{b)a}B_{pq}, \\
2\partial_t B_{bd} - 2\epsilon^{ap}{}_{(d|}\bar{e}_a{}^{0}\partial_t E_{|b)p} - 2\epsilon^{ap}{}_{(d}\bar{e}_{|a}{}^{j}\partial_{j|}E_{b)p} &= -2\chi B_{bd} + 6\chi^{q}{}_{(b}B_{d)q} + 2\chi_{ac}B_{pq}\epsilon^{pa}{}_{(b}\epsilon_{d)}{}^{qc} \\
&\quad - 4a_a E_{p(b}\epsilon_{d)}{}^{pa} - 2\gamma^{q}{}_{pa}E_{q(b}\epsilon_{d)}{}^{ap} \\
&\quad - 2\epsilon^{ap}{}_{(b}\gamma^{q}{}_{d)a}E_{pq}, \\
\partial_t\gamma'^{a}{}_{bd} &= B_{dp}\epsilon^{pa}{}_{b} - \chi_d{}^{p}\bar{\gamma}^{a}{}_{bp} + 2h^{ap}\chi_{d[p}a_{b]}, \\
\partial_t\bar{e}_b{}^{i} &= -\chi_b{}^{c}\bar{e}_c{}^{i}, \\
\partial_t\bar{e}_b{}^{0} &= -\chi_b{}^{c}\bar{e}_c{}^{0} + a_b.
\end{aligned}
\tag{4.55}
$$

This system has clearly the form given by equation (4.52). If one writes

$$
\mathbf{u}^T = \left(\phi,\ \psi,\ \mathbf{z}^T,\ \mathbf{w}^T,\ \mathbf{x}^T,\ \mathbf{y}^T\right), \tag{4.56}
$$

where

$$
\begin{aligned}
\mathbf{z}^T &= \left(\chi_{11},\ \chi_{22},\ \chi_{33},\ \chi_{12},\ \chi_{13},\ \chi_{23},\ a_1,\ a_2,\ a_3\right), \\
\mathbf{w}^T &= \left(E_{12},\ E_{13},\ E_{23},\ E_{11},\ E_{22},\ E_{33},\ B_{12},\ B_{13},\ B_{23},\ B_{11},\ B_{22},\ B_{33}\right), \\
\mathbf{x}^T &= \left(e_1{}^0,\ e_2{}^0,\ e_3{}^0,\ e_1{}^1,\ e_1{}^2,\ e_1{}^3,\ e_2{}^1,\ e_2{}^2,\ e_2{}^3,\ e_3{}^1,\ e_3{}^2,\ e_3{}^3\right), \\
\mathbf{y}^T &= \left(\gamma^1{}_{22},\ \gamma^1{}_{33},\ \gamma^1{}_{23},\ \gamma^2{}_{11},\ \gamma^2{}_{33},\ \gamma^2{}_{31},\ \gamma^3{}_{11},\ \gamma^3{}_{22},\ \gamma^3{}_{12}\right),
\end{aligned}
$$

then the matrices given in equations (4.52) and (4.53) have the explicit form

$$
\tilde{\mathbf{A}}^0(\mathbf{u}) = \begin{pmatrix} \mathbf{I}_{2\times2} & 0 & 0 & 0 \\ 0 & \tilde{\mathbf{A}}^0_{9\times9} & 0 & 0 \\ 0 & 0 & \tilde{\mathbf{A}}^0_{12\times12} & 0 \\ 0 & 0 & 0 & \mathbf{I}_{21\times21} \end{pmatrix}, \qquad \mathbf{A}^j(\mathbf{u}) = \begin{pmatrix} \mathbf{0}_{2\times2} & 0 & 0 & 0 \\ 0 & \mathbf{A}^j_{9\times9} & 0 & 0 \\ 0 & 0 & \mathbf{A}^j_{12\times12} & 0 \\ 0 & 0 & 0 & \mathbf{0}_{21\times21} \end{pmatrix}, \tag{4.57}
$$

with

$$
\tilde{\mathbf{A}}^0_{9\times9} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & -e_1{}^0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & -e_2{}^0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -e_3{}^0 \\
0 & 0 & 0 & 2 & 0 & 0 & -e_2{}^0 & -e_1{}^0 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 & -e_3{}^0 & 0 & -e_1{}^0 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & -e_3{}^0 & -e_2{}^0 \\
-e_1{}^0 & 0 & 0 & -e_2{}^0 & -e_3{}^0 & 0 & 1 & 0 & 0 \\
0 & -e_2{}^0 & 0 & -e_1{}^0 & 0 & -e_3{}^0 & 0 & 1 & 0 \\
0 & 0 & -e_3{}^0 & 0 & -e_1{}^0 & -e_2{}^0 & 0 & 0 & 1
\end{pmatrix},
$$

$$
\tilde{\mathbf{A}}^0_{12\times 12} = \begin{pmatrix}
2 & 0 & 0 & 0 & 0 & 0 & 0 & -{e_1}^0 & {e_2}^0 & {e_3}^0 & -{e_3}^0 & 0 \\
0 & 2 & 0 & 0 & 0 & 0 & {e_1}^0 & 0 & -{e_3}^0 & -{e_2}^0 & 0 & {e_2}^0 \\
0 & 0 & 2 & 0 & 0 & 0 & -{e_2}^0 & {e_3}^0 & 0 & 0 & {e_1}^0 & -{e_1}^0 \\
0 & 0 & 0 & 1 & 0 & 0 & -{e_3}^0 & {e_2}^0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & {e_3}^0 & 0 & -{e_1}^0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & -{e_2}^0 & {e_1}^0 & 0 & 0 & 0 \\
0 & {e_1}^0 & -{e_2}^0 & -{e_3}^0 & {e_3}^0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\
-{e_1}^0 & 0 & {e_3}^0 & {e_2}^0 & 0 & -{e_2}^0 & 0 & 2 & 0 & 0 & 0 & 0 \\
{e_2}^0 & -{e_3}^0 & 0 & 0 & -{e_1}^0 & {e_1}^0 & 0 & 0 & 2 & 0 & 0 & 0 \\
{e_3}^0 & -{e_2}^0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
-{e_3}^0 & 0 & {e_1}^0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & {e_2}^0 & -{e_1}^0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},
$$

$$
\mathbf{A}^j_{9\times 9} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & {e_1}^j & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & {e_2}^j & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & {e_3}^j \\
0 & 0 & 0 & 0 & 0 & 0 & {e_2}^j & {e_1}^j & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & {e_3}^j & 0 & {e_1}^j \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & {e_3}^j & {e_2}^j \\
{e_1}^j & 0 & 0 & {e_2}^j & {e_3}^j & 0 & 0 & 0 & 0 \\
0 & {e_2}^j & 0 & {e_1}^j & 0 & {e_3}^j & 0 & 0 & 0 \\
0 & 0 & {e_3}^j & 0 & {e_1}^j & {e_2}^j & 0 & 0 & 0
\end{pmatrix},
$$

and

$$
\mathbf{A}^j_{12\times 12} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & {e_1}^j & -{e_2}^j & -{e_3}^j & {e_3}^j & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -{e_1}^j & 0 & {e_3}^j & {e_2}^j & 0 & -{e_2}^j \\
0 & 0 & 0 & 0 & 0 & 0 & {e_2}^j & -{e_3}^j & 0 & 0 & -{e_1}^j & {e_1}^j \\
0 & 0 & 0 & 0 & 0 & 0 & {e_3}^j & -{e_2}^j & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -{e_3}^j & 0 & {e_1}^j & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & {e_2}^j & -{e_1}^j & 0 & 0 & 0 \\
0 & -{e_1}^j & {e_2}^j & {e_3}^j & -{e_3}^j & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
{e_1}^j & 0 & -{e_3}^j & -{e_2}^j & 0 & {e_2}^j & 0 & 0 & 0 & 0 & 0 & 0 \\
-{e_2}^j & {e_3}^j & 0 & 0 & {e_1}^j & -{e_1}^j & 0 & 0 & 0 & 0 & 0 & 0 \\
-{e_3}^j & {e_2}^j & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
{e_3}^j & 0 & -{e_1}^j & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -{e_2}^j & {e_1}^j & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
$$

It can, therefore, be verified that $\tilde{\mathbf{A}}^0(\mathbf{u})$, $\mathbf{A}^j(\mathbf{u})$ are symmetric and that, furthermore, $\tilde{\mathbf{A}}^0(\mathbf{u})$ is positive definite as long as (4.54) is satisfied. We summarise the results of this section in the following theorem:

Theorem 1. *The Einstein-Friedrich-nonlinear scalar field (EFsf) system consisting of the equations in* (4.55) *forms a quasi-linear first-order symmetric hyperbolic (FOSH) system for the scalar field, its momentum-density, the frame coefficients, the connection coefficients and the electric and magnetic parts of the Weyl tensor, relatively to the slices of constant time t, as long as the quadratic form*

$$
\sum_{a=1,2,3} {\theta^a}_i {\theta^a}_j - \frac{\partial_i \phi}{\psi} \frac{\partial_j \phi}{\psi},
$$

is positive definite.

Using the standard theory of symmetric hyperbolic systems one can then conclude the local existence in time and uniqueness of smooth solutions for the evolution equations implied by the Einstein-nonlinear scalar field. In order to conclude the existence of solutions to the full Einstein-scalar field equations one has to verify that the constraint equations are satisfied during the evolution if these hold initially. This will not be discussed here as the purpose of the present work is to study the evolution of the perturbations. In any case, it follows from general arguments that those constraints are preserved during evolution, see e.g. [86, 165, 166].

Remark 5. *As part of the procedure to turn the system* (4.55) *explicitly symmetric hyperbolic one has to divide the evolution equation for the acceleration* (4.49) *by* ψ^2*. This could imply that the system is not well behaved when* $\psi \to 0$*. An inspection shows that the potentially troublesome term is the one containing the first derivative of the potential, which, by virtue of equations* (4.38)-(4.39) *must be zero in this limit (possibly at* $t \to +\infty$*). Thus, we must require the coefficient* $\mathcal{V}'/\psi$ *to be finite.*

4.4 Stability Analysis (preliminary results)

In this section, we shall show preliminary results and the main ideias to use the symmetric hyperbolic system derived in last section to show that, for some classes of potentials, the evolution of sufficiently small nonlinear perturbations of a FL-nonlinear scalar field background prescribed on a Cauchy hypersurface with the topology of a 3-torus $\mathbb{T}^3$, have an asymptotic exponential decay.

4.4.1 The background solution

The Robertson-Walker spacetime has metric given by (2.3). Since the metric is locally conformally flat, it follows that

$$\mathring{E}_{bd} = \mathring{B}_{bd} = 0.$$

Now, the gauge conditions for the frame are satisfied if one sets

$$\mathring{e}_0{}^\mu = \delta_0{}^\mu, \quad \mathring{e}_b{}^\mu = \left(\frac{\omega}{a}\right)\delta_b{}^\mu, \quad b = 1,2,3,$$

so that the spatial connection coefficients are given by

$$\mathring{\gamma}^c{}_{bd} = \frac{k}{2a^2}\left(h_{db}x^c - h_d{}^c x_b\right), \quad b,c,d = 1,2,3,$$

with $x^\mu = (\omega/a)\delta^\mu{}_c x^c$. The remaining non-vanishing connection coefficients are

$$\mathring{\gamma}^0{}_{bd} = \mathring{\chi}_{db} = Hh_{bd} \quad , \quad \mathring{\gamma}^b{}_{0d} = \mathring{\chi}_d{}^b = Hh_d{}^b, \quad b,d = 1,2,3.$$

where we recall that $H(t) \equiv \dot{a}/a$ is the Hubble function and the dot $\dot{}$ denotes differentiation with respect to time t. In particular

$$\mathring{\chi}_{[bd]} = a_b = 0, \quad \text{and} \quad \mathring{\chi}_{(bd)} = 0 \quad \text{for} \quad b \neq d,$$

and for such background metrics

$$\mathring{\chi} = 3\dot{a}/a \equiv 3H.$$

Therefore in the case of a FL-nonlinear scalar field models, the Einstein-scalar field system (4.55) reduces to the evolution and equations (2.6) and, the constraints should reduce to (2.7), with $N = 1$.

4.4.2 Linearised evolution equations

In this subsection, we derive the linearised system, corresponding to the nonlinear equations of Theorem 1, for the case of a FL background with a self-interacting scalar field. In order to perform the linearisation procedure we compute

$$\left.\frac{\mathrm{d}\mathbf{u}^{\epsilon}}{\mathrm{d}\epsilon}\right|_{\epsilon=0}$$

and drop all (nonlinear) terms of coupled perturbations. In this way, we obtain the following system

$$\begin{aligned}
\partial_t\breve{\phi} =& \breve{\psi}, \\
\partial_t\breve{\psi} =& -\left(\frac{\mathrm{d}^2\mathring{\mathcal{V}}}{\mathrm{d}\mathring{\phi}^2}\right)\breve{\phi} - 3H\breve{\psi} - \mathring{\psi}\breve{\chi}, \\
2\partial_t\breve{\chi}_{(bd)} - 2\left(\frac{\omega}{a}\right)\delta_{(d}{}^{j}\partial_j\breve{a}_{b)} =& \frac{2}{3}\left(\left(\frac{\mathrm{d}\mathring{\mathcal{V}}}{\mathrm{d}\mathring{\phi}}\right)\breve{\phi} - 2\mathring{\psi}\breve{\psi}\right)h_{bd} - 4H\breve{\chi}_{(bd)} + 2\breve{E}_{bd} \\
& - \frac{k}{a^2}\left(h_{bd}x^p\breve{a}_p - x_{(b}\breve{a}_{d)}\right), \\
\partial_t\breve{a}_c - \left(\frac{\omega}{a}\right)\delta_p{}^{j}\partial_j\breve{\chi}_c{}^{p} =& \left(2H + \frac{2}{\mathring{\psi}}\frac{\mathrm{d}\mathring{\mathcal{V}}}{\mathrm{d}\mathring{\phi}}\right)\breve{a}_c + \left(\frac{\mathrm{d}H}{\mathrm{d}t}\right)\breve{e}_c{}^{0} \\
& + \frac{k}{2a^2}x_c\breve{\chi} - \frac{3k}{2a^2}x^q\breve{\chi}_{(qc)}, \\
2\partial_t\breve{E}_{bd} - 2\left(\frac{\omega}{a}\right)\epsilon^{pa}{}_{(b|}\delta_a{}^{j}\partial_j\breve{B}_{p|d)} =& -\mathring{\psi}^2\left(\breve{\chi}_{(bd)} - \frac{h_{bd}}{3}\breve{\chi}\right) - 2H\breve{E}_{bd} + \frac{k}{a^2}x_p\epsilon^{pq}{}_{(b}\breve{B}_{d)q}, \\
2\partial_t\breve{B}_{bd} - 2\left(\frac{\omega}{a}\right)\epsilon_{(d|}{}^{ap}\delta_a{}^{j}\partial_j\breve{E}_{p|b)} =& -2H\breve{B}_{bd} + \frac{k}{a^2}x_p\epsilon^{qp}{}_{(b}\breve{E}_{d)q}, \\
\partial_t\breve{e}_b{}^{0} =& -H\breve{e}_b{}^{0} + \breve{a}_b, \\
\partial_t\breve{e}_b{}^{i} =& -H\breve{e}_b{}^{i} - \left(\frac{\omega}{a}\right)\delta_c{}^{i}\breve{\chi}_b{}^{c}, \\
\partial_t\breve{\gamma}^a{}_{bd} =& -H\breve{\gamma}^a{}_{bd} - \frac{k}{2a^2}\left(x^a\breve{\chi}_{db} - x_b\breve{\chi}_d{}^{a}\right) \\
& + H\left(\delta^a_d\breve{a}_b - h_{bd}\breve{a}^a\right) + \breve{B}_{dp}\epsilon^{pa}{}_{b}.
\end{aligned} \tag{4.58}$$

As a consequence, the linearised system has the following form

$$\mathring{\mathbf{A}}^0\partial_t\breve{\mathbf{u}} - \mathring{\mathbf{A}}^j(t,\boldsymbol{x})\partial_j\breve{\mathbf{u}} = \mathring{\mathbf{B}}(t,\boldsymbol{x})\breve{\mathbf{u}}. \tag{4.59}$$

If one defines $\breve{\mathbf{u}}$ in the same way as in equation (4.56), one obtains the linearised matrix $\mathring{\mathbf{B}}(t,\boldsymbol{x})$ as

$$\begin{pmatrix}
\mathbf{B}^{(1)} & -\mathring{\psi}\mathbf{B}^{(2)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\mathbf{B}^{(3)} & -2H\mathbf{I} & 0 & -\frac{k}{2a\omega}\mathbf{B}^{(4)} & 0 & \mathbf{I} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -4H\mathbf{I} & \frac{k}{2a\omega}\mathbf{B}^{(5)} & 2\mathbf{I} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{k}{2a\omega}\mathbf{B}^{(6)} & -\frac{3k}{2a\omega}\left(\mathbf{B}^{(5)}\right)^T & \left(\frac{2}{\mathring{\psi}}\mathring{\mathcal{V}}'+2H\right)\mathbf{I} & 0 & 0 & 0 & 0 & \frac{dH}{dt}\mathbf{I} & 0 & 0 \\
0 & 0 & -\mathring{\psi}^2\mathbf{I} & 0 & -2H\mathbf{I} & 0 & \frac{k}{2a\omega}\mathbf{B}^{(7)} & \frac{k}{2a\omega}\mathbf{B}^{(8)} & 0 & 0 & 0 \\
0 & -\frac{\mathring{\psi}^2}{2}\mathbf{B}^{(9)} & 0 & 0 & 0 & -H\mathbf{I} & \frac{k}{2a\omega}\mathbf{B}^{(10)} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{k}{2a\omega}\left(\mathbf{B}^{(7)}\right)^T & \frac{k}{2a\omega}\left(\mathbf{B}^{(10)}\right)^T & -2H\mathbf{I} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{k}{2a\omega}\left(\mathbf{B}^{(8)}\right)^T & 0 & 0 & -H\mathbf{I} & 0 & 0 & 0 \\
0 & 0 & 0 & \mathbf{I} & 0 & 0 & 0 & 0 & -H\mathbf{I} & 0 & 0 \\
0 & -\frac{\omega}{a}\mathbf{B}^{(11)} & -\frac{\omega}{a}\mathbf{B}^{(12)} & 0 & 0 & 0 & 0 & 0 & 0 & -H\mathbf{I} & 0 \\
0 & -\frac{k}{2a\omega}\mathbf{B}^{(13)} & \frac{k}{2a\omega}\mathbf{B}^{(14)} & -H\mathbf{B}^{(15)} & 0 & 0 & \mathbf{B}^{(16)} & \mathbf{B}^{(17)} & 0 & 0 & -H\mathbf{I}
\end{pmatrix}$$

where

$$\mathbf{B}^{(1)}_{2\times 2} = \begin{pmatrix} 0 & 1 \\ -\mathring{\mathcal{V}}'' & -3H \end{pmatrix}, \quad \mathbf{B}^{(2)}_{2\times 3} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \quad \mathbf{B}^{(3)}_{3\times 2} = \begin{pmatrix} \frac{1}{3}\mathring{\mathcal{V}}' & -\frac{2}{3}\mathring{\psi} \\ \frac{1}{3}\mathring{\mathcal{V}}' & -\frac{2}{3}\mathring{\psi} \\ \frac{1}{3}\mathring{\mathcal{V}}' & -\frac{2}{3}\mathring{\psi} \end{pmatrix},$$

$$\mathbf{B}^{(4)}_{3\times 3} = \begin{pmatrix} 0 & x^j\delta^2_j & x^j\delta^3_j \\ x^j\delta^1_j & 0 & x^j\delta^3_j \\ x^j\delta^1_j & x^j\delta^2_j & 0 \end{pmatrix}, \quad \mathbf{B}^{(5)}_{3\times 3} = \begin{pmatrix} x^j\delta^2_j & x^j\delta^1_j & 0 \\ x^j\delta^3_j & 0 & x^j\delta^1_j \\ 0 & x^j\delta^3_j & x^j\delta^2_j \end{pmatrix}, \quad \mathbf{B}^{(6)}_{3\times 3} = \begin{pmatrix} -2x^j\delta^1_j & x^j\delta^1_j & x^j\delta^1_j \\ x^j\delta^2_j & -2x^j\delta^2_j & x^j\delta_j \\ x^j\delta^3_j & x^j\delta^3_j & -2x^j\delta^3_j \end{pmatrix},$$

$$\mathbf{B}^{(7)}_{3\times 3} = \begin{pmatrix} 0 & -x^j\delta^1_j & x^j\delta^2_j \\ x^j\delta^1_j & 0 & -x^j\delta^3_j \\ -x^j\delta^2_j & x^j\delta^3_j & 0 \end{pmatrix}, \quad \mathbf{B}^{(8)}_{3\times 3} = \begin{pmatrix} x^j\delta^3_j & -x^j\delta^3_j & 0 \\ -x^j\delta^2_j & 0 & x^j\delta^2_j \\ 0 & x^j\delta^1_j & -x^j\delta^1_j \end{pmatrix},$$

$$\mathbf{B}^{(9)}_{3\times 3} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{pmatrix}, \quad \mathbf{B}^{(10)}_{3\times 3} = \begin{pmatrix} -x^j\delta^3_j & x^j\delta^2_j & 0 \\ x^j\delta^3_j & 0 & -x^j\delta^1_j \\ 0 & -x^j\delta^2_j & x^j\delta^1_j \end{pmatrix},$$

$$\mathbf{B}^{(11)}_{9\times 3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{B}^{(12)}_{9\times 3} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{B}^{(13)}_{9\times 3} = \begin{pmatrix} 0 & \delta^1_j x^j & 0 \\ 0 & 0 & \delta^1_j x^j \\ 0 & 0 & 0 \\ \delta^2_j x^j & 0 & 0 \\ 0 & 0 & \delta^2_j x^j \\ 0 & 0 & 0 \\ \delta^3_j x^j & 0 & 0 \\ 0 & \delta^3_j x^j & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{B}^{(15)}_{9\times 3} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{B}^{(14)}_{9\times 3} = \begin{pmatrix} \delta^j_2 x_j & 0 & 0 \\ 0 & \delta^j_3 x_j & 0 \\ 0 & \delta^j_2 x_j & -\delta^1_j x^j \\ \delta^j_1 x_j & 0 & 0 \\ 0 & 0 & \delta^j_3 x_j \\ \delta^j_3 x_j & -\delta^2_j x^j & 0 \\ 0 & \delta^j_1 x_j & 0 \\ 0 & 0 & \delta^j_2 x_j \\ -\delta^3_j x^j & 0 & \delta^j_1 x_j \end{pmatrix}, \quad \mathbf{B}^{(16)}_{9\times 3} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{B}^{(17)}_{9\times 3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

4.4.3 Exponential decay of nonlinear perturbations of FL-nonlinear scalar field models

In this subsection, we present results of the theory of symmetric hyperbolic systems required to analyse the exponential decay of nonlinear perturbations of a reference solution to *the Einstein-scalar field system with flat spatial sections* —i.e. $k = 0$, $\omega = 1$. The analysis carried out in the previous sections shows that the Cauchy problem for these nonlinear perturbations takes the form

$$\begin{aligned} &\left(\mathring{\mathbf{A}}^0 - \epsilon\breve{\mathbf{A}}^0(t,\breve{\mathbf{u}},\epsilon)\right)\partial_t\breve{\mathbf{u}} - \left(\mathring{\mathbf{A}}^j(t) + \epsilon\breve{\mathbf{A}}^j(t,\breve{\mathbf{u}},\epsilon)\right)\partial_j\breve{\mathbf{u}} = \left(\mathring{\mathbf{B}}(t) + \epsilon\breve{\mathbf{B}}(t,\breve{\mathbf{u}},\epsilon)\right)\breve{\mathbf{u}}, \\ &\breve{\mathbf{u}}(\boldsymbol{x},0) = \breve{\mathbf{u}}_0(\boldsymbol{x}), \end{aligned} \tag{4.60}$$

where $\mathring{\mathbf{A}}^0$ is a constant diagonal matrix with positive entries, while $\mathring{\mathbf{A}}^j$ and $\mathring{\mathbf{B}}$ are matrices whose entries are smooth functions of the time coordinate t. The matrices $\mathring{\mathbf{A}}^j$, $j = 1, 2, 3$, are symmetric.

We now introduce some notation which is used in the stability analysis. Let $H^k(\mathbb{T}^n; \mathbb{R}^s)$ be the space of all summable functions $\mathbf{u}(\cdot, t) : \mathbb{T}^n \to \mathbb{R}^s$ such that for each multi-index $|\alpha| \leq k$, $\partial^\alpha \mathbf{u}(\boldsymbol{x}, t)$ exists in the weak sense and belongs to $L^2(\mathbb{T}^n)$. The norm in $H^k(\mathbb{T}^n; \mathbb{R}^s)$ is defined by

$$\|\mathbf{u}(t)\|_{H^k(\mathbb{T}^n)} \equiv \left(\sum_{|\alpha|=0}^{k} \int_{\mathbb{T}^n} |\partial^\alpha \mathbf{u}|^2 \mathrm{d}\boldsymbol{x} \right)^{1/2} ,$$

where

$$\partial^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} ... \partial x_n^{\alpha_n}} .$$

see e.g. [76]. When the coefficients of the linearised system are constant matrices, the exponential decay of a solution to the nonlinear perturbations is given by Theorem 5.2 in [120]. The main assumption is that the the linearised matrix of the non-principal part of the system satisfies the so-called *stability eigenvalue condition*, i.e. there exists a constant $\delta > 0$ such that all the eigenvalues λ of $\mathring{\mathbf{B}}$ satisfy

$$\mathrm{Re}\,(\lambda) \leq -\delta \, .$$

If $\mathring{\mathbf{B}}$ is diagonal or normal, then this conditon can be directly translated into the following crucial inequality for the energy estimates

$$\mathring{\mathbf{B}} + \mathring{\mathbf{B}}^T \leq -2\delta \mathbf{I_d}.$$

Otherwise, there exists a positive definite symmetric matrix $\mathbf{L}$ such that

$$\mathbf{L}\mathring{\mathbf{B}} + \mathring{\mathbf{B}}^T \mathbf{L} \leq -2\delta_1 \mathbf{L},$$

with $\delta_1 > \delta > 0$, see Theorem 2.2 in [120]. Then, since $\mathbf{L}$ is positive definite, a new scalar product and norm can be defined, which to a constant, is equivalent to the L^2 norm (see pag. 211 and 212 in [120]).

In the case of a flat background ($k = 0$, $\omega = 1$), the linearized matrices $\mathring{\mathbf{A}}$ and $\mathring{\mathbf{B}}$ are functions of cosmic time t only. If $\mathring{\mathbf{B}}(t)$ was diagonal, then it would suffice to obtain the conditons that make its eigenvalues negative and bounded away from zero for all times, so that Theorem 5.2 in [120] would apply directly.

However, this is not the case and we need to construct the matrix $\mathbf{L}(t)$, which would now depend on time. Thus it is necessary to understand how to obtain a unifom bound, so that an equivalent norm can be defined. Moreover an explicit theorem for the nonlinear theorem must be written. For instance there are new terms appearing as

$$\frac{d\mathbf{L}(t)}{dt}$$

which must be handle carefully. See also pag. 244 in [120] for other problems appearing in the nonlinear case when the L^2 norm must be replaced by an equivalent norm.

Nevertheless, an analysis of negativity of the eigenvalues is always a necessary step.

4.4.4 Negativity of the eigenvalues

Because of the block structure of the matrix, the characteristic polynomial of $\mathring{\mathbf{B}}$ is the product of the characteristic polynomials of the block matrices around the diagonal:

$$
\begin{aligned}
&(\lambda+H)^{21}(\lambda+2H)^3\left(\lambda^2+6H\lambda+2\mathring{\psi}^2+8H^2\right)^3\left(\lambda^2-\left(H+2\frac{\mathring{\mathcal{V}}'}{\mathring{\psi}}\right)\lambda-\left(\frac{\mathrm{d}H}{\mathrm{d}t}+2H^2+2H\frac{\mathring{\mathcal{V}}'}{\mathring{\psi}}\right)\right)^3\\
&\times\Big[\lambda^8+12H\lambda^7+\left(\mathring{\mathcal{V}}''-\mathring{\psi}^2+60H^2\right)\lambda^6+\left(9H\left(\mathring{\mathcal{V}}''-\frac{5}{9}\mathring{\psi}^2+18H^2\right)+\mathring{\psi}^2\left(\frac{\mathring{\mathcal{V}}'}{\mathring{\psi}}\right)\right)\lambda^5\\
&+\left(\mathring{\psi}^2\left(\mathring{\mathcal{V}}''-\frac{7}{4}\mathring{\psi}^2\right)+H^2\left(33\mathring{\mathcal{V}}''-7\mathring{\psi}^2+255H^2\right)+7H\mathring{\psi}^2\left(\frac{\mathring{\mathcal{V}}'}{\mathring{\psi}}\right)\right)\lambda^4\\
&+\left(H\left(\mathring{\psi}^2\left(6\mathring{\mathcal{V}}''-\frac{13}{2}\mathring{\psi}^2+H^2\right)+H^2\left(63\mathring{\mathcal{V}}''+234H^2\right)\right)+\mathring{\psi}^2\left(\frac{\mathring{\mathcal{V}}'}{\mathring{\psi}}\right)\left(\mathring{\psi}^2+19H^2\right)\right)\lambda^3\\
&+\Bigg(\mathring{\psi}^2H\left(\frac{\mathring{\mathcal{V}}'}{\mathring{\psi}}\right)\left(4\mathring{\psi}^2+25H^2\right)+H^2\mathring{\psi}^2\left(13\mathring{\mathcal{V}}''-\frac{29}{4}\mathring{\psi}^2\right)+\mathring{\psi}^4\left(\frac{\mathring{\mathcal{V}}''}{4}-\frac{1}{2}\mathring{\psi}^2\right)\\
&+H^4\left(66\mathring{\mathcal{V}}''+8\mathring{\psi}^2+116H^2\right)\Bigg)\lambda^2+\\
&\Bigg(H\left(H^4\left(36\mathring{\mathcal{V}}''+4\mathring{\psi}^2\right)+\mathring{\psi}^4\left(\frac{3}{4}\mathring{\mathcal{V}}''-\frac{1}{2}\mathring{\psi}^2-\frac{5}{2}H^2\right)+H^2\mathring{\psi}^2\left(12\mathring{\mathcal{V}}''\right)+24H^6\right)\\
&+H^2\mathring{\psi}^2\left(\frac{1}{4}\mathring{\psi}^4+5\mathring{\psi}^2+16H^2\right)\left(\frac{\mathring{\mathcal{V}}'}{\mathring{\psi}}\right)\Bigg)\lambda+2H\left(\mathring{\psi}^2\left(\frac{\mathring{\mathcal{V}}'}{\mathring{\psi}}\right)+2H\mathring{\mathcal{V}}''\right)\left(\frac{\mathring{\psi}^4}{8}+H^2\mathring{\psi}^2+2H^4\right)\Big]
\end{aligned}
\tag{4.61}
$$

The first two terms in (4.61) correspond to the characteristic polynomial of the block related with the perturbation variables $(\breve{B}_{bd},\ \breve{\bar{e}}_b{}^j,\ \bar{\gamma}^c{}_{ab})$. The third term is related to the variables $(\breve{\chi}_{bd},\ \breve{E}_{bd})$, $b\neq d$, and the fourth one to $(\breve{a}_c,\ \breve{\bar{e}}_b{}^0)$. Finally, the last term —the polynomial of degree 8— arises from the block of unknowns $(\breve{\phi},\ \breve{\psi},\ \breve{\chi}_{bd},\ \breve{E}_{bd})$, $b=d$.

In order to obtain conditions from the characteristic polynomial, we will make use of the *Liénard-Chipart theorem*. The latter gives necessary and sufficient conditions for a polynomial with real coefficients to have roots with negative real part, see e.g. [154]. These polynomials are called *Hurwitz polynomials.*

Theorem 2 (Liénard-Chipart)**.** *Let*

$$f(z)=a_0z^n+a_1z^{n-1}+a_2z^{n-2}+\cdots+a_n,$$

be a polynomial with real constant coefficients satisfying $a_0>0, a_1>0,...,a_n>0$. *Then the following statements are equivalent:*

(i) the polynomial is a Hurwitz polynomial;

(ii) The coefficients of f *are positive and* $\delta_2>0,\ \delta_4>0,...,\delta_n>0,\quad n$ *even;*

(iii) The coefficients of f *are positive and* $\delta_1>0,\ \delta_3>0,...,\ \delta_n>0,\quad n$ *odd,*

where the Hurwitz determinants are defined by

$$\delta_0\equiv 1,\qquad \delta_i\equiv\det\begin{pmatrix}a_1 & a_3 & a_5 & \cdots & a_{2i-1}\\ a_0 & a_2 & a_4 & \cdots & a_{2i-2}\\ 0 & a_1 & a_3 & \cdots & a_{2i-3}\\ \vdots & \vdots & \vdots & & \vdots\\ 0 & 0 & 0 & \cdots & a_i\end{pmatrix}\qquad i=1,\ldots,n.$$

If the coefficients in (4.61) and the resulting Hurwitz determinants are positive, then from the convergence of the background quantities as $t \to +\infty$, one could easily see that the necessary and sufficient conditions for the global in time result, would follow by imposing that $a_\infty > 0$ and $\delta_\infty > 0$ alone.

However, by making an inspection of the coefficients in (4.61) as well as its resulting Hurwitz determinants, we cannot say that they are positive and, as we shall see, there exist negative terms wich are impossible to control. To circunvent this problem we define

$$A_i \equiv \inf_{t\in[0,T)} a_i(t) \quad \text{and} \quad \Delta_i \equiv \inf_{t\in[0,T)} \delta_i(t) \ , \tag{4.62}$$

and impose that $A_i > 0$ and $\Delta_i > 0$, for all finite $t < T$. This then implies that

$$\lambda \equiv \sup_{t\in[0,T)} \ Re\ \lambda(t) < 0,$$

and, in the given time interval, the stability eigenvalue condition is satisfied and exponential decay follows. If this was not the case, that is, if at a finite time the eigenvalues are zero, then no exponential decay would follow for later times. In particular, given a background Cosmological solution to the Einstein-nonlinear scalar field system — calculated, say, numerically — one can verify explicitly whether the conditions are satisfied or not.

Thus, at the end, all we are left with is the task of finding conditions in order to ensure that as $t \to +\infty$ the eigenvalues are bounded away from zero.

We shall now analyse each term of the characteristic polynomial (4.61). The first two terms in (4.61) require

$$H(t) > 0.$$

But we know from Rendall results [159] that for initially expanding $H_0 > 0$ backgrounds, $H(t)$ is monotonically decreasing tending to a limit as $t \to +\infty$, given by

$$H_\infty = \sqrt{\frac{\mathcal{V}_\infty}{3}}.$$

Therefore, in order to have $H_\infty > 0$ as required, the potential must have a positive lower bound

$$\mathcal{V}_\infty > 0. \tag{4.63}$$

The first, second-order polynomial in (4.61), have all its coefficients positive and in particular it remains positive in the limit $t \to +\infty$, if the above condition is satisfied. In turn, the coefficients of the other second-order polynomial must satisfy

$$-\frac{\mathring{\mathcal{V}}'}{\mathring{\psi}} - \frac{H}{2} > 0 \quad \text{and} \quad -\left(2H\frac{\mathring{\mathcal{V}}'}{\mathring{\psi}} + \frac{dH}{dt}\right) - 2H^2 > 0.$$

Notice that, by construction, we have that $\mathring{\psi} < 0$, see equation (4.38), so that the first condition implies the first derivative of the scalar field potential to be positive, i.e. $\mathring{\mathcal{V}}'(\mathring{\phi}) > 0$. Moreover, since the coefficients are not clearly positive, one must impose that for all finite times they satisfy the positivity conditions, and that they stay bounded away from zero as $t \to +\infty$. This is achieved by

$$-\left(\frac{\mathring{\mathcal{V}}'}{\mathring{\psi}}\right)_\infty > \sqrt{\frac{\mathcal{V}_\infty}{3}}\ . \tag{4.64}$$

From the coefficients in (4.61), one can show that as $t \to \infty$ one has $a_i > 0$ if

$$\mathring{\mathcal{V}}''_\infty > 0, \tag{4.65}$$

for all $i = 0, 1, ..., 8$, in particular is a_8 which gives this bound. We list the simplest expression for $\delta_i(t)$ from the 8th degree polynomial of Section 5.4, see equations (4.61) and (4.62), and the definitions of the Hurwitz determinants as:

$$
\begin{aligned}
\delta_1(t) &= 12H \\
\delta_2(t) &= \left(H\left(3\mathring{\mathcal{V}}'' - 7\mathring{\psi}^2 + 558H^2\right) - \mathring{\psi}^2\left(\frac{\mathring{\mathcal{V}}'}{\mathring{\psi}}\right)\right) \\
\delta_3(t) &= \left(27H^2\left(\mathring{\mathcal{V}}''\right)^2 + H^2\left(1512H^2 - 150\mathring{\psi}^2 - 6\frac{\mathring{\psi}^2}{H}\left(\frac{\mathring{\mathcal{V}}'}{\mathring{\psi}}\right)\right)\mathring{\mathcal{V}}'' - \left(384H^2 - 10\mathring{\psi}^2 + \frac{\mathring{\psi}^2}{H}\left(\frac{\mathring{\mathcal{V}}'}{\mathring{\psi}}\right)\right)H\mathring{\psi}^2\left(\frac{\mathring{\mathcal{V}}'}{\mathring{\psi}}\right)\right. \\
&\quad \left. +H^2\left(56484H^4 - 2904H^2\mathring{\psi}^2 + 209\mathring{\psi}^4\right)\right) \\
\delta_4(t) &= \left(H^2\left(702H^2 + 9\mathring{\psi}^2\right)\left(\mathring{\mathcal{V}}''\right)^3 + H^2\left(32832H^4 - 4620H^2\mathring{\psi}^2 - 75\mathring{\psi}^4 - 3\left(H^2 + \mathring{\psi}^2\right)\frac{\mathring{\psi}^2}{H}\left(\frac{\mathring{\mathcal{V}}'}{\mathring{\psi}}\right)\right)\left(\mathring{\mathcal{V}}''\right)^2 \right. \\
&\quad + H^2\left[649512H^6 - 137172H^4\mathring{\psi}^2 + 9043H^2\mathring{\psi}^4 + \frac{825}{4}\mathring{\psi}^6 - \left(3108H^4 + 553H^2\mathring{\psi}^2 - 15\mathring{\psi}^4 + 56H\mathring{\psi}^2\left(\frac{\mathring{\mathcal{V}}'}{\mathring{\psi}}\right)\right)\frac{\mathring{\psi}^2}{H}\left(\frac{\mathring{\mathcal{V}}'}{\mathring{\psi}}\right)\right]\mathring{\mathcal{V}}'' \\
&\quad + H^2\left(7993728H^8 - \frac{9369}{2}\mathring{\psi}^6H^2 - \frac{375}{2}\mathring{\psi}^8 + 122094\,\mathring{\psi}^4H^4 - 915840\,\mathring{\psi}^2H^6\right) \\
&\quad \left. -7H\mathring{\psi}^6\left(\frac{\mathring{\mathcal{V}}'}{\mathring{\psi}}\right)^3 - \left(\frac{75}{4}\mathring{\psi}^6 - 1379\,\mathring{\psi}^4H^2 + 98604\,H^6 + 7206\,\mathring{\psi}^2H^4\right)H\mathring{\psi}^2\left(\frac{\mathring{\mathcal{V}}'}{\mathring{\psi}}\right) + \left(121\,\mathring{\psi}^2 - 852\,H^2\right)\left(\frac{\mathring{\mathcal{V}}'}{\mathring{\psi}}\right)^2\mathring{\psi}^4H^2\right)
\end{aligned}
$$

Similarly, one finds from $\delta_i(t)$ that, as $t \to \infty$, these are positive if $\mathring{\mathcal{V}}''_\infty > 0$, for $i = 0, 1, 2, 3, 4$. The expressions for $\delta_5(t), \delta_6(t), \delta_7(t)$ and $\delta_8(t)$ are too lengthy to be written here, although they can be computed explicitly and attain the limit

$$
\begin{aligned}
\delta_5 =& 29160H_\infty^7\left(\mathring{\mathcal{V}}''_\infty\right)^4 + \left(1273968H_\infty^9 - 648H_\infty^4\right)\left(\mathring{\mathcal{V}}''_\infty\right)^3 + \left(22680000H_\infty^{11} - 36288H_\infty^6\right)\left(\mathring{\mathcal{V}}''_\infty\right)^2 \\
&+ \left(212761728H_\infty^{13} - 1355616H_\infty^8\right)\left(\mathring{\mathcal{V}}''_\infty\right) + 1046587392H_\infty^{15} > 0 \\
\delta_6 =& \left(10656H_\infty^6 + 117H_\infty\right)\left(\mathring{\mathcal{V}}''_\infty\right)^5 + \left(444192H_\infty^8 + 6834H_\infty^3\right)\left(\mathring{\mathcal{V}}''_\infty\right)^4 \\
&+ \left(7373168H_\infty^{10} + 167516H_\infty^5 - 16\right)\left(\mathring{\mathcal{V}}''_\infty\right)^3 + \left(62196768H_\infty^{12} + 2040792H_\infty^7 - 2976H_\infty^2\right)\left(\mathring{\mathcal{V}}''_\infty\right)^2 \\
&+ \left(243606528H_\infty^{14} + 21396096H_\infty^9\right)\left(\mathring{\mathcal{V}}''_\infty\right) + 746496000H_\infty^{16} > 0 \\
\delta_7 =& \left(127872\,H_\infty^{10} - 8532H_\infty^5 - 27\right)\left(\mathring{\mathcal{V}}''_\infty\right)^6 + \left(5415552\,H_\infty^{12} - 367200\,H_\infty^7 - 1728\,H_\infty^2\right)\left(\mathring{\mathcal{V}}''_\infty\right)^5 \\
&+ \left(92031552\,H_\infty^{14} - 6292080\,H_\infty^9 - 49896\,H_\infty^4\right)\left(\mathring{\mathcal{V}}''_\infty\right)^4 \\
&+ \left(805346560\,H_\infty^{16} - 56244288\,H_\infty^{11} - 676608\,H_\infty^6 + 512\,H_\infty\right)\left(\mathring{\mathcal{V}}''_\infty\right)^3 \\
&+ \left(3420852480\,H_\infty^{18} - 147699072\,H_\infty^{13} - 11716272\,H_\infty^8\right)\left(\mathring{\mathcal{V}}''_\infty\right)^2 \\
&+ \left(10906804224\,H_\infty^{20} - 1159059456\,H_\infty^{15}\right)\left(\mathring{\mathcal{V}}''_\infty\right) + 5971968000\,H_\infty^{22} > 0.
\end{aligned}
$$

Part III

Spherically symmetric spacetimes with a massless scalar field and positive cosmological constant

Chapter 5

Einstein-Λ-scalar field system in Bondi coordinates

Although there are now robust mathematical methods available to treat the EFEs for small initial data, much less is known in the large see eg. [173], where solutions may develop singularities through gravitational collapse. A key aspect on the nature of the singularities is the question whether these are hidden in a black hole, or if they are naked, leading to a deterministic breakdown. The *weak cosmic censorship* is the conjecture that, generically, singularities arising from gravitational collapse are contained in black holes. Since stronger global results for large initial data are known in 1 + 1 dimensions, we are led to study models exhibiting some kind of special symmetry. Here we will consider the spherically symmetric Einstein-scalar field system with positive cosmological constant. By the Birkhoff theorem, we know that solutions of the spherically symmetric Einstein-vacuum-(Λ) equations are isometric to the Schwarzschild-(de Sitter) family and thus are static up to the horizons. Therefore, in order to produce dynamics one has to add matter. The first physical system describing a black hole formation is the Oppenheimer-Snyder solution. This model consists of a homogeneous dust ball undergoing gravitational collapse. Introducing $\Lambda > 0$ there exist analogue solutions which do not undergo gravitational collapse and are thus asymptotically de Sitter (the Einstein cosmos being a borderline case). The asymptotically flat case, ($\Lambda = 0$), was generalised by Christodoulou's to the inhomogeneous setting, where he found a non-empty open set of initial data leading to black hole formation but also a non-empty open set giving rise to naked singularities [39]. However, dust is a pathological matter model, as it is known to form singularities even in Minkowski spacetime, i.e., in the absence of gravitation. Well behaved matter models include for example the scalar field or Vlasov matter. The weak cosmic censorship has been proved for the spherically symmetric Einstein-scalar field equations with $\Lambda = 0$ by Christodoulou [43], see also the introduction to [44] for a thorough review of Christodoulou's results on spherically symmetric self-gravitating scalar fields. Christodoulou's work has also inspired a considerable amount of numerical work, including Choptuik's discovery of critical phenomena [36] (see also [92] and references therein). Besides the asymptotically flat case very little has been done concerning the asymptotically de Sitter collapse, see however [26, 20].

The monumental work of Christodoulou started in the celebrated paper "*The problem of a self-gravitating scalar field*", where he proved that for sufficiently small data (in a precise sense), the solution disperses to the future, asymptotically approaching Minkowski spacetime [40]. In this chapter we review Christodoulou's framework for spherical waves, while generalising to the case when a positive cosmological constant is present.

We will say that a spacetime $(\mathcal{M}, \mathbf{g})$ is *Bondi-spherically symmetric* if it admits a global representation for the metric of the form

$$\mathbf{g} = -g(u,r)\tilde{g}(u,r)\mathrm{d}u^2 - 2g(u,r)\mathrm{d}u\mathrm{d}r + r^2\mathrm{d}\Omega^2 \;, \tag{5.1}$$

where

$$\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2 \ ,$$

is the round metric of the two-sphere, and

$$(u, r) \in [0, U) \times [0, R) \ \ , \quad U, R \in \mathbb{R}^+ \cup \{+\infty\} \ .$$

If U or R are finite these intervals can also be closed, which corresponds to adding a final light cone $\{u = U\}$ or a cylinder $[0, U) \times \mathbb{S}^2$ as a boundary, in addition to the initial light cone $\{u = 0\}$; the metric is assumed to be regular at the center $\{r = 0\}$, which is not a boundary.

The coordinates (u, r, θ, φ) will be called *Bondi coordinates*. For instance, the causal future of any point in de Sitter spacetime (2.2) may be covered by Bondi coordinates with the metric given by

$$\mathbf{g}_{dS} = -\left(1 - \frac{\Lambda}{3}r^2\right)\mathrm{d}u^2 - 2\mathrm{d}u\mathrm{d}r + r^2\mathrm{d}\Omega^2 \tag{5.2}$$

(see Figure 5.1). Note that this coordinate system does not cover the full de Sitter manifold (which strictly speaking is not Bondi-spherically symmetric), unlike in the asymptotically flat $\Lambda = 0$ case. The boundary of the region covered by Bondi coordinates, that is, the surface $u = -\infty$ (which for $\Lambda = 0$ would correspond to past null infinity), is an embedded null hypersurface (the cosmological horizon of the observer antipodal to the one at $r = 0$). Moreover, the lines of constant r, which for $\Lambda = 0$ approach timelike geodesics as $r \to +\infty$, here become spacelike for sufficiently large r.

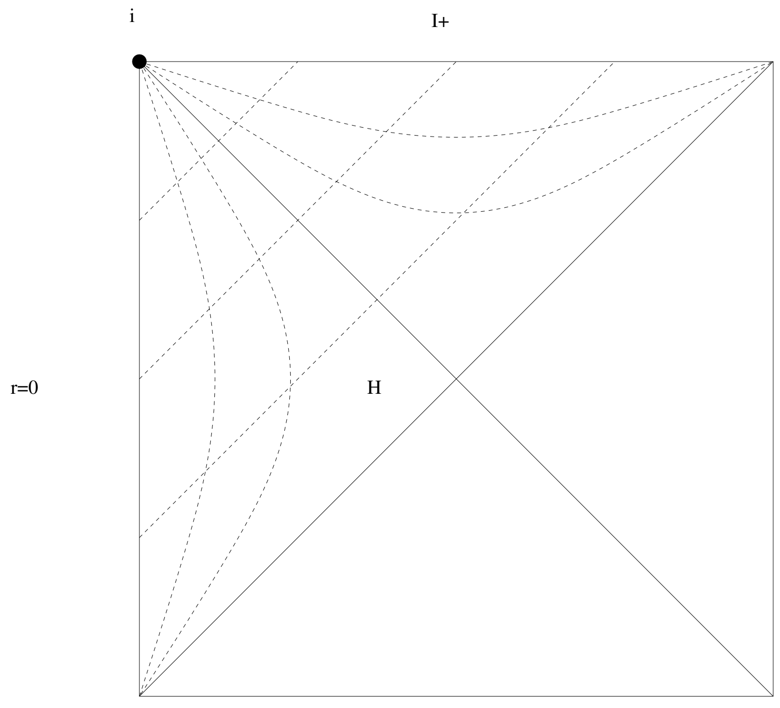

Figure 5.1: Penrose diagram of de Sitter spacetime. The lines $u =$ constant are the outgoing null geodesics starting at $r = 0$. The point i corresponds to $u = +\infty$, the cosmological horizon $\mathcal{H}$ to $r = \sqrt{\frac{3}{\Lambda}}$ and the future null infinity $\mathscr{I}^+$ to $r = \infty$.

Although the causal structures of Minkowski and de Sitter spacetimes are quite different, the existence of Bondi coordinates depends solely on certain common symmetries. More precisely, a global representation for the metric of the form (5.1) can be derived from the following geometrical hypotheses:

(i) the spacetime admits a $SO(3)$ action by isometries, whose orbits are either fixed points or 2-spheres;

(ii) the orbit space $Q = M/SO(3)$ is a 2-dimensional Lorentzian manifold with boundary, corresponding to the sets of fixed and boundary points in M;

(iii) the set of fixed points is a timelike curve (necessarily a geodesic), and any point in M is on the future null cone of some fixed point;

(iv) the *radius function*, defined by $r(p) := \sqrt{\mathrm{Area}(\mathcal{O}_p)/4\pi}$ (where $\mathcal{O}_p$ is the orbit through p), is monotonically increasing along the generators of these future null cones.[1]

Given a spacetime metric of the form (5.1) we have

$$
\begin{aligned}
g_{uu} = -g\tilde{g}\ , \quad g_{ur} = g_{ru} = -g\ , \quad g_{\theta\theta} = r^2\ , \quad g_{\varphi\varphi} = r^2\sin^2\theta = \sin^2\theta g_{\theta\theta}\ , \\
g^{rr} = \frac{\tilde{g}}{g}\ , \quad g^{ru} = g^{ur} = -\frac{1}{g}\ , \quad g^{\theta\theta} = \frac{1}{r^2}\ , \quad g^{\varphi\varphi} = \frac{1}{r^2\sin^2\theta} = \frac{g^{\theta\theta}}{\sin^2\theta}\ ;
\end{aligned}
$$

The nonvanishing Christoffel symbols (1.5) give

$$
\begin{aligned}
&\Gamma^u_{uu} = \frac{1}{g}\left[\frac{\partial}{\partial u}g - \frac{1}{2}\frac{\partial}{\partial r}(g\tilde{g})\right]\ , \quad \Gamma^u_{\theta\theta} = \frac{r}{g}\ , \quad \Gamma^u_{\varphi\varphi} = \frac{r\sin^2\theta}{g} = \sin^2\theta\Gamma^u_{\theta\theta}\ , \\
&\Gamma^r_{uu} = \frac{1}{2g}\frac{\partial}{\partial u}(g\tilde{g}) - \frac{\tilde{g}}{g}\left[\frac{\partial}{\partial u}g - \frac{1}{2}\frac{\partial}{\partial r}(g\tilde{g})\right]\ , \quad \Gamma^r_{ur} = \frac{1}{2g}\frac{\partial}{\partial r}(g\tilde{g})\ , \quad \Gamma^r_{rr} = \frac{1}{g}\frac{\partial}{\partial r}g\ , \\
&\Gamma^r_{\theta\theta} = -\frac{\tilde{g}}{g}r\ , \quad \Gamma^r_{\varphi\varphi} = -\frac{\tilde{g}}{g}r\sin^2\theta = \sin^2\theta\Gamma^r_{\theta\theta}\ , \\
&\Gamma^\theta_{\varphi\varphi} = -\sin\theta\cos\theta\ , \quad \Gamma^\theta_{\theta r} = \frac{1}{r} = \Gamma^\varphi_{\varphi r}\ , \quad \Gamma^\varphi_{\theta\varphi} = \frac{\cos\theta}{\sin\theta} = -\frac{1}{\sin^2\theta}\Gamma^\theta_{\varphi\varphi}\ ;
\end{aligned}
$$

and the nonvanishing components of the Ricci tensor are

$$
\begin{aligned}
R_{uu} &= \frac{1}{2g}\left(\frac{\partial^2}{\partial r\partial u}(g\tilde{g}) - \frac{\partial^2}{\partial u\partial r}(g\tilde{g}) + \tilde{g}\frac{\partial^2}{\partial r^2}(g\tilde{g}) - 2\tilde{g}\frac{\partial^2}{\partial r\partial u}g\right) + \frac{1}{2g^2}\left(2\tilde{g}\frac{\partial g}{\partial r}\frac{\partial g}{\partial u} - \tilde{g}\frac{\partial g}{\partial r}\frac{\partial}{\partial r}(g\tilde{g})\right) \\
&\quad + \frac{1}{rg}\left(\frac{\partial}{\partial u}(g\tilde{g}) + \tilde{g}\frac{\partial}{\partial r}(g\tilde{g}) - 2\tilde{g}\frac{\partial g}{\partial u}\right)\ , \\
R_{ur} &= \frac{1}{2g}\left(\frac{\partial^2}{\partial r^2}(g\tilde{g}) - 2\frac{\partial^2}{\partial u\partial r}g\right) - \frac{1}{2g^2}\left(\frac{\partial g}{\partial r}\frac{\partial}{\partial r}(g\tilde{g}) - 2\frac{\partial g}{\partial u}\frac{\partial g}{\partial r}\right) + \frac{1}{rg}\frac{\partial}{\partial r}(g\tilde{g})\ , \\
R_{rr} &= \frac{2}{r}\frac{1}{g}\frac{\partial g}{\partial r}\ , \\
R_{\theta\theta} &= -\frac{1}{g}\frac{\partial}{\partial r}(r\tilde{g}) + 1\ , \\
R_{\varphi\varphi} &= \sin^2\theta R_{\theta\theta}\ .
\end{aligned}
$$

Considering a spherically symmetric massless scalar field $\mathcal{V}(\phi) = 0$, the Einstein field equations with a positive cosmological constant (1.3) have the following nontrivial components:

$$
\begin{aligned}
&\frac{1}{2g}\left(\tilde{g}\frac{\partial^2}{\partial r^2}(g\tilde{g}) - 2\tilde{g}\frac{\partial^2}{\partial r\partial u}g\right) + \frac{1}{2g^2}\left(2\tilde{g}\frac{\partial g}{\partial r}\frac{\partial g}{\partial u} - \tilde{g}\frac{\partial g}{\partial r}\frac{\partial}{\partial r}(g\tilde{g})\right) \\
&\qquad + \frac{g}{r}\frac{\partial}{\partial u}\left(\frac{\tilde{g}}{g}\right) + \frac{\tilde{g}}{rg}\frac{\partial}{\partial r}(g\tilde{g}) = (\partial_u\phi)^2 - \Lambda g\tilde{g}\ ,
\end{aligned}
\tag{5.3}
$$

[1] These two last assumptions exclude the Nariai solution, for instance, from our analysis.

$$\frac{1}{2g}\left(\frac{\partial^2}{\partial r^2}(g\tilde{g})-2\frac{\partial^2}{\partial u\partial r}g\right)-\frac{1}{2g^2}\left(\frac{\partial g}{\partial r}\frac{\partial}{\partial r}(g\tilde{g})-2\frac{\partial g}{\partial u}\frac{\partial g}{\partial r}\right)$$
$$+\frac{1}{rg}\frac{\partial}{\partial r}(g\tilde{g})=(\partial_u\phi)(\partial_r\phi)-\Lambda g\ ,$$
$$\frac{2}{r}\frac{1}{g}\frac{\partial g}{\partial r}=(\partial_r\phi)^2\ ,$$
$$\frac{\partial}{\partial r}(r\tilde{g})=g\left(1-\Lambda r^2\right)\ ,$$

and the wave equation for the scalar field, reads

$$-\frac{2}{g}\left(\partial_u-\Gamma^r_{ru}\right)(\partial_r\phi)+\frac{\tilde{g}}{g}\left(\partial_r-\Gamma^r_{rr}\right)(\partial_r\phi)-\frac{2}{r^2}\Gamma^r_{\theta\theta}(\partial_r\phi)-\frac{2}{r^2}\Gamma^u_{\theta\theta}(\partial_u\phi)=0$$
$$\Leftrightarrow\quad\frac{1}{r}\left[\frac{\partial}{\partial u}-\frac{\tilde{g}}{2}\frac{\partial}{\partial r}\right]\frac{\partial}{\partial r}(r\phi)=\frac{1}{2}\left(\frac{\partial\tilde{g}}{\partial r}\right)\left(\frac{\partial\phi}{\partial r}\right)\ .$$

5.1 Christodoulou's framework for spherical waves

As shown in [40], the full content of the EFEs is encoded in the following three equations: the rr component of the field equations,

$$\frac{2}{r}\frac{1}{g}\frac{\partial g}{\partial r}=(\partial_r\phi)^2\ ;\tag{5.4}$$

the $\theta\theta$ component of the field equations,

$$\frac{\partial}{\partial r}(r\tilde{g})=g\left(1-\Lambda r^2\right)\ ;\tag{5.5}$$

and the wave equation,

$$\frac{1}{r}\left[\frac{\partial}{\partial u}-\frac{\tilde{g}}{2}\frac{\partial}{\partial r}\right]\frac{\partial}{\partial r}(r\phi)=\frac{1}{2}\left(\frac{\partial\tilde{g}}{\partial r}\right)\left(\frac{\partial\phi}{\partial r}\right)\ .\tag{5.6}$$

Integrating (5.4) with initial condition

$$g(u,r=0)=1$$

(so that we label the future null cones by the proper time of the free-falling observer at the center[2]) yields

$$g=e^{\frac{1}{2}\int_0^r s(\partial_s\phi)^2ds}.\tag{5.7}$$

Given any continuous function $f=f(u,r)$ we define its *average function* by

$$\bar{f}(u,r):=\frac{1}{r}\int_0^r f(u,s)\,ds\ ,\tag{5.8}$$

for which the following identity holds:

$$\frac{\partial\bar{f}}{\partial r}=\frac{f-\bar{f}}{r}\ .\tag{5.9}$$

Using the regularity condition

$$\lim_{r\to 0}r\tilde{g}=0\ ,$$

[2]This differs from Christodoulou's original choice, which was to use the proper time of observers at infinity.

implicit in our definition of Bondi-spherically symmetric spacetime, we obtain by integrating (5.5):

$$\tilde{g} = \frac{1}{r}\int_0^r g\left(1 - \Lambda s^2\right) ds = \overline{g\left(1 - \Lambda r^2\right)} = \bar{g} - \frac{\Lambda}{r}\int_0^r g s^2 ds \ . \tag{5.10}$$

Following [40] we introduce

$$h := \partial_r \left(r\phi\right) \ .$$

Assuming ϕ continuous, which implies

$$\lim_{r\to 0} r\phi = 0 \ ,$$

we have

$$\phi = \frac{1}{r}\int_0^r h\left(u,s\right) ds = \bar{h} \quad \text{and} \quad \frac{\partial \phi}{\partial r} = \frac{\partial \bar{h}}{\partial r} = \frac{h - \bar{h}}{r} \ , \tag{5.11}$$

and so (5.7) reads

$$g(u,r) = \exp\left(\frac{1}{2}\int_0^r \frac{\left(h - \bar{h}\right)^2}{s} ds\right) \ . \tag{5.12}$$

Now, defining the differential operator

$$D := \frac{\partial}{\partial u} - \frac{\tilde{g}}{2}\frac{\partial}{\partial r} \ ,$$

whose integral lines are the incoming light rays (with respect to the observer at the center $r = 0$), and using (5.11) together with (5.5), the wave-equation (5.6) is rewritten as the integro-differential equation

$$Dh = G\left(h - \bar{h}\right) \ , \tag{5.13}$$

where we have set

$$\begin{aligned} G &:= \frac{1}{2}\partial_r \tilde{g} && (5.14)\\ &= \frac{1}{2r}\left[(1 - \Lambda r^2)g - \overline{(1 - \Lambda r^2)g}\right] && (5.15)\\ &= \frac{(g - \bar{g})}{2r} + \frac{\Lambda}{2r^2}\int_0^r g s^2 ds - \frac{\Lambda}{2} r g \ . && (5.16) \end{aligned}$$

Thus we have derived the following:

Proposition 5.1. *For Bondi-spherically symmetric spacetimes* (5.1)*, the Einstein-scalar field system with cosmological constant* (1.3) *is equivalent to the integro-differential equation* (5.13)*, together with* (5.10)*,* (5.11)*,* (5.12) *and* (5.14)*.*

We will also need an evolution equation for $\partial_r h$ given a sufficiently regular solution of (5.13): using

$$[D, \partial_r] = G\partial_r \ ,$$

differentiating (5.13), and assuming that we are allowed to commute partial derivatives, we obtain

$$D\partial_r h - 2G\partial_r h = -J\,\partial_r \bar{h} \ , \tag{5.17}$$

where

$$\begin{aligned} J :&= G - r\partial_r G && (5.18)\\ &= 3G + \Lambda g r + (\Lambda r^2 - 1)\frac{1}{2}\frac{\partial g}{\partial r} \ . && (5.19) \end{aligned}$$

Chapter 6

Spherical linear waves in de Sitter spacetime

The study of the linear wave equation

$$\Box_{\mathbf{g}}\phi = 0 \tag{6.1}$$

on fixed backgrounds $(\mathcal{M}, \mathbf{g})$ is a stepping stone to the analysis of the nonlinearities of gravitation. In this paper we apply Christodoulou's framework, developed in [40], to spherically symmetric solutions of (6.1) on a de Sitter background, as a prerequisite to the study of the coupled Einstein-scalar field equations with positive cosmological constant in spherical symmetry (which will be pursued elsewhere). If the cosmological constant vanishes then the uncoupled problem (6.1) is trivial[1], a fact that Christodoulou explored in [40] to solve the coupled case for suitably small initial data. For positive cosmological constant, however, the uncoupled case is more complicated[2], and it is essential to understand it thoroughly in order to ascertain how much freedom is there when perturbing it to the nonlinear case, as well as to determine which decays to expect and which function spaces to use.

Following Christodoulou, we turn (6.1) into an integro-differential evolution equation, which we solve by taking initial data on a null cone. This step, which is trivial in the case of vanishing cosmological constant, turns out to be quite subtle for positive cosmological constant. As a corollary we obtain elementary derivations of expected properties of linear waves in de Sitter spacetime: boundedness in terms of (characteristic) initial data, and uniform exponential decay, in Bondi time, to a constant (from which exponential decay to a constant in the usual static time coordinate easily follows.Although widely expected from the stability of de Sitter spacetime [83], we are unaware of a written proof of uniform exponential decay, see also [160]. Similarly, the bound that we obtain for the solution in terms of the C^0 norm of the (characteristic) initial data is, to the best of our knowledge, original (notice that in particular the bound (6.4) involves no "loss of derivatives"). Another novel aspect of our work is the fact that our results apply to a domain containing the cosmological horizon in its interior, therefore including both a local and a cosmological regions (as opposed to considering only the local region). This allows us to determine decays for initial data which lead to uniform exponential decay in time of the solution.

Numerical evidence for exponential decay can be found in [26], and references therein, where higher spherical harmonics are also studied, as well as the non-linear system. Yagdjian and Galstian [210] constructed the fundamental solutions of (6.1) in de Sitter spacetime and proved exponential decay of certain homogeneous Sobolev L^p norms, $2 \leq p < \infty$. Also along these lines, Ringström [170] obtained exponential decay for non-linear perturbations of locally de

[1]This can be seen from the fact that operator $\mathcal{F}$ in equation (6.10), whose fixed points are the solutions of (6.1), is a constant operator for $\Lambda = 0$; when perturbing to the nonlinear problem this operator becomes a contraction for small initial data.

[2]In this case the operator $\mathcal{F}$ is not even a contraction in the full domain.

Sitter cosmological models in the context of the Einstein-nonlinear scalar field system with a positive potential. Finally, exponential pointwise decay in the local region between the black hole and the cosmological horizons in a Schwarzschild-de-Sitter spacetime follows from the papers by Dafermos and Rodnianski [53, 54] (see also [24]). Recently [53] was extended to cover also the *cosmological region* by Schlue in [177].

Given the metric (5.2), the wave-equation (5.13) reads

$$Dh = -\frac{\Lambda}{3} r(h - \bar{h}) \ , \tag{6.2}$$

where D is the differential operator given by

$$D := \frac{\partial}{\partial u} - \frac{1}{2}\left(1 - \frac{\Lambda}{3} r^2\right) \frac{\partial}{\partial r} \ .$$

6.1 Main result: statement and proof

The main result of this chpater is the following

Theorem 3. *Let $\Lambda > 0$. Given $h_0 \in \mathcal{C}^k([0,\infty))$, for some $k \geq 1$, the problem*

$$\begin{cases} Dh = -\frac{\Lambda}{3} r(h - \bar{h}) \\ h(0,r) = h_0(r) \end{cases} \tag{6.3}$$

has a unique solution $h \in \mathcal{C}^k([0,\infty) \times [0,\infty))$.

Moreover, if $\|h_0\|_{\mathcal{C}^0}$ is finite[3] *then*

$$\|h\|_{\mathcal{C}^0} = \|h_0\|_{\mathcal{C}^0} \ . \tag{6.4}$$

Also, if $\|(1+r)^p \partial_r h_0\|_{\mathcal{C}^0}$ is finite for some $0 \leq p \leq 4$ and $H \leq 2\sqrt{\frac{\Lambda}{3}}$ then

$$\|(1+r)^p e^{Hu} \partial_r h\|_{\mathcal{C}^0} \lesssim \|(1+r)^p \partial_r h_0\|_{\mathcal{C}^0} \ , \tag{6.5}$$

and, consequently, there exists $\underline{h} \in \mathbb{R}$ such that

$$|h(u,r) - \underline{h}| \lesssim (1+r)^{n(p)} e^{-Hu} \ , \tag{6.6}$$

with

$$n(p) = \begin{cases} 0 & , \quad 2 < p \leq 4 \\ 2 & , \quad 0 \leq p \leq 2 \end{cases} \ . \tag{6.7}$$

Remark 6. *The powers of $1+r$ obtained are far from optimal. Since we are mainly interested in understanding whether the decay in u obtained by this method is uniform in r, we were only careful in computing precise estimates for $2 < p \leq 4$, which is enough to establish uniform decay for $p > 2$ (if $p > 4$ the $p = 4$ result applies, and in fact it does not seem to be possible to obtain a stronger decay in r for $\partial_r h$). For $p \leq 2$ our method does not provide uniform decay, but it is not clear if this is an artifact of these techniques or an intrinsic property of spherical linear waves in de Sitter.*

Proof. For $h \in \mathcal{C}^0([0,\infty) \times [0,\infty))$, we have $r\bar{h} \in \mathcal{C}^0([0,\infty) \times [0,\infty))$, and so we can define $\mathcal{F}(h)$ to be the solution to the linear equation

$$\begin{cases} D(\mathcal{F}(h)) = -\frac{\Lambda}{3} r(\mathcal{F}(h) - \bar{h}) \\ \mathcal{F}(h)(0,r) = h_0(r) \end{cases} \ . \tag{6.8}$$

[3] Recall that if $f : X \to \mathbb{R}$ is continuous and bounded then $\|f\|_{\mathcal{C}^0} = \sup_{x \in X} |f(x)|$.

The integral lines of D (incoming light rays in de Sitter), which satisfy

$$\frac{dr}{du} = -\frac{1}{2}\left(1 - \frac{\Lambda}{3}r^2\right) , \tag{6.9}$$

are characteristics of the problem at hand. Integrating (6.8) along such characteristics we obtain

$$\mathcal{F}(h)(u_1, r_1) = h_0(r(0))e^{-\frac{\Lambda}{3}\int_0^{u_1} r(s)ds} + \frac{\Lambda}{3}\int_0^{u_1} r(v)\bar{h}(v, r(v))e^{-\frac{\Lambda}{3}\int_v^{u_1} r(s)ds}dv , \tag{6.10}$$

where, to simplify the notation, we denote the solution to (6.9) satisfying $r(u_1) = r_1$ simply by $s \mapsto r(s)$; we are dropping any explicit reference to the dependence on (u_1, r_1), but it should be noted, in particular, that $r(0)$ is an analytic function of (u_1, r_1).

Given $U, R > 0$, let $\mathcal{C}^0_{U,R}$ denote the Banach space $\left(\mathcal{C}^0([0,U]\times[0,R]), \|\cdot\|_{\mathcal{C}^0_{U,R}}\right)$, where

$$\|f\|_{\mathcal{C}^0_{U,R}} = \sup_{(u,r)\in[0,U]\times[0,R]} |f(u,r)| . \tag{6.11}$$

Let $r_c := \sqrt{\frac{3}{\Lambda}}$ be the unique non-negative zero of $1 - \frac{\Lambda}{3}r^2$ (see (5.2) and (6.9)). The non-decreasing behavior of the characteristics satisfying $r_1 \geq r_c$ shows that the restriction of $\mathcal{F}$ to $\mathcal{C}^0_{U,R}$ is well defined for all $R \geq r_c$. In fact:

Lemma 4. *Given $U > 0$ and $R \geq r_c := \sqrt{\frac{3}{\Lambda}}$, $\mathcal{F}$ contracts in $\mathcal{C}^0_{U,R}$.*

Proof. Fix $U > 0$ and $R \geq r_c$. Then

$$\begin{aligned}
\|\mathcal{F}(h_1) - \mathcal{F}(h_2)\|_{\mathcal{C}^0_{U,R}} &= \sup_{(u_1,r_1)\in[0,U]\times[0,R]} |\mathcal{F}(h_1)(u_1,r_1) - \mathcal{F}(h_2)(u_1,r_1)| \\
&\leq \sup_{(u_1,r_1)\in[0,U]\times[0,R]} \left\{\frac{\Lambda}{3}\int_0^{u_1} r(v)\left|\bar{h}_1(v,r(v)) - \bar{h}_2(v,r(v))\right| e^{-\frac{\Lambda}{3}\int_v^{u_1} r(s)ds}dv\right\} \\
&\leq \sup_{(u_1,r_1)\in[0,U]\times[0,R]} \left\{\int_0^{u_1} \frac{\Lambda}{3}r(v)e^{-\frac{\Lambda}{3}\int_v^{u_1} r(s)ds}dv\right\} \cdot \|\bar{h}_1 - \bar{h}_2\|_{\mathcal{C}^0_{U,R}} \\
&\leq \sup_{(u_1,r_1)\in[0,U]\times[0,R]} \left\{\left[e^{-\frac{\Lambda}{3}\int_v^{u_1} r(s)ds}\right]_{v=0}^{u_1}\right\} \cdot \sup_{(u,r)\in[0,U]\times[0,R]} \left\{\frac{1}{r}\int_0^r |h_1(u,s) - h_2(u,s)|ds\right\} \\
&\leq \underbrace{\sup_{(u_1,r_1)\in[0,U]\times[0,R]} \left\{1 - e^{-\frac{\Lambda}{3}\int_0^{u_1} r(s)ds}\right\}}_{:=\sigma} \cdot \|h_1 - h_2\|_{\mathcal{C}^0_{U,R}} .
\end{aligned}$$

Throughout, to obtain estimates, and in particular to estimate σ, one needs to consider three (causally) separate regions, naturally corresponding to the bifurcations of (6.9): the local region ($r < r_c$), the cosmological horizon ($r = r_c$), and the cosmological region ($r > r_c$). However, since the computations are similar we will only present the details concerning the most delicate case, $r > r_c$.

The solution to (6.9) satisfying $r_1 = r(u_1) > r_c := \sqrt{\frac{3}{\Lambda}}$, is given by

$$r(u) = \sqrt{\frac{3}{\Lambda}}\coth\left(\frac{1}{2}\sqrt{\frac{\Lambda}{3}}(c - u)\right) , \tag{6.12}$$

where

$$c = u_1 + 2\sqrt{\frac{3}{\Lambda}}\operatorname{arcoth}\left(\sqrt{\frac{\Lambda}{3}}r_1\right)$$

(in particular $c > u_1$, and so (6.12) is well defined for $0 \le u \le u_1$). It follows that

$$-\frac{\Lambda}{3}\int_0^{u_1} r(s)ds = \int_0^{u_1} -\sqrt{\frac{\Lambda}{3}}\coth\left(\frac{1}{2}\sqrt{\frac{\Lambda}{3}}(c-s)\right)ds$$

$$= \int_0^{u_1} 2\frac{d}{ds}\ln\left[\sinh\left(\frac{1}{2}\sqrt{\frac{\Lambda}{3}}(c-s)\right)\right]ds = \ln\left[\frac{\sinh\left(\frac{1}{2}\sqrt{\frac{\Lambda}{3}}(c-u_1)\right)}{\sinh\left(\frac{1}{2}\sqrt{\frac{\Lambda}{3}}c\right)}\right]^2 ,$$

and consequently

$$e^{-\frac{\Lambda}{3}\int_0^{u_1} r(s)ds} = \frac{\sinh^2\left(\frac{1}{2}\sqrt{\frac{\Lambda}{3}}(c-u_1)\right)}{\sinh^2\left(\frac{1}{2}\sqrt{\frac{\Lambda}{3}}c\right)} = \frac{\cosh^2\left(\frac{1}{2}\sqrt{\frac{\Lambda}{3}}(c-u_1)\right)}{\sinh^2\left(\frac{1}{2}\sqrt{\frac{\Lambda}{3}}c\right)\coth^2\left(\frac{1}{2}\sqrt{\frac{\Lambda}{3}}(c-u_1)\right)}$$

$$= \left[\frac{\cosh\left(\alpha(c-u_1)\right)}{\sinh\left(\alpha c\right)}\frac{1}{2\alpha r_1}\right]^2 = \left[\frac{e^{\alpha(c-u_1)}+e^{-\alpha(c-u_1)}}{e^{\alpha c}-e^{-\alpha c}}\right]^2\frac{1}{4\alpha^2 r_1^2}$$

$$\ge \frac{e^{-2\alpha u_1}}{4\alpha^2 r_1^2},$$

where $\alpha := \frac{1}{2}\sqrt{\frac{\Lambda}{3}}$. Define

$$\sigma_{cosm}(U,R) := \sup_{(u_1,r_1)\in[0,U]\times(r_c,R]}\left(1-e^{-\frac{\Lambda}{3}\int_0^{u_1} r(s)ds}\right)$$

$$\le \sup_{(u_1,r_1)\in[0,U]\times(r_c,R]}\left(1-\frac{e^{-2\alpha u_1}}{4\alpha^2 r_1^2}\right) \le \left(1-\frac{3}{\Lambda}\frac{e^{-\sqrt{\frac{\Lambda}{3}}U}}{R^2}\right) < 1 .$$

Similar computations give

$$\sigma_{loc} := \sup_{(u_1,r_1)\in[0,U]\times[0,r_c)}\left(1-e^{-\frac{\Lambda}{3}\int_0^{u_1} r(s)ds}\right) \le \left(1-\frac{e^{-\sqrt{\frac{\Lambda}{3}}U}}{4}\right) < 1 ,$$

for the local region, and

$$\sigma_{hor} := \sup_{u_1\in[0,U]}\left(1-e^{-\frac{\Lambda}{3}\int_0^{u_1} r_c ds}\right) \le 1-e^{-\sqrt{\frac{\Lambda}{3}}U} < 1 ,$$

along the cosmological horizon. Finally $\sigma = \max\{\sigma_{loc},\sigma_{hor},\sigma_{cosm}\} < 1$, and the statement of the lemma follows.

□

By the contraction mapping theorem [89], given $U > 0$ and $R \ge r_c$, there exits a unique fixed point $h_{U,R} \in \mathcal{C}^0_{U,R}$ of $\mathcal{F}$. Uniqueness guarantees that in the intersection of two rectangles $[0,U_1]\times[0,R_1]\cap[0,U_2]\times[0,R_2]$ the corresponding h_{U_1,R_1} and h_{U_2,R_2} coincide. Consequently, there exists a unique continuous map $h : [0,\infty)\times[0,\infty)\to\mathbb{R}$ such that $h = \mathcal{F}(h)$, i.e.,

$$h(u_1,r_1) = h_0(r(0))e^{-\frac{\Lambda}{3}\int_0^{u_1} r(s)ds} + \frac{\Lambda}{3}\int_0^{u_1} r(v)\bar{h}(v,r(v))e^{-\frac{\Lambda}{3}\int_v^{u_1} r(s)ds}dv , \qquad (6.13)$$

in $[0,\infty)\times[0,\infty)$. Continuity of h implies continuity of $r\bar{h}$, so we are allowed to differentiate (6.13) in the direction of D, which proves that h is in fact a ($\mathcal{C}^0$) solution of (6.3). Existence and uniqueness in $\mathcal{C}^0([0,\infty)\times[0,\infty))$ follow.

To see that a solution of (6.3) is as regular as its initial condition assume that $h_0 \in \mathcal{C}^{k+1}$, $k \geq 0$, and start by noticing that if $h \in \mathcal{C}^k$ then $r\bar{h}$ and $\partial_r(r\bar{h})$ are also in $\mathcal{C}^k$. In particular for $h \in \mathcal{C}^0$ we can differentiate (6.13) with respect to u_1 to obtain

$$\begin{aligned}\frac{\partial h}{\partial u_1} &= \frac{\partial}{\partial u_1}\left(h_0(r(0))e^{-\frac{\Lambda}{3}\int_0^{u_1} r(s)ds}\right) + \frac{\Lambda}{3}\left(r\bar{h}\right)(u_1,r_1) \\ &+ \frac{\Lambda}{3}\int_0^{u_1} \frac{\partial(r\bar{h})}{\partial r}(v,r(v))\frac{\partial r}{\partial u_1}(v)e^{-\frac{\Lambda}{3}\int_v^{u_1} r(s)ds}dv \\ &+ \frac{\Lambda}{3}\int_0^{u_1} r(v)\bar{h}(v,r(v))\frac{\partial}{\partial u_1}\left(e^{-\frac{\Lambda}{3}\int_v^{u_1} r(s)ds}\right) .\end{aligned} \tag{6.14}$$

This last expression shows that $h_0 \in \mathcal{C}^{k+1}$ and $h \in \mathcal{C}^k$ implies $\frac{\partial h}{\partial u_1} \in \mathcal{C}^k$. The same reasoning works for the derivative with respect to r_1 (note that although $r(v)$ is given by different expressions according to whether $r_1 < r_c$, $r_1 = r_c$ or $r_1 > r_c$, it is still the solution of a smooth ODE satisfying $r(u_1) = r_1$, and as such depends smoothly on the data (u_1, r_1)). Consequently, if $h_0 \in C^{k+1}$ and $h \in \mathcal{C}^k$, then h is in fact in C^{k+1} and the regularity statement follows by induction.

To establish (6.4) first note that:

Lemma 5. *If $\|h_0\|_{\mathcal{C}^0} \leq y_0$ and $\|h\|_{\mathcal{C}^0} \leq y_0$, for some $y_0 \geq 0$, then $\|\mathcal{F}(h)\|_{\mathcal{C}^0} \leq y_0$.*

Proof. From (6.10) we see that

$$\begin{aligned}|\mathcal{F}(h)(u_1,r_1)| &\leq \|h_0\|_{\mathcal{C}^0}\, e^{-\frac{\Lambda}{3}\int_0^{u_1} r(s)ds} + \|\bar{h}\|_{\mathcal{C}^0}\,\frac{\Lambda}{3}\int_0^{u_1} r(v)e^{-\frac{\Lambda}{3}\int_v^{u_1} r(s)ds}dv \\ &\leq y_0 \underbrace{\left(e^{-\frac{\Lambda}{3}\int_0^{u_1} r(s)ds} + \frac{\Lambda}{3}\int_0^{u_1} r(v)e^{-\frac{\Lambda}{3}\int_v^{u_1} r(s)ds}dv\right)}_{\equiv 1} = y_0 .\end{aligned}$$

The last step follows by a direct computation, as before, or by noticing that since $h \equiv 1$ is a solution to (6.3), with $h_0 \equiv 1$, one has $\mathcal{F}(1) \equiv 1$.

□

Now consider the sequence

$$\begin{cases} h_0(u,r) = h_0(r) \\ h_{n+1} = \mathcal{F}(h_n) \end{cases} .$$

We have already established that, for any $U > 0$ and $R \geq r_c$, h_n converges in $\mathcal{C}^0_{U,R}$ to h, the solution of (6.3). Lemma 5 then tells us that

$$\|h_n\|_{\mathcal{C}^0_{U,R}} \leq \|h_n\|_{\mathcal{C}^0} \leq \|h_0\|_{\mathcal{C}^0}\ , \qquad \text{and so} \qquad \|h\|_{\mathcal{C}^0_{U,R}} = \lim_{n\to\infty}\|h_n\|_{\mathcal{C}^0_{U,R}} \leq \|h_0\|_{\mathcal{C}^0}\ .$$

Since this holds for arbitrarily large U and R, the bound (6.4) follows.

We will now show that the estimate (6.5) holds. First of all if $h \in \mathcal{C}^1$ we see that Dh and $\partial_r Dh$ are both continuous, and consequently $D\partial_r h$ exists and its equal to $\partial_r Dh + [D,\partial_r]h$ [4].

[4] Here we are using the following generalized version of the Schwarz Lemma: if X and Y are two nonvanishing $\mathcal{C}^1$ vector fields in $\mathbb{R}^2$ and f is a $\mathcal{C}^1$ function such that $X\cdot(Y\cdot f)$ exists and is continuous then $Y\cdot(X\cdot f)$ also exists and is equal to $X\cdot(Y\cdot f) - [X,Y]\cdot f$.

Using this last fact and equations (5.11) while differentiating (6.2) with respect to r we obtain an evolution equation for $\partial_r h$:

$$D\partial_r h = -2\frac{\Lambda}{3} r\, \partial_r h \ . \tag{6.15}$$

Integrating the last equation along the (ingoing) characteristics, as before, yields

$$\partial_r h(u_1, r_1) = \partial_r h_0(r_0) e^{-\frac{2\Lambda}{3}\int_0^{u_1} r(s)ds}. \tag{6.16}$$

It is then clear that initial data controls the supremum norm of $\partial_r h$. In fact, let

$$d_0 = \|(1+r)^p \partial_r h_0\|_{\mathcal{C}^0} \ .$$

In the cosmological region ($r > r_c$), one has, after recalling (6.12),

$$\begin{aligned} \left|(1+r_1)^p e^{Hu_1}\partial_r h(u_1,r_1)\right| &= \left|(1+r_1)^p e^{Hu_1}\partial_r h_0(r_0) e^{-\frac{2\Lambda}{3}\int_0^{u_1} r(s)ds}\right| \\ &\le d_0 \left(\frac{1+r_1}{1+r_0}\frac{\sinh(\alpha(c-u_1))}{\sinh(\alpha c)}\right)^p e^{Hu_1}\left(\frac{\sinh(\alpha(c-u_1))}{\sinh(\alpha c)}\right)^{4-p}, \end{aligned} \tag{6.17}$$

where $\alpha = \frac{1}{2}\sqrt{\frac{\Lambda}{3}}$ as before. Now, since $c - u_1 \le c$, then $e^{-2\alpha(c-u_1)} \ge e^{-2\alpha c}$, and

$$\begin{aligned} \frac{\sinh(\alpha(c-u_1))}{\sinh(\alpha c)} &= \frac{e^{\alpha(c-u_1)} - e^{-\alpha(c-u_1)}}{e^{\alpha c} - e^{-\alpha c}} \\ &= e^{-\alpha u_1}\frac{1-e^{-2\alpha(c-u_1)}}{1-e^{-2\alpha c}} \\ &\le e^{-\alpha u_1} \ . \end{aligned} \tag{6.18}$$

Also

$$\begin{aligned} \frac{1+r_1}{1+r_0}\frac{\sinh(\alpha(c-u_1))}{\sinh(\alpha c)} &= \frac{1+\frac{1}{2\alpha}\coth(\alpha(c-u_1))}{1+\frac{1}{2\alpha}\coth(\alpha c)}\frac{\sinh(\alpha(c-u_1))}{\sinh(\alpha c)} \\ &= \frac{\sinh(\alpha(c-u_1)) + \frac{1}{2\alpha}\cosh(\alpha(c-u_1))}{\sinh(\alpha(c)) + \frac{1}{2\alpha}\cosh(\alpha c)} \\ &\le \frac{1+\frac{1}{2\alpha}}{\frac{1}{2\alpha}}\cdot\frac{\cosh(\alpha(c-u_1))}{\cosh(\alpha c)} \\ &\le (2\alpha+1)\, 2e^{-\alpha u_1}. \end{aligned} \tag{6.19}$$

Therefore, if $0 \le p \le 4$ and $H \le 4\alpha = 2\sqrt{\Lambda/3}$, we plug (6.18) and (6.19) into (6.17) to obtain

$$\begin{aligned} \sup_{(u_1,r_1)\in[0,U]\times[r_c,R]}\left|(1+r_1)^p e^{Hu_1}\partial_r h(u_1,r_1)\right| &\le d_0 \sup_{(u_1,r_1)\in[0,U]\times[r_c,R]}\left|2^p(2\alpha+1)^p e^{(H-4\alpha)u_1}\right| \\ &\le 2^p(2\alpha+1)^p d_0 \ . \end{aligned} \tag{6.20}$$

Similar, although simpler, computations yield

$$\sup_{(u_1,r_1)\in[0,U]\times[0,r_c]}\left|(1+r_1)^p e^{Hu_1}\partial_r h(u_1,r_1)\right| \le 16 \sup_{r_1\in[0,r_c]}\left|(1+r_1)^p\partial_r h_0(r_1)\right| \le 16 d_0 \tag{6.21}$$

for the local region. This proves (6.5).

To finish the proof of Theorem 3 all is left is to establish the uniform decay statement (6.6). Start with

$$\begin{aligned}\left|h(u,r)-\bar{h}(u,r)\right| &\le \frac{1}{r}\int_0^r |h(u,r)-h(u,s)|\,ds \\ &\le \frac{1}{r}\int_0^r\int_s^r |\partial_\rho h(u,\rho)|\,d\rho\,ds \\ &\lesssim \frac{1}{r}\int_0^r\int_s^r \frac{e^{-Hu}}{(1+\rho)^p}d\rho\,ds \lesssim \begin{cases} \frac{e^{-Hu}}{1+r} & , \quad 2<p\le 4 \\ re^{-Hu} & , \quad 0\le p\le 2\end{cases} .\end{aligned}$$

These estimates for $2<p\le 4$ are obtained by direct computation; they seem to be the optimal results which follow from this method. The remaining cases, with the exception of $p=0$, are far from optimal. In fact, since we are mainly interested in a qualitative analysis, namely if the decay obtained is or not uniform in r (see Remark 6), the results for $p\le 2$ were obtained simply using $\frac{1}{(1+r)^p}\le 1$.

Using (6.2) we then see that

$$\begin{aligned}|\partial_u h| &= \left|Dh+\frac{1}{2}\left(1-\frac{\Lambda}{3}r^2\right)\partial_r h\right| \\ &\le \left|-\frac{\Lambda}{3}r\left(h-\bar{h}\right)\right|+\frac{1}{2}\left|\left(1-\frac{\Lambda}{3}r^2\right)\partial_r h\right| \lesssim (1+r)^{n(p)}e^{-Hu} ,\end{aligned}$$

with $n(p)$ as in the statement of the theorem.

Now since $\partial_u h$ is integrable with respect to u, by the fundamental theorem of calculus, we see that there exists

$$\lim_{u\to\infty} h(u,r)=\underline{h}(r) .$$

But

$$\begin{aligned}|\underline{h}(r_2)-\underline{h}(r_1)| &= \lim_{u\to\infty}|h(u,r_2)-h(u,r_1)| \\ &\le \lim_{u\to\infty}\left|\int_{r_1}^{r_2}|\partial_r h(u,r)|dr\right| \\ &\lesssim \lim_{u\to\infty}|r_2-r_1|e^{-Hu}=0 ,\end{aligned}$$

and, consequently, there exists $\underline{h}\in\mathbb{R}$ such that

$$\underline{h}(r)\equiv\underline{h} .$$

Finally

$$\begin{aligned}|h(u,r)-\underline{h}| &\le \int_u^\infty |\partial_v h(v,r)|\,dv \\ &\lesssim \int_u^\infty (1+r)^{n(p)}e^{-Hv}dv \lesssim (1+r)^{n(p)}e^{-Hu} .\end{aligned}$$

□

Remark 7. *The same calculation shows that given $R>0$ the solutions of* (6.3) *satisfy $|h(u,r)-\underline{h}|\lesssim e^{-Hu}$ uniformly for $r\in[0,R]$, even if $\|(1+r)^p\partial_r h_0\|_{\mathcal{C}^0}$ is not finite.*

6.2 Boundedness and uniform exponential decay in Bondi time

We now translate part of the results in Theorem (3) back into results concerning linear waves in de Sitter.

Theorem 4. *Let* $(\mathcal{M}, \mathbf{g}_{ds})$ *be de Sitter spacetime with cosmological constant* $\Lambda > 0$ *and* (u, r, θ, φ) *Bondi coordinates as in Chapter (5). Let* $\phi = \phi(u, r) \in \mathcal{C}^2([0, \infty) \times [0, \infty))$ *be a solution* [5] *to*

$$\Box_g \phi = 0 \; .$$

Then

$$|\phi| \leq \sup_{r \geq 0} |\partial_r (r\phi(0, r))| \; . \tag{6.22}$$

Moreover, if for some $0 \leq p \leq 4$

$$\sup_{r \geq 0} \left| (1 + r)^p \frac{\partial^2}{\partial r^2} (r\phi(0, r)) \right| < \infty \; , \tag{6.23}$$

then there exists $\underline{\phi} \in \mathbb{R}$ *such that, for* $H \leq 2\sqrt{\frac{\Lambda}{3}}$ *,*

$$\left| \phi(u, r) - \underline{\phi} \right| \lesssim (1 + r)^{n(p)} e^{-Hu} \; , \tag{6.24}$$

where

$$n(p) = \begin{cases} 0 & , \quad 2 < p \leq 4 \\ 2 & , \quad 0 \leq p \leq 2 \end{cases} \; . \tag{6.25}$$

Proof. Since ϕ is a spherically symmetric $\mathcal{C}^2$ solution of (6.1) we saw in Chapter 5 that $h = \partial_r(r\phi)$ satisfies (6.2), with $\phi = \bar{h}$. Applying Theorem (3) the results easily follow. □

Remark 8. *Note that the bound* (6.22) *for* ϕ*, unlike the bound* (6.4) *for* h*, depends on the derivative of the initial data, so we do have "loss of derivatives" in this case.*

Remark 9. *Once more that the powers of* $1 + r$ *obtained are far from optimal, see Remark 6.*

Remark 10. *It should be emphasized that the boundedness and decay results are logically independent. In fact* (6.22) *follows from* (6.4)*, which in turn is a consequence of a fortunate trick (see proof of Lemma 5) relying on the non-positivity of the factor of the zeroth order term in* (6.2) *(here, non-negativity of* Λ*) and the fact that* $\mathcal{F}$ (6.10) *is a contraction in appropriate function spaces; in some sense one is required to prove existence and uniqueness of* (6.3) *in the process. That is no longer the case for obtaining* (6.6)*, from which uniform decay of* ϕ *follows.*

6.3 Conclusions

We have applied Christodoulou's framework, developed in [40], to spherically symmetric solutions of the wave equation in de Sitter spacetime, as a prerequisite to the study of the Einstein-scalar field equations with positive cosmological constant and spherically symmetric initial data. We obtained an integro-differential evolution equation, which we solved by taking initial data on a null cone. As a corollary we obtained elementary derivations of expected properties of linear waves in de Sitter spacetime: boundedness in terms of (characteristic) initial data, and uniform exponential decay, in Bondi time, to a constant (from which exponential decay to a constant in the usual de Sitter time coordinate easily follows).

[5] Alternatively one might consider a general solution and infer results about its zeroth spherical harmonic.

Chapter 7

Global existence and exponential decay in Bondi time for small data

As was already clear from the study of the uncoupled case of last chapter, the presence of a positive cosmological constant increases the difficulty of the problem at hand considerably. In fact, a global solution for the zero cosmological constant case was obtained in [40] by constructing a sequence of functions which, for an appropriate choice of Banach space, was a contraction in the full domain; such direct strategy does not work (at least for analogous choices of function spaces) when a positive cosmological constant is considered, since a global contraction is no longer available even in the uncoupled case, see Lemma 4.

Moreover, new difficulties appear in the non-linear problem when passing from zero to a positive cosmological constant: first of all, the incoming light rays (characteristics), whose behavior obviously depends of the unknown, bifurcates into three distinct families, with different, sometimes divergent, asymptotics;[1] this is in contrast with the $\Lambda = 0$ case, where all the characteristics approach the center of symmetry at a similar rate.

Also, for a vanishing cosmological constant the coefficient of the integral term of the equation decays radially, which is of crucial importance in solving the problem; on the contrary, for $\Lambda > 0$ such coefficient grows linearly with the radial coordinate.

To overcome these difficulties we were forced to differ from Christodoulou's original strategy considerably. The cornerstone of our analysis is a remarkable a priori estimate, the aforementioned result of boundedness in terms of initial data, whose inspiration comes from the uncoupled case of last chpater. We can then establish a local existence result with estimates for the solution and its radial derivative solely in terms of initial data and constants not depending on the time of existence, which allows us to extend a given local solution indefinitely. The decay results, which in the vanishing cosmological setting are an immediate consequence of the choice of function spaces and the existence of the already mentioned global contraction, here follow by establishing "energy inequalities", where the "energy function" is given by the supremum norm of the radial derivative of the unknown (7.47).

To make this strategy work we were forced to restrict our analysis to a finite range of the radial coordinate; one should note nonetheless, that although finite, the results here hold for arbitrarily large radial domains. At a first glance one would expect the need to impose boundary conditions at $r = R$, for R the maximal radius; this turns out to be unnecessary, since for sufficiently large radius the radial coordinate of the characteristics becomes an increasing function of time, and consequently the data at the boundary $r = R$ is completely determined by the initial data (see

[1] The use of double null coordinates (u, v), also introduced by Christodoulou for the study of the Einstein-scalar field equations in [42], would facilitate the handling of the characteristics, which in such coordinates take the form $v = const.$, but in doing so we are no longer able to reduce the full system to a single scalar equation. See also [52]

Figure 7.1). This situations parallels that of [170], where local information in space (here, in a light cone) allows to obtain global information in time.

A natural consequence of the introduction of a positive cosmological constant is the appearance of a cosmological horizon. In fact, although the small data assumptions do not allow the formation of a black hole event horizon, a cosmological apparent horizon is present from the start, and a cosmological horizon formed; this is of course related to the difficulties mentioned above concerning the dynamics of the characteristics.

Our main results may be summarized in the following:

Theorem 5. *Let $\Lambda > 0$ and $R > \sqrt{3/\Lambda}$. There exists $\epsilon_0 > 0$, depending on Λ and R, such that for $\phi_0 \in \mathcal{C}^{k+1}([0,R])$ $(k \geq 1)$ satisfying*

$$\sup_{0\leq r\leq R} |\phi_0(r)| + \sup_{0\leq r\leq R} |\partial_r \phi_0(r)| < \epsilon_0 \ ,$$

there exists a unique Bondi-spherically symmetric $\mathcal{C}^k$ solution[2] $(\mathcal{M}, \mathbf{g}, \phi)$ of the Einstein-Λ-scalar field system (1.3)*, with the scalar field ϕ satisfying the characteristic condition*

$$\phi_{|u=0} = \phi_0 \ .$$

The Bondi coordinates for $\mathcal{M}$ have range $[0,+\infty) \times [0,R] \times \mathbb{S}^2$, and the metric takes the form (5.1)*. Moreover, we have the following bound in terms of initial data:*

$$|\phi| \leq \sup_{0\leq r\leq R} |\partial_r (r\phi_0(r))| \ .$$

Regarding the asymptotics, there exists $\underline{\phi} \in \mathbb{R}$ such that

$$\left|\phi(u,r) - \underline{\phi}\right| \lesssim e^{-Hu} \ , \tag{7.1}$$

and

$$|\mathbf{g}_{\mu\nu} - \mathring{\mathbf{g}}_{\mu\nu}| \lesssim e^{-Hu} \ , \tag{7.2}$$

where $H := 2\sqrt{\Lambda/3}$ and $\mathring{\mathbf{g}}$ is de Sitter's metric in Bondi coordinates, as given in (5.2)*. Finally, the spacetime $(\mathcal{M}, \mathbf{g})$ is causally geodesically complete towards the future[3] and has vanishing final Bondi mass[4].*

This result is an immediate consequence of Proposition 5.1 and Theorems 7 and 8. Note that, as is the case with the characteristic initial value problem for the wave equation, only ϕ needs to be specified on the initial characteristic hypersurface[5] (as opposed to, say, ϕ and $\partial_u\phi$). There is no initial data for the metric functions, whose initial data is fixed by the choice of ϕ_0. A related issue that may cause confusion is that the vanishing of $\partial_r\phi_0(0)$ is not required to ensure regularity at the center: in fact, the precise condition for ϕ to be regular at the center is $\partial_u\phi(u,0) = \partial_r\phi(u,0)$, which is an automatic consequence of the wave equation. The reader unfamiliar with these facts should note that, for example, the solution of the spherically symmetric wave equation in Minkowski spacetime, $\partial_t^2(r\phi) - \partial_r^2(r\phi) = 0$, with initial data $\phi(r,r) = r$, is the smooth function $\phi(t,r) = t$ for $t > r$.

[2]See Chapter 5 for the precise meaning of a $\mathcal{C}^1$ solution of the Einstein-Λ-scalar field system in Bondi-spherical symmetry.

[3]A manifold with boundary is geodesically complete towards the future if the only geodesics which cannot be continued for all values of the affine parameter are those with endpoints on the boundary.

[4]See Section 7.1 for the definition of the final Bondi mass in this context.

[5]Notice however that uniqueness is not expected to hold towards the past.

7.1 The mass equation

Consider a Bondi-spherically symmetric $\mathcal{C}^k$ solution of (1.3) on a domain $(u,r) \in [0,U) \times [0,R]$ (with $R > \sqrt{3/\Lambda}$). From equations (5.5) and (5.12) it is clear that $r\tilde{g}$ is increasing in r for $r < \sqrt{1/\Lambda}$ and decreasing for $r > \sqrt{1/\Lambda}$. On the other hand, equation (5.10) implies that $\tilde{g}(u,r)$ approaches $-\infty$ as $r \to +\infty$. Therefore there exists a unique $r = r_c(u) > \sqrt{1/\Lambda}$ where $\tilde{g}(u,r)$ vanishes. This defines precisely the set of points where $\frac{\partial}{\partial u}$ is null, and hence the curve $r = r_c(u)$ determines an apparent (cosmological) horizon. Since g is increasing in r, we have from (5.10)

$$\tilde{g}(u,r) \leq g(u,\sqrt{1/\Lambda})\frac{1}{r}\int_0^r \left(1-\Lambda s^2\right) ds = g(u,\sqrt{1/\Lambda})\left(1-\frac{\Lambda r^2}{3}\right).$$

Therefore the radius of the apparent cosmological horizon is bounded by

$$\sqrt{\frac{1}{\Lambda}} < r_c(u) \leq \sqrt{\frac{3}{\Lambda}}$$

for all u. From (5.5) it is then clear that $\frac{\partial \tilde{g}}{\partial r} < 0$ for $r = r_c(u)$, and so by the implicit function theorem the function $r_c(u)$ is $\mathcal{C}^k$. From the uu component of (1.3) (equation (5.3)), we obtain

$$\frac{g}{r}\frac{\partial}{\partial u}\left(\frac{\tilde{g}}{g}\right) = (\partial_u \phi)^2$$

when $\tilde{g} = 0$, showing that $\frac{\tilde{g}}{g}$ is nondecreasing in u, and so $r_c(u)$ must also be nondecreasing. In particular the limit

$$r_1 := \lim_{u\to U} r_c(u)$$

exists, and $\sqrt{1/\Lambda} < r_1 \leq \sqrt{3/\Lambda}$. We introduce the renormalized Hawking mass function[6] [141, 135]

$$m(u,r) = \frac{r}{2}\left(1 - \frac{\tilde{g}}{g} - \frac{\Lambda}{3}r^2\right), \tag{7.3}$$

which measures the mass contained within the sphere of radius r at retarded time u, renormalized so as to remove the contribution of the cosmological constant and make it coincide with the mass parameter in the case of the Schwarzschild-de Sitter spacetime. This function is zero at $r = 0$, and from (5.4), (5.5) we obtain

$$\frac{\partial m}{\partial r} = \frac{r^2 \tilde{g}}{4g}(\partial_u \phi)^2,$$

implying that $m(u,r) \geq 0$ for $r \leq r_c(u)$. We have

$$m(u,r_c(u)) = \frac{r_c(u)}{2}\left(1 - \frac{\Lambda}{3}r_c(u)^2\right),$$

whence

$$\frac{d}{du}m(u,r_c(u)) = \frac{\dot{r}_c(u)}{2}\left(1 - \Lambda r_c(u)^2\right) \leq 0,$$

and so $m(u,r_c(u))$ is a nonincreasing function of u. Therefore the limit

$$M_1 := \lim_{u\to U} m(u,r_c(u)) = \frac{r_1}{2}\left(1 - \frac{\Lambda}{3}r_1^2\right)$$

exists, and from $\sqrt{1/\Lambda} < r_1 \leq \sqrt{3/\Lambda}$ we have $0 \leq M_1 < 1/\sqrt{9\Lambda}$. We call this limit the *final Bondi mass.* Note that, unlike the usual definition in the asymptotically flat case, where the limit is taken at $r = +\infty$, here we take the limit along the apparent cosmological horizon; the reason for doing this is that $r \leq R$ in our case.

[6] This function is also known as the "generalized Misner-Sharp mass".

7.2 Basic Estimates

Given $U, R > 0$, let $\mathcal{C}^0_{U,R}$ denote the Banach space $\left(\mathcal{C}^0([0,U]\times[0,R]), \|\cdot\|_{\mathcal{C}^0_{U,R}}\right)$, where

$$\|f\|_{\mathcal{C}^0_{U,R}} := \sup_{(u,r)\in[0,U]\times[0,R]} |f(u,r)|\,,$$

and let $X_{U,R}$ denote the Banach space of functions which are continuous and have continuous partial derivative with respect to r, normed by

$$\|f\|_{X_{U,R}} := \|f\|_{\mathcal{C}^0_{U,R}} + \|\partial_r f\|_{\mathcal{C}^0_{U,R}}\,.$$

For functions defined on $[0,R]$ we will denote $C^0([0,R])$ by C^0_R, $C^1([0,R])$ by X_R, and will also use these notations for the corresponding norms. For $h \in \mathcal{C}^0_{U,R}$ we have

$$\left|\bar{h}(u,r)\right| \le \frac{1}{r}\int_0^r |h(u,s)|\,ds \le \frac{1}{r}\int_0^r \|h\|_{\mathcal{C}^0_{U,R}}\,ds = \|h\|_{\mathcal{C}^0_{U,R}}$$

and if $h \in X_{U,R}$ we can estimate

$$\begin{aligned} \left|(h-\bar{h})(u,r)\right| &= \left|\frac{1}{r}\int_0^r (h(u,r)-h(u,s))\,ds\right| = \left|\frac{1}{r}\int_0^r\int_s^r \frac{\partial h}{\partial\rho}(u,\rho)d\rho\,ds\right| \\ &\le \frac{1}{r}\int_0^r\int_s^r \|\partial_r h\|_{\mathcal{C}^0_{U,R}}\,d\rho\,ds = \frac{r}{2}\,\|\partial_r h\|_{\mathcal{C}^0_{U,R}}\,. \end{aligned} \tag{7.4}$$

Thus

$$\frac{1}{2}\int_0^R \frac{\left(h-\bar{h}\right)^2}{r}dr \le \frac{1}{16}\|\partial_r h\|^2_{\mathcal{C}^0_{U,R}} R^2\,,$$

and by (5.12) we get

$$g(u,0) = 1 \le g(u,r) \le K := \exp\left(\frac{1}{16}\|\partial_r h\|^2_{\mathcal{C}^0_{U,R}} R^2\right). \tag{7.5}$$

7.2.1 The characteristics of the problem

The integral curves of D, which are the incoming light rays, are the characteristics of the problem. These satisfy the ordinary differential equation,

$$\frac{dr}{du} = -\frac{1}{2}\tilde{g}(u,r). \tag{7.6}$$

To simplify the notation we shall denote simply by $u \mapsto r(u)$, the solution to (7.6), satisfying $r(u_1) = r_1$. However it should be always kept in mind that $r(u) = r(u; u_1, r_1)$.

Using (7.5) we can estimate $\tilde{g}$, given by (5.10), and consequently the solutions to the characteristic equation (7.6): In fact, for $r \le \frac{1}{\sqrt{\Lambda}} \Rightarrow 1 - \Lambda r^2 \ge 0$ we get

$$\tilde{g} \ge \frac{1}{r}\int_0^r (1-\Lambda s^2)ds = 1 - \frac{\Lambda}{3}r^2 \ge 1 - K\frac{\Lambda}{3}r^2\,.$$

For $r \ge \frac{1}{\sqrt{\Lambda}}$ we have

$$\begin{aligned} \tilde{g}(u,r) &\ge \frac{1}{r}\int_0^{\frac{1}{\sqrt{\Lambda}}} \left(1-\Lambda s^2\right) ds + \frac{K}{r}\int_{\frac{1}{\sqrt{\Lambda}}}^r \left(1-\Lambda s^2\right) ds \\ &= \frac{2}{3\sqrt{\Lambda}r}(1-K) + K\left(1 - \frac{\Lambda}{3}r^2\right) \\ &\ge \frac{2}{3}(1-K) + K\left(1 - \frac{\Lambda}{3}r^2\right)\,. \end{aligned}$$

We then see that the following estimate holds for all $r \geq 0$:

$$\tilde{g} \geq 1 - \frac{K\Lambda}{3} r^2 \,. \tag{7.7}$$

The same kind of reasoning also provides the upper bound

$$\tilde{g} \leq K - \frac{\Lambda}{3} r^2 \,. \tag{7.8}$$

From (7.6) and (7.7) we now obtain the following differential inequality

$$\frac{dr}{du} \leq -\frac{1}{2} + \frac{\Lambda K}{6} r^2 \,. \tag{7.9}$$

Denoting

$$\alpha = \frac{1}{2}\sqrt{\frac{\Lambda K}{3}} \quad \text{and} \quad r_c^- = \sqrt{\frac{3}{\Lambda K}} \,,$$

where r_c^- is the positive root of the polynomial in (7.7), the solution $r^-(u)$ of the differential equation obtained from (7.9) (by replacing the inequality with an equality) satisfying $r^-(u_1) = r_1 < r_c^-$ is given by

$$r^-(u) = \frac{1}{2\alpha} \tanh\left\{\alpha(c^- - u)\right\} \,,$$

for some $c^- = c^-(u_1, r_1)$; by a basic comparison principle it then follows that whenever $r(u_1) = r_1 < r_c^-$ we have

$$r(u) \geq \frac{1}{2\alpha} \tanh\left\{\alpha(c^- - u)\right\} \quad , \quad \forall u \leq u_1 \,. \tag{7.10}$$

Denote the positive root of the polynomial in (7.8) by

$$r_c^+ = \sqrt{\frac{3K}{\Lambda}} \,,$$

then, for appropriate choices (differing in each case) of $c^- = c^-(u_1, r_1)$ and $c^+ = c^+(u_1, r_1)$, similar reasonings based on comparison principles give the following global estimates for the characteristics (see also Fig. 7.1):

- **Local region** ($r_1 < r_c^-$):

$$\frac{1}{2\alpha} \tanh\left\{\alpha(c^- - u)\right\} \leq r(u) \leq \frac{K}{2\alpha} \tanh\left\{\alpha(c^+ - u)\right\} \quad , \quad \forall u \leq u_1 \,. \tag{7.11}$$

- **Intermediate region** ($r_c^- \leq r_1 < r_c^+$):

$$\frac{1}{2\alpha} \coth\left\{\alpha(c^- - u)\right\} \leq r(u) \leq \frac{K}{2\alpha} \tanh\left\{\alpha(c^+ - u)\right\} \quad , \quad \forall u \leq u_1 \,. \tag{7.12}$$

- **Cosmological region** ($r \geq r_c^+$):

$$\frac{1}{2\alpha} \coth\left\{\alpha(c^- - u)\right\} \leq r(u) \leq \frac{K}{2\alpha} \coth\left\{\alpha(c^+ - u)\right\} \quad , \quad \forall u \leq u_1 \,. \tag{7.13}$$

In particular, for $r(u_1) = r_1 \geq r_c^-$ we obtain

$$r(u) \geq r_c^- > 0 \quad , \quad \forall u \leq u_1. \tag{7.14}$$

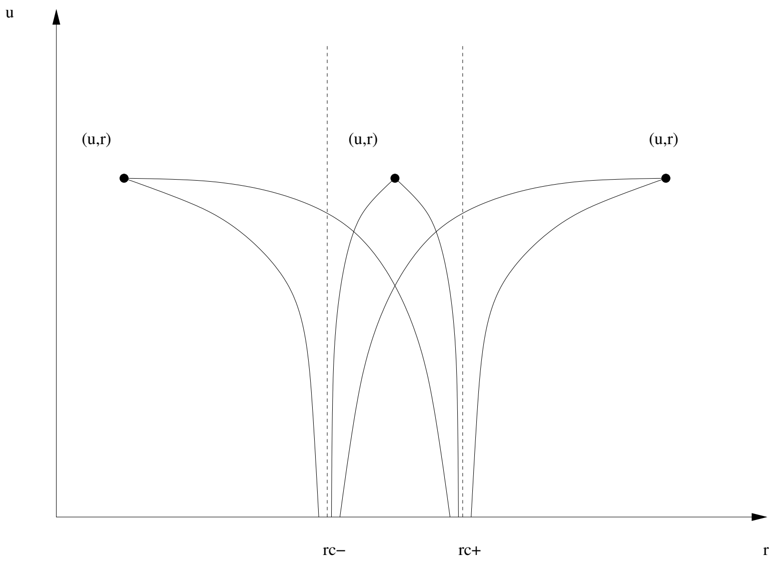

Figure 7.1: Bounds for the characteristics through the point (u_1, r_1) in the local $(r_1 < r_c^-)$, intermediate $(r_c^- \leq r_1 < r_c^+)$ and cosmological $(r_1 \geq r_c^+)$ regions.

7.2.2 Main Lemma

The purpose of this section is to prove the following lemma:

Lemma 6. *Let $\Lambda > 0$ and $R > 0$.*

There exists $x^* = x^*(\Lambda, R) > 0$ and constants $C_i = C_i(x^*, \Lambda, R) > 0$, such that if $\|h\|_{X_{U,R}} \leq x^*$, then[7]

$$G < -C_1 r \ , \ \ C_1 = \frac{\Lambda}{3} + O(x^*) \ , \tag{7.15}$$

$$|G| < C_2 r \ , \tag{7.16}$$

$$|J| < C_3 r \ , \ \ C_3 = O(x^*), \tag{7.17}$$

and, for any $u_1 \geq 0$ and $r_1 \leq R$,

$$\int_0^{u_1} \exp\left(\int_u^{u_1} 2G(v, r(v))dv\right) du \leq C_4 \, , \tag{7.18}$$

where $r(u) = r(u; u_1, r_1)$ is the characteristic through (u_1, r_1).

Remark 11. *We stress the fact that while allowed to depend on R the constants do not depend on any parameter associated with the u-coordinate.*

Proof. We have, from (7.5),

$$1 \leq g \leq K^* := \exp\left(\frac{(x^*)^2}{16} R^2\right) \ . \tag{7.19}$$

Differentiating (5.12) while using (7.4) and (7.19) leads to[8]

[7] As usual, $O(x^*)$ means a bounded function of x^* times x^* in some neighborhood of $x^* = 0$.

[8] From now on we will use the notation $f \lesssim g$ meaning that $f \leq Cg$, for $C \geq 0$ only allowed to depend on the fixed parameters Λ and R.

$$0 \leq \frac{\partial g}{\partial r} \lesssim g \frac{\left(h-\bar{h}\right)^2}{r} \lesssim K^*(x^*)^2 r, \tag{7.20}$$

and consequently

$$\begin{aligned} 0 \leq (g-\bar{g})(u,r) &= \frac{1}{r}\int_0^r (g(u,r)-g(u,s))\, ds \\ &= \frac{1}{r}\int_0^r \left\{\int_s^r \frac{\partial g}{\partial \rho}(u,\rho) d\rho\right\} ds \\ &\lesssim \frac{1}{r}\int_0^r \int_s^r K^*(x^*)^2 \rho \, d\rho \, ds \\ &\lesssim K^*(x^*)^2 r^2. \end{aligned}$$

From this estimate, (5.16) and (7.19) we see that

$$\left(\frac{\Lambda}{6} - \frac{\Lambda}{2}K^*\right) r \leq G \leq \left[K^*\left(C(x^*)^2 + \frac{\Lambda}{6}\right) - \frac{\Lambda}{2}\right] r \tag{7.21}$$

for some constant $C > 0$ depending only on Λ and R. Since $K^* \to 1$ as $x^* \to 0$, (7.15) then follows by choosing x^* appropriately small. Also, inequality (7.16) is immediate. From (5.19), (7.20) and (7.21) we now obtain (7.17).

To prove (7.18) we start by using (7.15) to obtain

$$\int_0^{u_1} e^{\int_u^{u_1} 2G(v,r(v))dv} du \leq \int_0^{u_1} e^{-2C_1 \int_u^{u_1} r(v)dv} du \; .$$

If $r_1 < r_c^- = \sqrt{\frac{3}{\Lambda K}}$ then (7.10) holds and we then have

$$\begin{aligned} -2C_1 \int_u^{u_1} r(v) dv &\leq -\frac{C_1}{\alpha}\int_u^{u_1} \tanh\left(\alpha(c^- - v)\right) dv \\ &= \frac{C_1}{\alpha^2} \ln\left(\frac{\cosh\left(\alpha(c^- - u_1)\right)}{\cosh\left(\alpha(c^- - u)\right)}\right). \end{aligned}$$

Since

$$\frac{\cosh\left(\alpha(c^- - u_1)\right)}{\cosh\left(\alpha(c^- - u)\right)} \leq 2e^{\alpha(u-u_1)}$$

and

$$\frac{1}{2}\sqrt{\frac{\Lambda}{3}} \leq \alpha = \frac{1}{2}\sqrt{\frac{\Lambda K}{3}} \lesssim \sqrt{K^*} \; ,$$

we obtain

$$\begin{aligned} \int_0^{u_1} e^{-2C_1 \int_u^{u_1} r(v)dv} du &\leq 2^{C_1/\alpha^2} \int_0^{u_1} e^{\frac{C_1}{\alpha}(u-u_1)} du \\ &\leq 2^{C_1/\alpha^2} \frac{\alpha}{C_1}\left[1 - e^{-\frac{C_1}{\alpha}u_1}\right] \leq 2^{C_1/\alpha^2}\frac{\alpha}{C_1} \leq C_4(x^*, \Lambda, R) \; , \end{aligned} \tag{7.22}$$

as desired. If $r_1 \geq r_c^-$, we have (7.14) which gives

$$\int_0^{u_1} e^{-2C_1 \int_u^{u_1} r(v)dv} du \leq \int_0^{u_1} e^{-2C_1 r_c^-(u_1-u)} du \leq \frac{1}{2C_1 r_c^-}\left[1 - e^{-2C_1 r_c^- u_1}\right] \leq \frac{2\alpha}{C_1} \leq C_4(x^*, \Lambda, R) \; ,$$

which completes the proof of the lemma. □

7.3 Controlled local existence

Local existence will be proven by constructing a contracting sequence of solutions to related linear problems. Given a sequence $\{h_n\}$ we will write $g_n := g(h_n)$, $G_n := G(h_n)$, etc, for the quantities (5.12), (5.14), etc, obtained from h_n; for a given h_n the corresponding differential operator will be denoted by

$$D_n = \partial_u - \frac{\tilde{g}_n}{2}\partial_r \ ,$$

and the associated characteristic through (u_1, r_1) by $\chi_n = \chi_n(u) = (u, r_n(u; u_1, r_1))$; as before, we will drop the explicit dependence on initial conditions when confusion is unlikely to arise. With these notational issues settled we are ready to prove the following fundamental result:

Lemma 7. *Let $\Lambda > 0$, $R > \sqrt{\frac{3}{\Lambda}}$ and $h_0 \in \mathcal{C}^1([0,R])$. There exists $x^* = x^*(\Lambda, R) > 0$ and $C^* = C^*(x^*, \Lambda, R) > 0$ such that if*

$$\|h_0\|_{X_R} \leq \frac{x^*}{1+C^*} \ ,$$

then the sequence $\{h_n\}_{n\in\mathbb{N}_0}$ defined by $h_0(u,r) = h_0(r)$ and

$$\begin{cases} D_n h_{n+1} - G_n h_{n+1} = -G_n \bar{h}_n \\ h_{n+1}(0,r) = h_0(r) \ , \end{cases}$$

is in $C^1([0,R]\times[0,U])$ and satisfies

$$G_n \leq 0 \ , \tag{7.23}$$

$$\|h_n\|_{C^0_{U,R}} = \|h_0\|_{C^0_R} \ , \tag{7.24}$$

$$\|h_n\|_{X_{U,R}} \leq (1+C^*)\|h_0\|_{X_R} \ , \tag{7.25}$$

for all $n \in \mathbb{N}_0$ and all $U \geq 0$.

Remark 12. *We stress the fact that C^* does not depend on either U or n.*

Proof. The proof is by induction. That the conclusions follow for the 0^{th} term is immediate, with (7.23) obtained from Lemma 6 by setting x^* accordingly small. Assume that h_n satisfies all the conclusions of the lemma.

In particular, since we have $h_n \in C^1([0,U]\times[0,R])$ we see, from the respective definitions, that $\bar{h}_n$, g_n and $\tilde{g}_n$ are C^1 for $r \neq 0$; regularity at the origin then follows by inserting the first order Taylor expansion in r of h_n, centered at $r = 0$, in the definitions of $\bar{h}_n$, then g_n and finally $\tilde{g}_n$. Later in the proof we will also need $\partial_r G_n$ to be well defined and continuous in the domain under consideration; this follows by using the previous referred expansions in equations (5.18) and (5.19).

Note that, as a consequence of the regularity for $\tilde{g}_n$, we also obtain well posedness and differentiability with respect to the initial datum r_1 for the characteristics given by (7.6); in particular we are allowed to integrate the linear equation for h_{n+1} along such characteristics to obtain

$$h_{n+1}(u_1, r_1) = h_0(r_n(0))e^{\int_0^{u_1} G_{n|\chi_n} dv} - \int_0^{u_1} \left(G_n\bar{h}_n\right)_{|\chi_n} e^{\int_u^{u_1} G_{n|\chi_n} dv} du \ . \tag{7.26}$$

This defines a function $h_{n+1} : \mathcal{R}_{n+1} \subset [0,U]\times[0,R] \to \mathbb{R}$ where

$$\mathcal{R}_{n+1} = \{(u,r) \mid \chi_n(u) = (u, r_n(u)) = (u,r) \text{ and } r_n(0) \in [0,R]\} \ .$$

Since the problem for the characteristics is well posed, there is a characteristic through every $(u,r) \in [0,U] \times [0,R]$; in particular $\mathcal{R}_{n+1}$ is non empty, but nonetheless, integrating backwards in u, the characteristics may leave the fixed rectangle before reaching $u=0$, which in turn would lead to $\mathcal{R}_{n+1} \neq [0,U]\times[0,R]$. We may rule out this undesirable possibility by a choice of appropriately small x^*; in fact, it suffices to guarantee that the r_n component of all characteristics with sufficiently large initial datum r_1 are nondecreasing in u: given $R > \sqrt{\frac{3}{\Lambda}}$, since (7.25) and the smallness condition on the initial data imply

$$\|h_n\|_{X_{U,R}} \leq (1+C^*)\|h_0\|_{X_R} \leq x^* ,$$

we see that (recall (7.5))

$$K_n \leq K^* = e^{C(x^*)^2R^2} ,$$

and from the global characterization (7.11)-(7.13) the desired monotonicity property follows if

$$r^+_{c,n} = \sqrt{\frac{K_n\Lambda}{3}} < R ,$$

which can be arranged by choosing x^* sufficiently small (see also Figure 7.1).

We have already showed that $\bar{h}_n$, χ_n and G_n have continuous partial derivatives with respect to r; we are then allowed to differentiate (7.26) with respect to r_1 and, since $D_n h_{n+1}$ is clearly continuous, we conclude that

$$h_{n+1} \in C^1([0,U]\times[0,R]) .$$

From the previous discussion $\partial_r D h_{n+1}$ is continuous, so differentiating equation (7.26) with respect to r, and using the fact that[9]

$$[D_n, \partial_r] = G_n \partial_r$$

we obtain the following differential equation for $\partial_r h_{n+1}$ (recall (5.18)):

$$\begin{aligned} D_n(\partial_r h_{n+1}) - 2G_n\partial_r h_{n+1} &= \partial_r G_n (h_{n+1} - \bar{h}_n) - G_n \partial_r \bar{h}_n \\ &= -J_n \frac{\partial \bar{h}_n}{\partial r} - (J_n - G_n)\frac{(h_{n+1}-h_n)}{r} . \end{aligned}$$

Using the initial conditions

$$\partial_r h_{n+1}(0,r) = \partial_r h_0(r)$$

and integrating along the characteristics leads to

$$\begin{aligned} \partial_r h_{n+1}(u_1,r_1) =& \partial_r h_0(\chi_n(0))\, e^{\int_0^{u_1} 2G_{n|\chi_n} dv} \\ & - \int_0^{u_1} \left[J_n \partial_r \bar{h}_n + (J_n - G_n)\frac{(h_{n+1}-h_n)}{r}\right]_{|\chi_n} e^{\int_u^{u_1} 2G_{n|\chi_n} dv} du . \end{aligned} \tag{7.27}$$

By the induction hypothesis we have $G_n \leq 0$ and $\|\bar{h}_n\|_{\mathcal{C}^0_{U,R}} \leq \|h_n\|_{\mathcal{C}^0_{U,R}} \leq \|h_0\|_{\mathcal{C}^0_R}$; therefore

$$\begin{aligned} |h_{n+1}(u_1,r_1)| &\leq \|h_0\|_{\mathcal{C}^0_R}\; e^{\int_0^{u_1} G_n|_{\chi_n} dv} + \|\bar{h}_n\|_{\mathcal{C}^0_{U,R}} \int_0^{u_1} -G_n|_{\chi_n} e^{\int_u^{u_1} G_n|_{\chi_n} dv} du \\ &\leq \|h_0\|_{\mathcal{C}^0_R} \underbrace{\left(e^{\int_0^{u_1} G_n|_{\chi_n} dv} - \int_0^{u_1} G_n|_{\chi_n} e^{\int_u^{u_1} G_n|_{\chi_n} dv} dv \right)}_{\equiv 1} = \|h_0\|_{\mathcal{C}^0_R} . \end{aligned}$$

[9]Here we are using the following generalized version of the Schwarz Lemma: if X and Y are two nonvanishing $\mathcal{C}^1$ vector fields in $\mathbb{R}^2$ and f is a $\mathcal{C}^1$ function such that $X\cdot(Y\cdot f)$ exists and is continuous then $Y\cdot(X\cdot f)$ also exists and is equal to $X\cdot(Y\cdot f) - [X,Y]\cdot f$.

Then

$$|h_{n+1} - h_n| \leq 2\|h_0\|_{C^0_R} \quad \text{and} \quad |\partial_r \bar{h}_n| = \frac{|h_n - \bar{h}_n|}{r} \leq \frac{2\|h_0\|_{C^0_R}}{r} \,, \tag{7.28}$$

so that, relying once more on Lemma 6,

$$\begin{aligned}
|\partial_r h_{n+1}(u_1, r_1)| &\leq \|\partial_r h_0\|_{\mathcal{C}^0_R} e^{\int_0^{u_1} 2G_n|_{\chi_n} dv} \\
&\quad + 2(C_2 + 2C_3)\|h_0\|_{C^0_R} \int_0^{u_1} e^{\int_u^{u_1} 2G_n|_{\chi_n} dv} du \\
&\leq \|\partial_r h_0\|_{\mathcal{C}^0_R} + 2(C_2 + 2C_3)C_4\|h_0\|_{C^0_R} \,.
\end{aligned}$$

Setting $C^* := 2(C_2 + 2C_3)C_4$ it now follows that

$$\begin{aligned}
\|h_{n+1}\|_{X_{U,R}} &= \|h_{n+1}\|_{\mathcal{C}^0_{U,R}} + \|\partial_r h_{n+1}\|_{\mathcal{C}^0_{U,R}} \\
&\leq (1 + C^*)\|h_0\|_{C^0_R} + \|\partial_r h_0\|_{C^0_R} \\
&\leq (1 + C^*)\|h_0\|_{X_R} \,.
\end{aligned}$$

Thus, if $\|h_0\|_{X_R} \leq \frac{x^*}{1+C^*}$, then

$$\|h_{n+1}\|_{X_{U,R}} \leq x^*$$

and by Lemma 6

$$G_{n+1} \leq 0 \,,$$

which completes the proof. ☐

Lemma 7 will now allow us to establish a local existence theorem for small data, while controlling the previously defined supremum norms of the solutions in terms of initial data.

Theorem 6. *Let $\Lambda > 0$, $R > \sqrt{\frac{3}{\Lambda}}$ and $h_0 \in \mathcal{C}^k([0, R])$ for $k \geq 1$. There exists $x^* = x^*(\Lambda, R) > 0$ and $C^*(x^*, \Lambda, R) > 0$, such that, if $\|h_0\|_{X_R} \leq \frac{x^*}{1+C^*}$, then the initial value problem*

$$\begin{cases} Dh = G\left(h - \bar{h}\right) \\ h(0, r) = h_0(r) \end{cases} \tag{7.29}$$

has a unique solution $h \in \mathcal{C}^k([0, U] \times [0, R])$, for $U = U(x^*/(1 + C^*); R, \Lambda)$ sufficiently small. Moreover,

$$\|h\|_{\mathcal{C}^0_{U,R}} = \|h_0\|_{\mathcal{C}^0_R} \tag{7.30}$$

and

$$\|h\|_{X_{U,R}} \leq (1 + C^*) \, \|h_0\|_{X_R} \,. \tag{7.31}$$

Proof. Fix x^* as in Lemma 6 and consider a sequence $\{h_n\}$ as defined in Lemma 7, with $U < 1$. From (7.4) and Lemma 7 we have

$$\begin{aligned}
\left|(h_n - \bar{h}_n) + (h_{n-1} - \bar{h}_{n-1})\right| &\leq \frac{r}{2}\left(\|\partial_r h_n\|_{\mathcal{C}^0_{U,R}} + \|\partial_r h_{n-1}\|_{\mathcal{C}^0_{U,R}}\right) \\
&\leq (1 + C^*) \, r \, \|h_0\|_{X_R} \leq x^* r \,,
\end{aligned}$$

and

$$\left|(h_n - \bar{h}_n) - (h_{n-1} - \bar{h}_{n-1})\right| = \left|(h_n - h_{n-1}) - \overline{(h_n - h_{n-1})}\right| \leq 2 \, \|h_n - h_{n-1}\|_{\mathcal{C}^0_{U,R}}$$

so that

$$\begin{aligned}\left|(h_n-\bar{h}_n)^2-(h_{n-1}-\bar{h}_{n-1})^2\right| &\le \left|(h_n-\bar{h}_n)+(h_{n-1}-\bar{h}_{n-1})\right|\,\left|(h_n-\bar{h}_n)-(h_{n-1}-\bar{h}_{n-1})\right| \\ &\le 2x^*\,r\,\|h_n-h_{n-1}\|_{\mathcal{C}^0_{U,R}} \\ &= C\,r\,\|h_n-h_{n-1}\|_{\mathcal{C}^0_{U,R}}\end{aligned} \tag{7.32}$$

(we will, until the end of this proof, allow the constants to depend on x^*, besides the fixed parameters Λ and R). The mean value theorem yields the following elementary inequality

$$|e^x-e^y|\le\max\{e^x,e^y\}|x-y|\;, \tag{7.33}$$

from which (recall (7.19))

$$\begin{aligned}|g_n-g_{n-1}| &= \left|\exp\left(C\int_0^r\frac{(h_n-\bar{h}_n)^2}{s}\right)-\exp\left(C\int_0^r\frac{(h_{n-1}-\bar{h}_{n-1})^2}{s}\right)\right| \\ &\lesssim K^*\int_0^r\frac{\left|(h_n-\bar{h}_n)^2-(h_{n-1}-\bar{h}_{n-1})^2\right|}{s}ds \\ &\le C\,r\;\|h_n-h_{n-1}\|_{\mathcal{C}^0_{U,R}}\;.\end{aligned} \tag{7.34}$$

Then

$$\begin{aligned}|\tilde{g}_n-\tilde{g}_{n-1}| &= \left|\frac{1}{r}\int_0^r(g_n-g_{n-1})(1-\Lambda s^2)ds\right| \\ &\le C\,r\;\|h_n-h_{n-1}\|_{\mathcal{C}^0_{U,R}}\;,\end{aligned} \tag{7.35}$$

and using (5.15),

$$\begin{aligned}|G_n-G_{n-1}| &= \frac{1}{2r}\left|(g_n-g_{n-1})(1-\Lambda r^2)-\frac{1}{r}\int_0^r(g_n-g_{n-1})(1-\Lambda s^2)ds\right| \\ &\le C\;\|h_n-h_{n-1}\|_{\mathcal{C}^0_{U,R}}\;.\end{aligned}$$

Note that, since $r\le R$, the r factors in the previous estimates may be absorbed by the corresponding constants. Until now we have been estimating the difference between consecutive terms of sequences with both terms evaluated at the same point (u,r), but we will also need to estimate differences between consecutive terms evaluated at the corresponding characteristics; more precisely, for a given sequence f_n we will estimate

$$\begin{aligned}|f_{n|\chi_n}-f_{n-1|\chi_{n-1}}| &= |f_n(u,r_n(u))-f_{n-1}(u,r_{n-1}(u))| \\ &\le |f_n(u,r_n(u))-f_n(u,r_{n-1}(u))|+|f_n(u,r_{n-1}(u))-f_{n-1}(u,r_{n-1}(u))|\;.\end{aligned}$$

If for the second term we have, as before, a uniform estimate of the form $C\|h_n-h_{n-1}\|_{\mathcal{C}^0_{U,R}}$, and for the first one of the form $C|r_n-r_{n-1}|$, then, by (7.37) below, we will obtain, since $u_1\le U<1$,

$$|f_{n|\chi_n}-f_{n-1|\chi_{n-1}}|\le C\|f_n-f_{n-1}\|_{\mathcal{C}^0_{U,R}}\;. \tag{7.36}$$

Also, if $\|\partial_r f_n\|_{\mathcal{C}^0_{U,R}}\le C$ then the desired

$$|f_n(u,r_2)-f_n(u,r_1)|\le\left|\int_{r_1}^{r_2}\partial_r f_n(r)dr\right|\le C|r_2-r_1|\;,$$

follows immediately. We have (see (7.4))

$$|\partial_r\bar{h}_n|=\left|\frac{h_n-\bar{h}_n}{r}\right|\le C\;,$$

and from(7.20)

$$|\partial_r g_n| \leq Cr \ .$$

By Lemma 6 we have $\|G_n\|_{\mathcal{C}^0_{U,R}} \leq C$, which in view of (5.14) is equivalent to $\|\partial_r \tilde{g}_n\|_{\mathcal{C}^0_{U,R}} \leq C$; since (5.18), (5.19) and (7.16) together with the above bounds yield $\|\partial_r G_n\|_{\mathcal{C}^0_{U,R}} \leq C$, the desired estimates, of the form (7.36), follow for the sequences h_n, $\bar{h}_n$, g_n, $\tilde{g}_n$ and G_n once we have proved (7.37). To do this, start from equation (7.6) for the characteristics to obtain

$$r_n(u) = r_n(u_1) + \frac{1}{2}\int_u^{u_1} \tilde{g}_n(s, r_n(s))ds \ ,$$

so that the difference between two consecutive characteristics through (u_1, r_1) satisfies

$$\begin{aligned} r_n(u) - r_{n-1}(u) &= \frac{1}{2}\int_u^{u_1} \{\tilde{g}_n(s, r_n(s)) - g_{n-1}(s, r_{n-1}(s))\}\, ds \\ &= \frac{1}{2}\int_u^{u_1} \{\tilde{g}_n(s, r_n(s)) - \tilde{g}_n(s, r_{n-1}(s))\}\, ds + \frac{1}{2}\int_u^{u_1} \{\tilde{g}_n(s, r_{n-1}(s)) - \tilde{g}_{n-1}(s, r_{n-1}(s))\}\, ds \ . \end{aligned}$$

From the previously obtained bounds $\|\partial_r \tilde{g}_n\|_{\mathcal{C}^0_{U,R}} \leq C$ and (7.35), we then have

$$|r_n(u) - r_{n-1}(u)| \leq C\int_u^{u_1} |r_n(s) - r_{n-1}(s)|ds + C'(u_1 - u)\|h_n - h_{n-1}\|_{\mathcal{C}^0_{U,R}} \ ,$$

from which[10]

$$|r_n(u) - r_{n-1}(u)| \leq \frac{C'}{C}\|h_n - h_{n-1}\|_{\mathcal{C}^0_{U,R}}\left(e^{C(u_1-u)} - 1\right) \ , \qquad (7.37)$$

as desired. Now, from (7.26) and the elementary identity

$$a_2b_2c_2 - a_1b_1c_1 = (a_2 - a_1)b_2c_2 + (b_2 - b_1)a_1c_2 + (c_2 - c_1)a_1b_1$$

we get

$$\begin{aligned} |(h_{n+1} - h_n)(u_1, r_1)| \leq \|h_0\|_{\mathcal{C}^0_R} & \underbrace{\left|\exp\left(\int_0^{u_1} G_{n|\chi_n} dv\right) - \exp\left(\int_0^{u_1} G_{n-1|\chi_{n-1}} dv\right)\right|}_{I} \\ & + \underbrace{\int_0^{u_1} \left|G_{n|\chi_n} - G_{n-1|\chi_{n-1}}\right| \left|\bar{h}_{n|\chi_n}\right| \exp\left(\int_u^{u_1} G_{n|\chi_n} dv\right) du}_{II} \\ & + \underbrace{\int_0^{u_1} \left|\bar{h}_{n|\chi_n} - \bar{h}_{n-1|\chi_{n-1}}\right| \left|G_{n-1|\chi_{n-1}}\right| \exp\left(\int_u^{u_1} G_{n|\chi_n} dv\right) du}_{III} \\ & + \underbrace{\int_0^{u_1} \left|\exp\left(\int_u^{u_1} G_{n|\chi_n} dv\right) - \exp\left(\int_u^{u_1} G_{n-1|\chi_{n-1}} dv\right)\right| \left|G_{n-1|\chi_{n-1}}\bar{h}_{n-1|\chi_{n-1}}\right| du}_{IV} \ . \end{aligned}$$

Using (7.23), (7.33) and (7.36), which holds for the sequence G_n as discussed earlier, gives

$$I \leq \left|\left(\int_0^{u_1} G_{n|\chi_n} dv\right) - \left(\int_0^{u_1} G_{n-1|\chi_{n-1}} dv\right)\right| \leq Cu_1 \|h_n - h_{n-1}\|_{\mathcal{C}^0_{U,R}} \ ,$$

and, in view also of (7.24),

$$II \leq \int_0^{u_1} C\|h_0\|_{\mathcal{C}^0_R} \|h_n - h_{n-1}\|_{\mathcal{C}^0_{U,R}} du \leq Cu_1 \|h_n - h_{n-1}\|_{\mathcal{C}^0_{U,R}} \ .$$

[10] Here we used the following comparison principle: if $y, z \in \mathcal{C}^0([t_0, t_1])$ satisfy $y(t) \leq f(t) + C\int_t^{t_1} y(s)ds$ and $z(t) = f(t) + C\int_t^{t_1} z(s)ds$, then $y(t) \leq z(t)$, $\forall t \in [t_0, t_1]$.

In a similar way (recall that (7.36) also holds for the sequence $\bar{h}_n$)

$$III \leq Cu_1 \|h_n - h_{n-1}\|_{\mathcal{C}^0_{U,R}} ,$$

and, using the bound for I,

$$IV \leq Cu_1^2 \|h_n - h_{n-1}\|_{\mathcal{C}^0_{U,R}} .$$

Putting all the pieces together yields (recall that we have imposed the restriction $u_1 \leq U < 1$)

$$\|h_{n+1} - h_n\|_{\mathcal{C}^0_{U,R}} \leq C\,U \|h_n - h_{n-1}\|_{\mathcal{C}^0_{U,R}}. \tag{7.38}$$

Now, applying the same strategy to (7.27) leads to

$$\begin{aligned}
|(\partial_r h_{n+1} - \partial_r h_n)(u_1, r_1)| \leq \|\partial_r h_0\|_{\mathcal{C}^0_R} & \underbrace{\left|\exp\left(\int_0^{u_1} 2G_{n|\chi_n} dv\right) - \exp\left(\int_0^{u_1} 2G_{n-1|\chi_{n-1}} dv\right)\right|}_{(i)} \\
& + \underbrace{\int_0^{u_1} \left|J_{n|\chi_n} - J_{n-1|\chi_{n-1}}\right| \left|\partial_r \bar{h}_{n|\chi_n}\right| \exp\left(\int_u^{u_1} 2G_{n|\chi_n} dv\right) du}_{(ii)} \\
& + \underbrace{\int_0^{u_1} \left|\partial_r \bar{h}_{n|\chi_n} - \partial_r \bar{h}_{n-1|\chi_{n-1}}\right| \left|J_{n-1|\chi_{n-1}}\right| \exp\left(\int_u^{u_1} 2G_{n|\chi_n} dv\right) du}_{(iii)} \\
& + \underbrace{\int_0^{u_1} \left|\exp\left(\int_u^{u_1} 2G_{n|\chi_n} dv\right) - \exp\left(\int_u^{u_1} 2G_{n-1|\chi_{n-1}} dv\right)\right| \left|J_{n-1|\chi_{n-1}} \partial_r \bar{h}_{n-1|\chi_{n-1}}\right| du}_{(iv)} \\
& + \underbrace{\int_0^{u_1} \left[|J_n - G_n| \frac{|h_{n+1} - h_n|}{r}\right]_{|\chi_n} e^{\int_u^{u_1} 2G_{n|\chi_n} dv} du}_{(v)} \\
& + \underbrace{\int_0^{u_1} \left[|J_{n-1} - G_{n-1}| \frac{|h_n - h_{n-1}|}{r}\right]_{|\chi_{n-1}} e^{\int_u^{u_1} 2G_{n-1|\chi_{n-1}} dv} du}_{(vi)} .
\end{aligned}$$

We have

$$\begin{aligned}
|\partial_r g_n - \partial_r g_{n-1}| &\lesssim \left| g_n \frac{(h_n - \bar{h}_n)^2}{r} - g_{n-1} \frac{(h_{n-1} - \bar{h}_{n-1})^2}{r} \right| \\
&\lesssim |g_n| \frac{\left|(h_n - \bar{h}_n)^2 - (h_{n-1} - \bar{h}_{n-1})^2\right|}{r} + |g_n - g_{n-1}| \frac{(h_{n-1} - \bar{h}_{n-1})^2}{r} \\
&\lesssim \|h_n - h_{n-1}\|_{\mathcal{C}^0_{U,R}} ,
\end{aligned}$$

where we have used (7.19), (7.32) and (7.34). Similarly

$$\begin{aligned}
|\partial_r g_n(u, r_2) - \partial_r g_n(u, r_1)| &\lesssim |g_n(u, r_2)| \left| \frac{(h_n - \bar{h}_n)^2(u, r_2)}{r_2} - \frac{(h_n - \bar{h}_n)^2(u, r_1)}{r_1} \right| + \\
&\quad + |g_n(u, r_2) - g_n(u, r_1)| \left| \frac{(h_{n-1} - \bar{h}_{n-1})^2(u, r_1)}{r_1} \right| \\
&\lesssim |r_2 - r_1| .
\end{aligned}$$

We conclude that (7.36) holds for the sequence $\partial_r g_n$ and since it also holds for the sequences g_n and G_n we obtain from (5.19)

$$\left|J_{n|\chi_n} - J_{n-1|\chi_{n-1}}\right| \leq C \|h_n - h_{n-1}\|_{\mathcal{C}^0_{U,R}} .$$

As an immediate consequence one obtains for $(i) - (iv)$ estimates similar to the ones derived for $I - IV$ (recall (5.9), (7.4) and (7.17)). Using (7.16) and (7.17) we also have

$$(vi) \leq CU \|h_n - h_{n-1}\|_{\mathcal{C}^0_{U,R}} ,$$

and (7.38) provides

$$(v) \leq CU^2 \|h_n - h_{n-1}\|_{\mathcal{C}^0_{U,R}} .$$

We finally obtain

$$\begin{aligned} \|h_{n+1} - h_n\|_{X_{U,R}} &= \|h_{n+1} - h_n\|_{\mathcal{C}^0_{U,R}} + \|\partial_r h_{n+1} - \partial_r h_n\|_{\mathcal{C}^0_{U,R}} \\ &\leq CU \|h_n - h_{n-1}\|_{\mathcal{C}^0_{U,R}} \\ &\leq CU \|h_n - h_{n-1}\|_{X_{U,R}} . \end{aligned}$$

So, for U sufficiently small, $\{h_n\}$ contracts, and consequently converges, with respect to $\|\cdot\|_{X_{U,R}}$. The previous estimates show that the convergence of h_n lead to the uniform convergence of all the sequences appearing in (7.26) and (7.27). Taking the limit of (7.26) leads to

$$h(u_1, r_1) = h_0(\chi(0)) e^{\int_0^{u_1} G|_\chi dv} - \int_0^{u_1} \left(G\bar{h}\right)|_\chi e^{\int_u^{u_1} G|_\chi dv} du , \tag{7.39}$$

where we denote the limiting functions by removing the indices. Equation (7.39) shows that h is a continuous solution to (7.41), the limit of (7.27) shows that $\partial_r h$ solves (5.17) and is continuous, and we see that $h \in \mathcal{C}^1$, since Dh is also clearly continuous.

Now let $1 \leq m < k$ be an integer, and assume that $h \in \mathcal{C}^m$. As in the proof of Lemma 7, but using the Taylor expansion of order m, we can show that $\bar{h}$, $\tilde{g}$ (which controls the characteristics), G and $\partial_r G$ are also $\mathcal{C}^m$, from which it follows that $\partial_r(G\bar{h})$ is $\mathcal{C}^m$. Taking the partial derivatives of (7.39) as in last chapter (using the assumed regularity of the initial data) we then see that actually $h \in \mathcal{C}^{m+1}$, and so $h \in \mathcal{C}^k$.

To establish uniqueness consider two solutions of (7.41) and derive the following evolution equation for their difference:

$$D_1(h_2 - h_1) - G_1(h_2 - h_1) = \frac{1}{2}\left(\tilde{g}_2 - \tilde{g}_1\right)\partial_r h_2 + \left(G_2 - G_1\right)\left(h_2 - \bar{h}_2\right) - G_1\left(\bar{h}_2 - \bar{h}_1\right) . \tag{7.40}$$

Integrating it along the characteristics associated to h_1 yields

$$|(h_2 - h_1)(u_1, r_1)| \leq \int_0^{u_1} \left[\frac{1}{2}|\tilde{g}_2 - \tilde{g}_1| |\partial_r h_2| + |G_2 - G_1| \left|h_2 - \bar{h}_2\right| + |G_1| \left|\bar{h}_2 - \bar{h}_1\right|\right]_{|\chi_1} e^{\int_u^{u_1} G_{1|\chi_1} dv} du .$$

Setting

$$\delta(u) = \|(h_2 - h_1)(u, \cdot)\|_{\mathcal{C}^0_R} ,$$

then, arguing as in the beginning of the proof of this theorem, we obtain, from the previous inequality,

$$\delta(u_1) \leq C \int_0^{u_1} \delta(u) e^{\int_u^{u_1} G_{1|\chi_1} dv} du .$$

Applying Gronwall's inequality we conclude that

$$\delta(u) \leq 0 ,$$

and uniqueness follows.

The estimates (7.30) and (7.31) are now an immediate consequence of Lemma 7. □

7.4 Global existence in Bondi time

Theorem 7. *Let $\Lambda > 0$, $R > \sqrt{\frac{3}{\Lambda}}$ and $h_0 \in \mathcal{C}^k([0,R])$ for $k \geq 1$. There exists $x^* = x^*(\Lambda, R) > 0$ and $C^*(x^*, \Lambda, R) > 0$, such that, if $\|h_0\|_{X_R} \leq \frac{x^*}{(1+C^*)^2}$, then the initial value problem*

$$\begin{cases} Dh = G\left(h - \bar{h}\right) \\ h(0,r) = h_0(r) \end{cases} \tag{7.41}$$

has a unique solution $h \in \mathcal{C}^k([0,\infty] \times [0,R])$. Moreover,

$$\|h\|_{\mathcal{C}^0([0,\infty)\times[0,R])} = \|h_0\|_{\mathcal{C}^0([0,R])} , \tag{7.42}$$

and

$$\|h\|_{X([0,\infty)\times[0,R])} \leq (1 + C^*)\|h_0\|_{X([0,R])} . \tag{7.43}$$

Also, solutions depend continuously on initial data in the following precise sense: if h^1 and h^2 are two solutions with initial data h_0^1 and h_0^2, respectively, then

$$\|h^1 - h^2\|_{\mathcal{C}^0_{U,R}} \leq C(U, R, \Lambda)\|h_0^1 - h_0^2\|_{\mathcal{C}^0_R} ,$$

for all $U > 0$.

Proof. From Theorem 6 there exists a unique $h^1 \in \mathcal{C}^k([0,U_1] \times [0,R])$ solving (7.41), with existence time $U_1 = U(x^*/(1 + C^*)^2)$. Moreover

$$\|h^1(U_1, \cdot)\|_{X_R} \leq \|h^1\|_{X_{U_1,R}} \leq (1 + C^*)\|h_0\|_{X_R} \leq \frac{x^*}{1 + C^*} .$$

So Theorem 6 provides a solution $h^2 \in \mathcal{C}^k([0,U_2] \times [0,R])$ with initial data $h^2(0,r) = h^1(U_1,r)$ and existence time $U_2 = U(x^*/(1 + C^*))$. Now,

$$h : [0, U_1 + U_2] \times [0,R] \to \mathbb{R}$$

defined by

$$h(u,r) := \begin{cases} h^1(u,r) & , \quad u \in [0,U_1] \\ h^2(u,r) & , \quad u \in [U_1, U_1 + U_2] \, . \end{cases}$$

is the unique solution of our problem in $\mathcal{C}^k([0,U_1 + U_2] \times [0,R])$. Since (7.30) applies to both h^1 and h^2 we see that:

$$\|h^1\|_{U_1,R} = \|h_0\|_{\mathcal{C}^0_R} ,$$

so that

$$\|h^2\|_{\mathcal{C}^0_{U_2,R}} = \|h^1(U_1,\cdot)\|_{\mathcal{C}^0_R} \leq \|h_0\|_{\mathcal{C}^0_R} ,$$

and hence

$$\|h\|_{\mathcal{C}^0_{U_1+U_2,R}} = \|h_0\|_{\mathcal{C}^0_R} . \tag{7.44}$$

Arguing as in the proof of Lemma 7, we see that $\partial_r Dh$ is continuous and consequently $\partial_r h$ solves (5.17) so that:

$$\partial_r h(u_1, r_1) = \partial_r h_0(\chi(0))\, e^{\int_0^{u_1} 2G_{|\chi} dv} - \int_0^{u_1} \left(J\partial_r \bar{h}\right)_{|\chi} e^{\int_u^{u_1} 2G_{|\chi} dv} du \, . \tag{7.45}$$

Consequently,

$$\begin{aligned}|\partial_r h(u_1,r_1)| &\le |\partial_r h_0(r_0)| e^{\int_0^{u_1} 2G dv} + \int_0^{u_1} |J||\partial_r \bar{h}| e^{\int_u^{u_1} 2G dv} du \\ &\le \|\partial_r h_0\|_{\mathcal{C}^0_R} + 2C_3 C_4 \|h_0\|_{\mathcal{C}^0_R} \\ &\le \|\partial_r h_0\|_{\mathcal{C}^0_R} + C^* \|h_0\|_{\mathcal{C}^0_R} \ ,\end{aligned}$$

where we have used an estimate analogous to (7.28), the fact that Lemma 6 applies to h (with the same notation for the constants), and the fact that we may choose $C^* := 2(2C_2 + C_3)C_4$, which can be traced back to the proof of Lemma 7.

Combining the last two estimates with the smallness condition on the initial data leads to:

$$\|h\|_{X_{U_1+U_2,R}} \le (1+C^*)\|h_0\|_{X_R} \le \frac{x^*}{1+C^*} \ . \tag{7.46}$$

So, by Theorem 6, we can extend the solution by the same amount $U_2 = U(x^*/(1+C^*))$ as before; the global (in time) existence then follows, with the bounds (7.42) and (7.43) a consequence of (7.44) and (7.46). The continuous dependence statement follows by applying Gronwall's inequality to the integral inequality obtained integrating equation (7.40) and using the estimates derived in the beginning of the proof of Theorem 6. □

7.5 Exponential decay in Bondi time

Theorem 8. *Let $\Lambda > 0$, $R > \sqrt{\frac{3}{\Lambda}}$ and set $H = 2\sqrt{\frac{\Lambda}{3}}$. Then, for $\|h_0\|_{X_R}$ sufficiently small, the solution, $h \in \mathcal{C}^k([0,\infty] \times [0,R])$, of (7.41) satisfies*

$$\sup_{0\le r\le R} |\partial_r h(u,r)| \le \hat{C} e^{-Hu} \ ,$$

and, consequently, there exits $\underline{h} \in \mathbb{R}$ such that

$$|h(u,r) - \underline{h}| \le \bar{C} e^{-Hu} \ ,$$

with constants $\hat{C}$ and $\bar{C}$ depending on $\|h_0\|_{X_R}$, R and Λ .

Proof. Consider the solution provided by Theorem 7. Set

$$\mathcal{E}(u) := \|\partial_r h(u,\cdot)\|_{\mathcal{C}^0_R} \ , \tag{7.47}$$

and

$$E(u_0) := \sup_{u\ge u_0} \mathcal{E}(u) \ .$$

Arguing as in (7.4) we get

$$|(h-\bar{h})(u,r)| \le \frac{r}{2}\mathcal{E}(u) \ . \tag{7.48}$$

Lemma 6 applies and note that, for a fixed $x_0 \ge 0$, the estimates (7.15), (7.16) and (7.17) are still valid, with x^* replaced with $E(u_0)$, for the functions G and J restricted to $[u_0,\infty) \times [0,R]$. Integrating (5.17) with initial data on $u = u_0$ gives, for $u_1 \ge u_0$ (compare with (7.45))

$$\mathcal{E}(u_1) \le \mathcal{E}(u_0) e^{-2C_1 \int_{u_0}^{u_1} r(s)ds} + \frac{C_3 R}{2} \int_{u_0}^{u_1} \mathcal{E}(u) e^{-2C_1 \int_u^{u_1} r(v)dv} du \ ,$$

by using (7.48) and (5.11); once again we have used the notation for the constants set by Lemma 6. Recall that $r(u) = r(u; u_1, r_1)$ and that if $r_1 < r_c^- = \sqrt{\frac{3}{\Lambda K}}$ then, as in the calculations leading to (7.22), we have

$$\mathcal{E}(u_1) \leq \mathcal{E}(u_0) 2^{C_1/\alpha^2} e^{-\frac{C_1}{\alpha} u_1} + 2^{C_1/\alpha^2 - 1} C_3 R \int_{u_0}^{u_1} \mathcal{E}(u) e^{\frac{C_1}{\alpha}(u - u_1)} du \ ,$$

so that

$$e^{\frac{C_1}{\alpha} u_1} \mathcal{E}(u_1) \leq 2^{C_1/\alpha^2} \mathcal{E}(u_0) + 2^{C_1/\alpha^2 - 1} C_3 R \int_{u_0}^{u_1} \mathcal{E}(u) e^{\frac{C_1}{\alpha} u} du \ .$$

Applying Gronwall's Lemma to $\mathcal{F}(u_1) := e^{\frac{C_1}{\alpha} u_1} \mathcal{E}(u_1)$ then gives

$$e^{\frac{C_1}{\alpha} u_1} \mathcal{E}(u_1) \leq 2^{C_1/\alpha^2} \mathcal{E}(u_0) \exp\left(2^{C_1/\alpha^2 - 1} C_3 R (u_1 - u_0) \right) \ ,$$

so that finally

$$\mathcal{E}(u_1) \leq 2^{C_1/\alpha^2} \mathcal{E}(u_0) \exp\left\{ \left(2^{C_1/\alpha^2 - 1} C_3 R - \frac{C_1}{\alpha} \right) u_1 \right\} \ .$$

For $r_1 \geq r_c^-$ we have (7.14) instead and a similar, although simpler, derivation yields

$$\mathcal{E}(u_1) \leq \mathcal{E}(u_0) \exp\left\{ \left(\frac{C_3 R}{2} - 2 C_1 r_c^- \right) u_1 \right\} \ .$$

Observe that $K = e^{O(E(u_0))}$, $C_1 = \frac{\Lambda}{3} + O(E(u_0))$, $C_3 = O(E(u_0))$, uniformly in u_0 since $u_0 \mapsto E(u_0)$ is bounded. Using such boundedness once more, we can encode the previous estimates into

$$\mathcal{E}(u) \leq C e^{-\hat{H}(u_0) u} \ , \tag{7.49}$$

with

$$\hat{H}(u_0) = H + O(E(u_0)) \ . \tag{7.50}$$

Since $E(u_0)$ is controlled by $\|h_0\|_{X_R}$ (see (7.43)), choosing the later sufficiently small leads to

$$\hat{H}(u_0) \geq \mathring{H} > 0 \ ,$$

so that (7.49) implies

$$\mathcal{E}(u) \leq C e^{-\mathring{H} u} \ .$$

for $u \geq u_0$. Then clearly

$$E(u_0) \leq C e^{-\mathring{H} u_0} \ ,$$

so that (7.50) becomes

$$|H - \hat{H}(u_0)| \leq C e^{-\mathring{H} u_0} \ .$$

Finally, setting $u_0 = \frac{u}{2}$ yields

$$\begin{aligned} e^{Hu} \mathcal{E}(u) &\leq C \exp(Hu - \hat{H}(u/2) u) \\ &\leq C \exp(C e^{-\mathring{H} u/2} u) \leq \hat{C} \ , \end{aligned}$$

as desired; the remaining claims follow as in last chapter. □

It is now clear from (5.8), (5.12) and (5.10) that

$$|\bar{h}(u,r) - \underline{h}| \leq \bar{C}e^{-Hu} \, , \tag{7.51}$$

$$|g - 1| \leq \bar{C}e^{-Hu} \, , \tag{7.52}$$

$$|\tilde{g} - 1 + \Lambda r^2/3| \leq \bar{C}e^{-Hu} \, . \tag{7.53}$$

In particular, (7.3) implies that

$$m(u) \leq \bar{C}e^{-Hu} \, ,$$

and so the final Bondi mass M_1 vanishes. Finally, geodesic completeness is easily obtained from (7.51)-(7.53).

7.6 Conclusions and future work

We modified the framework developed in [40] to accommodate the presence of a cosmological constant, thus reducing the full content of the Einstein-scalar field system to a single integro-differential evolution equation. It is then natural, given both the structure of the equation and the domain of the Bondi coordinate system where the reduction is carried out, to consider a characteristic initial value problem by taking initial data on a truncated null cone.

For such an initial value problem we prove well posedness, global existence and exponential decay in (Bondi) time, for small data. From this, it follows that initial data close enough to de Sitter data evolves, according to the system under consideration, to a causally geodesically complete spacetime (with boundary), which approaches a region of de Sitter asymptotically at an exponential rate; this is a non-linear stability result for de Sitter within the class under consideration and can be seen as a realization of the cosmic no-hair conjecture[11]. Also, we note that the exponential decay rate obtained, $\lesssim e^{-Hu}$, with $H = 2\sqrt{\Lambda/3}$, is expected to be sharp[12] [160]. Moreover, an interesting side effect of the proof of our main results is the generalization, to this non-linear setting, of boundedness of the supremum norm of the scalar field in terms of its initial characteristic data.

It would be now interesting to extend these results to the noncompact case, where in this case results obtained in the linear problem, already show what decay to expect for the data, or by including scalar field potentials. A further step would consist in studying the gravitational collapse. This will need the introduction of regularised solutions which allow the study of the system for large initial data as in [41] as well as the methods of [42].

[11] Albeit in a limited sense, since our coordinates do not reach the whole of future infinity (see Figure 5.1). A precise statement of the cosmic no-hair conjecture can be found in [19], where it is shown that it follows from the existence of a smooth conformal future boundary.

[12] Although our retarded time coordinate u in (5.1) is different from the standard cosmic time coordinate t, it coincides with t along the center $r = 0$, and hence is close to t in our r-bounded domain, thus giving the same exponential decay. For instance, in de Sitter spacetime $u = t - \sqrt{3/\Lambda}\,\ln\left(1 + \sqrt{\Lambda/3}\,r\right)$.

Bibliography

[1] J. M. Aguirregabiria, A. Feinstein, and J. Ibáñez. Exponential-potential scalar field universes. II. Inhomogeneous models. *Physical Review D*, 48:4669–4675, November 1993.

[2] A. Alho. On the linear stability of a scalar-field with exponential potential in a flat robertson-walker background. In *Journal of Physics: Conference Series*, volume 229, page 012015. IOP Publishing, 2010.

[3] A. Alho, J.L. Costa, and J. Natário. Towards the einstein-λ-scalar field system in spherical symmetry. In *AIP Conference Proceedings*, volume 1460, page 149, 2012.

[4] A. Alho and F.C. Mena. Pre-inflationary homogenization of scalar field cosmologies. *Physics Letters B*, 703(5):537–542, 2011.

[5] A. Alho and F.C. Mena. Covariant and gauge-invariant scalar perturbations in multiple scalar field cosmologies. *in preparation*, 2012.

[6] A. Alho, F.C. Mena, and J.A.V. Kroon. The einstein-friedrich-nonlinear scalar field system and the stability of scalar field cosmologies. *arXiv preprint arXiv:1006.3778*, 2010.

[7] A. Alho, F.C. Mena, and J.A.V. Kroon. On the non-linear stability of scalar field cosmologies. In *Journal of Physics: Conference Series*, volume 314, page 012099. IOP Publishing, 2011.

[8] P. T. Allen and A. D. Rendall. Asymptotics of linearized cosmological perturbations. *Journal of Hyperbolic Differential Equations*, 7(2):255–277, 2010.

[9] R. A. Alpher and R. C. Herman. Evolution of the universe. *Nature*, 162:774–775, 1948.

[10] R. A. Alpher and R. C. Herman. Remarks on the evolution of the expanding universe. *Physical Review*, 75(7):1089, 1949.

[11] M. T. Anderson. Existence and Stability of Even-dimensional Asymptotically de Sitter Spaces. *Annales Henri Poincaré*, 6:801–820, October 2005.

[12] J. M. Bardeen. Gauge-invariant cosmological perturbations. *Physical Review D*, 22:1882–1905, October 1980.

[13] J. D. Barrow. Light elements and the isotropy of the Universe. *Monthly Notices of the Royal Astronomical Society*, 175:359–370, May 1976.

[14] J. D. Barrow. Graduated inflationary universes. *Physics Letters B*, 235:40–43, January 1990.

[15] V. A. Belinskii, L. P. Grishchuk, I. M. Khalatnikov, and Y. B. Zeldovich. Inflationary stages in cosmological models with a scalar field. *Physics Letters B*, 155(4):232–236, May 1985.

[16] V. A. Belinskii, L. P. Grishchuk, I. M. Khalatnikov, and Y. B. Zeldovich. Inflationary stages in cosmological models with a scalar field. In R. Ruffini, editor, *Fourth Marcel Grossmann Meeting on General Relativity*, pages 29–48, 1986.

[17] V. A. Belinskii, L. P. Grishchuk, Y. B. Zeldovich, and I. M. Khalatnikov. Inflationary stages in cosmological models with scalar fields. *Zhurnal Eksperimental noi i Teoreticheskoi Fiziki*, 89:346–360, August 1985.

[18] J. Bernstein and G. Feinberg. Cosmological constants: papers in modern cosmology. 1986.

[19] F. Beyer. Non-genericity of the nariai solutions: I. asymptotics and spatially homogeneous perturbations. *Classical and Quantum Gravity*, 26(23):235015, 2009.

[20] F. Beyer. Non-genericity of the nariai solutions: Ii. investigations within the gowdy class. *Classical and Quantum Gravity*, 26(23):235016, 2009.

[21] R. Bieli. Coupled quintessence and curvature-assisted acceleration. *Classical and Quantum Gravity*, 23:5983–5995, October 2006.

[22] A. P. Billyard and A. A. Coley. Interactions in scalar field cosmology. *Physical Review D*, 61(8):083503, April 2000.

[23] W. B. Bonnor. Jeans' formula for gravitational instability. *Monthly Notices of the Royal Astronomical Society*, 117:104, 1957.

[24] J.F. Bony and D. Häfner. Decay and non-decay of the local energy for the wave equation on the de sitter–schwarzschild metric. *Communications in Mathematical Physics*, 282(3):697–719, 2008.

[25] M. Bradley, P. K. S. Dunsby, M. Forsberg, and Z. Keresztes. Density growth in Kantowski-Sachs cosmologies with a cosmological constant. *Classical and Quantum Gravity*, 29(9):095023, May 2012.

[26] P.R. Brady, C.M. Chambers, W. Krivan, and P. Laguna. Telling tails in the presence of a cosmological constant. *Physical Review D*, 55(12):7538, 1997.

[27] M. Bruni. Stability of open universes. *Physical Review D*, 47(2):738–742, Jan 1993.

[28] M. Bruni, P. K. S. Dunsby, and G. F. R. Ellis. Cosmological perturbations and the physical meaning of gauge-invariant variables. *Astrophysical Journal*, 395:34–53, August 1992.

[29] M. Bruni, G. F. R. Ellis, and P. K. S. Dunsby. Gauge-invariant perturbations in a scalar field dominated universe. *Classical and Quantum Gravity*, 9:921–945, April 1992.

[30] M. Bruni, F. C. Mena, and R. Tavakol. Cosmic no-hair: nonlinear asymptotic stability of de Sitter universe. *Classical and Quantum Gravity*, 19:L23–L29, March 2002.

[31] M. Bruni and K. Piotrkowska. Dust/Radiation Universes - Stability Analysis. *Monthly Notices of the Royal Astronomical Society*, 270:630–+, October 1994.

[32] A. B. Burd and J. D. Barrow. Inflationary models with exponential potentials. *Nuclear Physics B*, 308:929–945, October 1988.

[33] B. J. Carr. The primordial black hole mass spectrum. *Astrophysical Journal*, 201:1–19, October 1975.

[34] B. J. Carr and S. W. Hawking. Black holes in the early Universe. *Monthly Notices of the Royal Astronomical Society*, 168:399–416, August 1974.

[35] A. Challinor and A. Lasenby. Relativistic Corrections to the Sunyaev-Zeldovich Effect. *Astrophysical Journal*, 499:1, May 1998.

[36] M.W. Choptuik. Universality and scaling in gravitational collapse of a massless scalar field. *Physical review letters*, 70(1):9–12, 1993.

[37] Y. Choquet-Bruhat and J. W. York. On H. Friedrich's formulation of Einstein's equations with fluid sources. *Topological Methods in Nonlinear Analysis*, 18(2):321–335, 2001.

[38] Y. Choquet-Bruhat and J. W. York. Constraints and Evolution in Cosmology. In S. Cotsakis and E. Papantonopoulos, editors, *Cosmological Crossroads*, volume 592 of *Lecture Notes in Physics, Berlin Springer Verlag*, page 29, 2002.

[39] D. Christodoulou. Violation of cosmic censorship in the gravitational collapse of a dust cloud. *Communications in Mathematical Physics*, 93(2):171–195, 1984.

[40] D. Christodoulou. The problem of a self-gravitating scalar field. *Communications in Mathematical Physics*, 105:337–361, September 1986.

[41] D. Christodoulou. The structure and uniqueness of generalized solutions of the spherically symmetric einstein-scalar equations. *Communications in Mathematical Physics*, 109(4):591–611, 1987.

[42] D. Christodoulou. The formation of black holes and singularities in spherically symmetric gravitational collapse. *Communications on Pure and Applied Mathematics*, 44(3):339–373, 1991.

[43] D. Christodoulou. The instability of naked singularities in the gravitational collapse of a scalar field. *Annals of Mathematics-Second Series*, 149(1):183–218, 1999.

[44] D. Christodoulou. *The formation of Black holes in General Relativity*, volume 4. European Mathematical Society, 2009.

[45] A. A. Coley, editor. *Dynamical Systems and Cosmology*, volume 291 of *Astrophysics and Space Science Library*. Kluwer, October 2003.

[46] A. A. Coley, J. Ibanez, and R. J. van den Hoogen. Homogeneous scalar field cosmologies with an exponential potential. *Journal of Mathematical Physics*, 38:5256–5271, October 1997.

[47] A. A. Coley and R. J. van den Hoogen. Dynamics of multi-scalar-field cosmological models and assisted inflation. *Physical Review D*, 62(2):023517, Jun 2000.

[48] E. J. Copeland, M. Sami, and S. Tsujikawa. Dynamics of Dark Energy. *International Journal of Modern Physics D*, 15:1753–1935, 2006.

[49] J.L. Costa, A. Alho, and J. Natário. Linear and non-linear waves in de sitter. In *AIP Conference Proceedings*, volume 1458, page 451, 2012.

[50] J.L. Costa, A. Alho, and J. Natario. The problem of a self-gravitating scalar field with positive cosmological constant. *Accepted for publication in Annales Henri Poincare, arXiv preprint arXiv:1206.4153*, 2012.

[51] J.L. Costa, A. Alho, and J. Natário. Spherical linear waves in de sitter spacetime. *Journal of Mathematical Physics*, 53:052501, 2012.

[52] M. Dafermos. Spherically symmetric spacetimes with a trapped surface. *Classical and Quantum Gravity*, 22(11):2221, 2005.

[53] M. Dafermos and I. Rodnianski. The wave equation on schwarzschild-de sitter spacetimes. *arXiv preprint arXiv:0709.2766*, 2007.

[54] M. Dafermos and I. Rodnianski. Lectures on black holes and linear waves. *arXiv preprint arXiv:0811.0354*, 2008.

[55] W. De Sitter. On einsteins theory of gravitation and its astronomical consequences. second paper. *Monthly Notices of the Royal Astronomical Society*, 77:155–184, 1916.

[56] W. De Sitter. On the relativity of rotation in einsteins theory. In *Proceedings of the Royal Academy of Sciences Amsterdam*, volume 19, pages 527–532, 1916.

[57] W. De Sitter. Einstein's theory of gravitation and its astronomical consequences. third paper. *Monthly Notices of the Royal Astronomical Society*, 78:3–28, 1917.

[58] W. De Sitter. On the curvature of space. In *Proc. Kon. Ned. Akad. Wet*, volume 20, pages 229–243, 1917.

[59] W. De Sitter. On the relativity of inertia. remarks concerning einsteins latest hypothesis. *Proc. Akad. Weteusch. Amsterdam*, 19:1217, 1917.

[60] W. De Sitter. Further remarks on the solutions of the field equations of einsteins theory of gravitation. *Koninklijke Nederlandse Akademie van Weteschappen Proceedings Series B Physical Sciences*, 20:1309–1312, 1918.

[61] P. K. S. Dunsby. Gauge invariant perturbations in multi-component fluid cosmologies. *Classical and Quantum Gravity*, 8:1785–1806, October 1991.

[62] P. K. S. Dunsby. Covariant perturbations of anisotropic cosmological models. *Physical Review D*, 48:3562–3576, October 1993.

[63] P. K. S. Dunsby, M. Bruni, and G. F. R. Ellis. Covariant perturbations in a multifluid cosmological medium. *Astrophysical Journal*, 395:54–74, August 1992.

[64] A.S. Eddington. The mathematical theory of relativity. *Cambridge, England*, 1924.

[65] J. Ehlers. Contributions to the relativistic mechanics of continuous media. *General Relativity and Gravitation*, 25:1225–1266, December 1993.

[66] A. Einstein. Kosmologische betrachtungen zur allgemeinen relativitätstheorie. *Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften (Berlin), Seite 142-152.*, 1:142–152, 1917.

[67] A. Einstein. Kritisches zu einer von hrn. de sitter gegebenen lösung der gravitationsgleichungen. *Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften (Berlin), Seite 270-272.*, 1:270–272, 1918.

[68] A. Einstein. Prinzipielles zur allgemeinen relativitätstheorie. *Annalen der Physik*, 360(4):241–244, 1918.

[69] G. F. R. Ellis. relativistic cosmology. In R. K. Sachs, editor, *General Relativity and Cosmology*, pages 104–182, 1971.

[70] G. F. R. Ellis and M. Bruni. Covariant and gauge-invariant approach to cosmological density fluctuations. *Physical Review D*, 40(6):1804–1818, Sep 1989.

[71] G. F. R. Ellis, M. Bruni, and J. Hwang. Density-gradient-vorticity relation in perfect-fluid robertson-walker perturbations. *Physical Review D*, 42(4):1035–1046, Aug 1990.

[72] G. F. R. Ellis, J. Hwang, and M. Bruni. Covariant and gauge-independent perfect-fluid robertson-walker perturbations. *Physical Review D*, 40(6):1819–1826, Sep 1989.

[73] G. F. R. Ellis and M. S. Madsen. Exact scalar field cosmologies. *Classical and Quantum Gravity*, 8:667–676, April 1991.

[74] G.F.R. Ellis. The expanding universe: a history of cosmology from 1917 to 1960. In *Einstein and the History of General Relativity*, volume 1, pages 367–431, 1989.

[75] R. I. Epstein and V. Petrosian. Effects of primordial fluctuations on the abundances of light elements. *Astrophysical Journal*, 197:281–284, April 1975.

[76] L. C. Evans. *Partial Differential Equations.*

[77] A. Feinstein and J. Ibanez. Exact inhomogeneous scalar field universes. *Classical and Quantum Gravity*, 10:L227–L231, December 1993.

[78] A. Feinstein, J. Ibáñez, and P. Labraga. Scalar field inhomogeneous cosmologies. *Journal of Mathematical Physics*, 36:4962–4974, September 1995.

[79] M. Forsberg, M. Bradley, and P. K. S. Dunsby. Density perturbations in Kantowski-Sachs models with a cosmological constant. *Journal of Physics Conference Series*, 314(1):012026, September 2011.

[80] P. Fosalba, E. Gaztañaga, and F. J. Castander. Detection of the Integrated Sachs-Wolfe and Sunyaev-Zeldovich Effects from the Cosmic Microwave Background-Galaxy Correlation. *The Astrophysical Journal Letters*, 597:L89–L92, November 2003.

[81] A. Friedman. On the curvature of space. *Zeitschrift fur Physik*, 10:377–386, 1922.

[82] A. Friedman. On the possibility of a world with constant negative curvature. *Zeitschrift fur Physik*, 21:326–332, 1924.

[83] H. Friedrich. On the existence of n-geodesically complete or future complete solutions of Einstein's field equations with smooth asymptotic structure. *Communications in Mathematical Physics*, 107:587–609, December 1986.

[84] H. Friedrich. Hyperbolic reductions for Einstein's equations. *Classical and Quantum Gravity*, 13:1451–1469, June 1996.

[85] H. Friedrich. Evolution equations for gravitating ideal fluid bodies in general relativity. *Physical Review D*, 57:2317–2322, February 1998.

[86] H. Friedrich and A. D. Rendall. The Cauchy Problem for the Einstein Equations. In B. G. Schmidt, editor, *Einstein's Field Equations and Their Physical Implications*, volume 540 of *Lecture Notes in Physics, Berlin Springer Verlag*, page 127, 2000.

[87] G. Gamow. Expanding universe and the origin of elements. *Physical Review*, 70(7-8):572–573, 1946.

[88] G. Gamow. The origin of elements and the separation of galaxies. *Physical Review*, 74:505–506, 1948.

[89] D. Gilbarg and N.S. Trudinger. *Elliptic partial differential equations of second order*, volume 224. Springer Verlag, 2001.

[90] G. R. Gisler, E. R. Harrison, and M. J. Rees. Variations in the primordial helium abundance. *Monthly Notices of the Royal Astronomical Society*, 166:663–672, March 1974.

[91] S. W. Goode. Analysis of spatially inhomogeneous perturbations of the frw cosmologies. *Physical Review D*, 39(10):2882, 1989.

[92] C. Gundlach and J.M. Martín-García. Critical phenomena in gravitational collapse. *Living Rev. Relativity*, 10, 2007.

[93] A.H. Guth. Inflationary universe: A possible solution to the horizon and flatness problems. *Physical Review D*, 23(2):347, 1981.

[94] J. J. Halliwell. Scalar fields in cosmology with an exponential potential. *Physics Letters B*, 185:341–344, February 1987.

[95] E. R. Harrison. Normal modes of vibrations of the universe. *Reviews of Modern Physics*, 39(4):862, 1967.

[96] E. R. Harrison. Standard Model of the Early Universe. *Annual Review of Astronomy and Astrophysics*, 11:155, 1973.

[97] S. W. Hawking. Perturbations of an Expanding Universe. *Astrophysical Journal*, 145:544–, August 1966.

[98] S. W. Hawking and G. F. R. Ellis. *The Large Scale Structure of Space-Time.* 1975.

[99] J. M. Heinzle and A. D. Rendall. Power-Law Inflation in Spacetimes Without Symmetry. *Communications in Mathematical Physics*, 269:1–15, January 2007.

[100] M. Heusler. Anisotropic asymptotic behavior in chaotic inflation. *Physics Letters B*, 253:33–37, January 1991.

[101] S. Hobbs and P. K. S. Dunsby. Dynamical systems approach to magnetized cosmological perturbations. *Physical Review D*, 62(12):124007, Nov 2000.

[102] E. Hubble. A relation between distance and radial velocity among extra-galactic nebulae. *Proceedings of the National Academy of Sciences*, 15(3):168–173, 1929.

[103] J. C. Hwang and E. T. Vishniac. Analyzing cosmological perturbations using the covariant approach. *Astrophysical Journal*, 353:1–20, April 1990.

[104] F. M. Ipavich. The Effect of Temperature Variations on the Primeval Production of Helium. *Astrophysical Journal*, 171:449, February 1972.

[105] W. M. Irvine. Local irregularities in an expanding universe. *Annals of Physics*, 32:322–347, April 1965.

[106] N. Itoh, Y. Kohyama, and S. Nozawa. Relativistic Corrections to the Sunyaev-Zeldovich Effect for Clusters of Galaxies. *Astrophysical Journal*, 502:7, July 1998.

[107] M. Janssen. The einstein-de sitter debate and its aftermath.

[108] P. Kerszberg. The invented universe: the einstein-de sitter controversy (1916-17) and the rise of relativistic cosmology. *Oxford [England]: Clarendon Press; New York: Oxford University Press, 1989.*, 1, 1989.

[109] S. Kind, J. Ehlers, and B. G. Schmidt. Relativistic stellar oscillations treated as an initial value problem. *Classical and Quantum Gravity*, 10(10):2137, 1999.

[110] A. R. King and G. F. R. Ellis. Tilted homogeneous cosmological models. *Communications in Mathematical Physics*, 31:209–242, 1973.

[111] V. V. Kiselev and S. A. Timofeev. Quasiattractor dynamics of lambda-phiˆ4-inflation. *ArXiv e-prints*, January 2008.

[112] V. V. Kiselev and S. A. Timofeev. Quasiattractor in models of new and chaotic inflation. *General Relativity and Gravitation*, 42:183–197, January 2010.

[113] Y. Kitada and K. Maeda. Cosmic no-hair theorem in homogeneous spacetimes. I. Bianchi models. *Classical and Quantum Gravity*, 10:703–734, April 1993.

[114] Y. Kitada and K. I. Maeda. Cosmic no-hair theorem in power-law inflation. *Physical Review D*, 45:1416–1419, February 1992.

[115] F. Klein. Bemerkungen über die beziehungen des de sitter'schen koordinatensystems b zu der allgemeinen welt konstanter positiver krümmung. *Koninklijke Nederlandse Akademie van Wetenschappen Amsterdam, Wis-en Natuurkundige Afdeeling*, 27:488–489, 1918.

[116] F. Klein. *Über die Integralform der Erhaltungssätze und die Theorie der räumlich-geschlossenen Welt.* Nachrichten von der K. Gesellschaft der Wissenschaften, 1918.

[117] L. Kofman, A. Linde, and A. A. Starobinsky. Reheating after inflation. *Physical Review Letters*, 73:3195–3198, December 1994.

[118] L. Kofman, A. Linde, and A. A. Starobinsky. Towards the theory of reheating after inflation. *Physical Review D*, 56:3258–3295, September 1997.

[119] E. Komatsu and T. Kitayama. Sunyaev-Zeldovich Fluctuations from Spatial Correlations between Clusters of Galaxies. *The Astrophysical Journal Letters*, 526:L1–L4, November 1999.

[120] H-O Kreiss and J. Lorenz. Stability for time-dependent differential equations. *Acta Numerica*, 7:203–285, January 1998.

[121] H-O Kreiss, G. B. Nagy, O. E. Ortiz, and O. A. Reula. Global existence and exponential decay for hyperbolic dissipative relativistic fluid theories. *Journal of Mathematical Physics*, 38:5272–5279, October 1997.

[122] H-O. Kreiss, O. E. Ortiz, and O. A. Reula. Stability of quasi-linear hyperbolic dissipative systems. *Journal of Differential Equations*, 142:78–96, 1998.

[123] J. H. Kung and R. H. Brandenberger. Chaotic inflation as an attractor in initial-condition space. *Physical Review D*, 42:1008–1015, August 1990.

[124] D. Langlois and F. Vernizzi. Non-linear perturbations of cosmological scalar fields. *Journal of Cosmology and Astroparticle Physics*, 2007(02):017, 2007.

[125] G. Lemaitre. *Ann Soc. Sci. Bruxelles*, A53:51, 1933.

[126] A. R. Liddle, A. Mazumdar, and F. E. Schunck. Assisted inflation. *Physical Review D*, 58(6):061301, September 1998.

[127] J. E. Lidsey. Multiple and anisotropic inflation with exponential potentials. *Classical and Quantum Gravity*, 9:1239–1253, May 1992.

[128] E. M. Lifshitz. On the gravitational stability of the expanding universe. *Journal Physics (USSR)*, 10(116), 1946.

[129] E. M. Lifshitz and I. M. Khalatnikov. Investigations in relativistic cosmology. *Advances in Physics*, 12:185–249, April 1963.

[130] A. Linde. Inflationary Cosmology. In M. Lemoine, J. Martin, and P. Peter, editors, *Inflationary Cosmology*, volume 738 of *Lecture Notes in Physics, Berlin Springer Verlag*, page 1, 2008.

[131] C. Lübbe and JA Kroon. A conformal approach for the analysis of the non-linear stability of pure radiation cosmologies. *arXiv preprint arXiv:1111.4691*, 2011.

[132] F. Lucchin and S. Matarrese. Power-law inflation. *Physical Review D*, 32:1316–1322, September 1985.

[133] M. S. Madsen. Scalar fields in curved spacetimes. *Classical and Quantum Gravity*, 5:627–639, 1988.

[134] M.S. Madsen. A note on the equation of state of a scalar field. *Astrophysics and space science*, 113(1):205–207, 1985.

[135] H. Maeda and M. Nozawa. Generalized misner-sharp quasilocal mass in einstein-gauss-bonnet gravity. *Physical Review D*, 77(6):064031, 2008.

[136] K. A. Malik and D. Wands. Dynamics of assisted inflation. *Physical Review D*, 59(12):123501, June 1999.

[137] M. Marklund, P. K. S. Dunsby, G. Betschart, M. Servin, and C. G. Tsagas. Charged multifluids in general relativity. *Classical and Quantum Gravity*, 20:1823–1834, May 2003.

[138] F. C. Mena. On the asymptotic stability of de Sitter spacetime: a nonlinear perturbative approach. *Journal of Physics A Mathematical General*, 43(16):165401, April 2010.

[139] I. Moss and V. Sahni. Anisotropy in the chaotic inflationary universe. *Physics Letters B*, 178:159–162, October 1986.

[140] V. Muller, H. J. Schmidt, and A. A. Starobinsky. Power-law inflation as an attractor solution for inhomogeneous cosmological models. *Classical and Quantum Gravity*, 7:1163–1168, July 1990.

[141] K. Nakao. On a quasi-local energy outside the cosmological horizon. *Arxiv preprint gr-qc/9507022*, 1995.

[142] J. Natario. Relativity and singularities-a short introduction for mathematicians. *arXiv preprint math/0603190*, 2006.

[143] J.D. North. *The measure of the universe.* Clarendon Press, 1965.

[144] K. A. Olive. Inflation. *Physics Reports*, 190:307–403, June 1990.

[145] D. W. Olson. Density perturbations in cosmological models. *Physical Review D*, 14:327–331, July 1976.

[146] B. O'neill. *Semi-Riemannian geometry: with applications to relativity*, volume 103. Academic Pr, 1983.

[147] O. E. Ortiz. Stability of nonconservative hyperbolic systems and relativistic dissipative fluids. *Journal of Mathematical Physics*, 42:1426–1442, March 2001.

[148] D. N. Page and S. W. Hawking. Gamma rays from primordial black holes. *Astrophysical Journal*, 206:1–7, May 1976.

[149] P. J. E. Peebles. The Gravitational-Instability Picture and the Nature of the Distribution of Galaxies. *Astrophysical Journal*, 189:L51, April 1974.

[150] P. J. E. Peebles. Large-scale background temperature and mass fluctuations due to scale-invariant primeval perturbations. *Astrophysical Journal*, 263:L1–L5, December 1982.

[151] P. J. E. Peebles and J. T. Yu. Primeval Adiabatic Perturbation in an Expanding Universe. *Astrophysical Journal*, 162:815, December 1970.

[152] A.A. Penzias and R.W. Wilson. A measurement of excess antenna temperature at 4080 mc/s. *The Astrophysical Journal*, 142:419–421, 1965.

[153] S. Perlmutter, G. Aldering, M. Della Valle, S. Deustua, RS Ellis, S. Fabbro, A. Fruchter, G. Goldhaber, DE Groom, IM Hook, et al. Discovery of a supernova explosion at half the age of the universe. *Nature*, 391(6662):51–54, 1998.

[154] Q.I. Rahman and G. Schmeisser. Analytic theory of polynomials: Critical points, zeros and extremal properties (london mathematical society monographs). 2002.

[155] M. J. Rees and D. W. Sciama. Large-scale Density Inhomogeneities in the Universe. *Nature*, 217:511–516, February 1968.

[156] M. J. Rees and D. W. Sciama. The Evolution of Density Fluctuations in the Universe: I. Dissipative Processes in the Early Stages. *Comments on Astrophysics and Space Physics*, 1:140, July 1969.

[157] M. J. Rees and D. W. Sciama. The Evolution of Density Fluctuations in the Universe: II. The Formation of Galaxies. *Comments on Astrophysics and Space Physics*, 1:153, September 1969.

[158] A. D. Rendall. Cosmological models and centre manifold theory. *General Relativity and Gravitation*, 34(8):1277–1294, August 2002.

[159] A. D. Rendall. Accelerated cosmological expansion due to a scalar field whose potential has a positive lower bound. *Classical and Quantum Gravity*, 21:2445–2454, May 2004.

[160] A. D. Rendall. Asymptotics of Solutions of the Einstein Equations with Positive Cosmological Constant. *Annales Henri Poincaré*, 5:1041–1064, December 2004.

[161] A. D. Rendall. Intermediate inflation and the slow-roll approximation. *Classical and Quantum Gravity*, 22:1655–1666, May 2005.

[162] A. D. Rendall. Mathematical Properties of Cosmological Models with Accelerated Expansion. In J. Frauendiener, D. J. W. Giulini, and V. Perlick, editors, *Analytical and Numerical Approaches to Mathematical Relativity*, volume 692 of *Lecture Notes in Physics, Berlin Springer Verlag*, pages 141–155, 2006.

[163] A. D. Rendall. Late-time oscillatory behaviour for self-gravitating scalar fields. *Classical and Quantum Gravity*, 24:667–677, February 2007.

[164] Y. Rephaeli. Comptonization Of The Cosmic Microwave Background: The Sunyaev-Zeldovich Effect. *Annual Review of Astronomy and Astrophysics*, 33:541–580, 1995.

[165] O. A. Reula. Hyperbolic Methods for Einstein's Equations. *Living Reviews in Relativity*, 1:3, January 1998.

[166] O. A. Reula. Exponential decay for small nonlinear perturbations of expanding flat homogeneous cosmologies. *Physical Review D*, 60(8):083507, October 1999.

[167] M. J. Reyes-Ibarra and L. A. Ureña-López. Attractor dynamics of inflationary monomial potentials. In H. A. Morales-Tecotl, L. A. Urena-Lopez, R. Linares-Romero, and H. H. Garcia-Compean, editors, *American Institute of Physics Conference Series*, volume 1256 of *American Institute of Physics Conference Series*, pages 293–300, July 2010.

[168] A.G. Riess, A.V. Filippenko, P. Challis, A. Clocchiatti, A. Diercks, P.M. Garnavich, R.L. Gilliland, C.J. Hogan, S. Jha, R.P. Kirshner, et al. Observational evidence from supernovae for an accelerating universe and a cosmological constant. *The Astronomical Journal*, 116(3):1009, 2007.

[169] W. Rindler. *Essential relativity: special, general, and cosmological.* Springer, 1969.

[170] H. Ringström. Future stability of the Einstein-non-linear scalar field system. *Inventiones Mathematicae*, 173:123–208, March 2008.

[171] H. Ringström. Power Law Inflation. *Communications in Mathematical Physics*, 290:155–218, August 2009.

[172] H.P. Robertson. On the foundations of relativistic cosmology. *Proceedings of the National Academy of Sciences of the United States of America*, 15(11):822, 1929.

[173] I. Rodnianski. The cauchy problem in general relativity. In *Proceedings oh the International Congress of Mathematicians: Madrid, August 22-30, 2006: invited lectures*, pages 421–442, 2006.

[174] I. Rodnianski and J. Speck. The stability of the irrotational euler-einstein system with a positive cosmological constant. *arXiv preprint arXiv:0911.5501*, 2009.

[175] R. K. Sachs and A. M. Wolfe. Perturbations of a Cosmological Model and Angular Variations of the Microwave Background. *Astrophysical Journal*, 147:73, January 1967.

[176] M. P. Savedoff and S. Vila. Growth of Fluctuations in a Uniformly Contracting or Expanding Cloud. *Astrophysical Journal*, 136:609, September 1962.

[177] V. Schlue. Global results for linear waves on expanding schwarzschild de sitter cosmologies. *arXiv preprint arXiv:1207.6055*, 2012.

[178] BP Schmidt. Measuring global curvature and cosmic acceleration with supernovae. *Classical and Quantum Gravity*, 19(13):3487, 2002.

[179] J. Silk. Cosmic Black-Body Radiation and Galaxy Formation. *Astrophysical Journal*, 151:459, February 1968.

[180] J. Silk and S. L. Shapiro. Primordial Variations in the Cosmic Abundance of Helium. *Astrophysical Journal*, 166:249, June 1971.

[181] J. Speck. The nonlinear future stability of the flrw family of solutions to the euler–einstein system with a positive cosmological constant. *Selecta Mathematica, New Series*, pages 1–83, 2012.

[182] J. Speck. The stabilizing effect of spacetime expansion on relativistic fluids with sharp results for the radiation equation of state. *arXiv preprint arXiv:1201.1963*, 2012.

[183] A. A. Starobinskii. Isotropization of arbitrary cosmological expansion given an effective cosmological constant. *JETP lett*, 37(1), 1983.

[184] J. M. Stewart. Perturbations of Friedmann-Robertson-Walker cosmological models. *Classical and Quantum Gravity*, 7:1169–1180, July 1990.

[185] J. M. Stewart and M. Walker. Perturbations of space-times in general relativity. *Royal Society of London Proceedings Series A*, 341:49–74, October 1974.

[186] R. A. Sunyaev and Y. B. Zeldovich. Small-Scale Fluctuations of Relic Radiation. *Astrophysics and Space Science*, 7:3–19, April 1970.

[187] R. A. Sunyaev and Y. B. Zeldovich. The interaction of matter and radiation in the hot model of the Universe, II. *Astrophysics and Space Science*, 7:20–30, April 1970.

[188] R. A. Sunyaev and Y. B. Zeldovich. Formation of Clusters of Galaxies; Protocluster Fragmentation and Intergalactic Gas Heating. *Astronomy and Astrophysics*, 20:189, August 1972.

[189] R. A. Sunyaev and Y. B. Zeldovich. Microwave background radiation as a probe of the contemporary structure and history of the universe. *In: Annual review of astronomy and astrophysics*, 18:537–560, 1980.

[190] C. G. Tsagas and J. D. Barrow. A gauge-invariant analysis of magnetic fields in general-relativistic cosmology. *Classical and Quantum Gravity*, 14:2539–2562, September 1997.

[191] C. G. Tsagas and J. D. Barrow. Gauge-invariant magnetic perturbations in perfect-fluid cosmologies. *Classical and Quantum Gravity*, 15:3523–3544, November 1998.

[192] C. G. Tsagas, A. Challinor, and R. Maartens. Relativistic cosmology and large-scale structure. *Physics Reports*, 465:61–147, August 2008.

[193] C. Uggla and J. Wainwright. Scalar cosmological perturbation. *Classical and Quantum Gravity*, 29(10):105002, 2012.

[194] L. A. Ureña-López and M. J. Reyes-Ibarra. On the Dynamics of a Quadratic Scalar Field Potential. *International Journal of Modern Physics D*, 18:621–634, 2009.

[195] R. J. van den Hoogen and I. Olasagasti. Isotropization of scalar field Bianchi type-IX models with an exponential potential. *Physical Review D*, 59(10):107302, May 1999.

[196] H. van Elst and C. Uggla. General relativistic orthonormal frame approach. *Classical and Quantum Gravity*, 14(9):2673, 1999.

[197] F. Vernizzi. Conservation of second-order cosmological perturbations in a scalar field dominated universe. *Physical Review D*, 71(6):061301, 2005.

[198] J. Wainwright and G. F. R. Ellis. *Dynamical Systems in Cosmology.* June 2005.

[199] J. Wainwright and L. Hsu. A dynamical systems approach to Bianchi cosmologies - Orthogonal models of class A. *Classical and Quantum Gravity*, 6:1409–1431, October 1989.

[200] J. Wainwright and W. C. Lim. Cosmological models from a dynamical systems perspective. *Journal of Hyperbolic Differential Equations*, 2:437–470, 2005.

[201] R. M. Wald. Asymptotic behavior of homogeneous cosmological models in the presence of a positive cosmological constant. *Physical Review D*, 28:2118–2120, October 1983.

[202] A. G. Walker. On milnes theory of world structure. *Proc. London Math. Soc.*, 42:90–217, 1936.

[203] H. Weyl. Uber die statischen kugelsymmetrischen lösungen von einsteins kosmologischen gravitationsgleichungen. *Phys. Z*, 20:31–34, 1919.

[204] H. Weyl. *Space, time, matter.* Dutton, 1922.

[205] M. White, D. Scott, and J. Silk. Anisotropies in the Cosmic Microwave Background. *Annual Review of Astronomy and Astrophysics*, 32:319–370, 1994.

[206] A. Woszczyna. A dynamical systems approach to cosmological structure formation - Newtonian universe. *Monthly Notices of the Royal Astronomical Society*, 255:701–706, April 1992.

[207] A. Woszczyna. Gauge-invariant cosmic structures: A dynamic systems approach. *Physical Review D*, 45(6):1982–1988, Mar 1992.

[208] A. Woszczyna. The Jeans mass versus the Lyapunov concept of stability. *Acta Cosmologica*, 21:71–77, 1995.

[209] A. Woszczyna and A. Kulak. Cosmological perturbations - Extension of Olson's gauge-invariant method. *Classical and Quantum Gravity*, 6:1665–1671, November 1989.

[210] K. Yagdjian and A. Galstian. Fundamental solutions for the klein-gordon equation in de sitter spacetime. *Communications in Mathematical Physics*, 285(1):293–344, 2009.

[211] W. Zimdahl. Cosmological perturbations in the inflationary Universe. *Physical Review D*, 56:3357–3364, September 1997.