Mathematical Relativity
2021/2022
2^{nd} Exam - 15 July 2022 - 10:00

1. Let (M,g) be a time-oriented spacetime and $p \in M$. For each of the following statements, give a proof if true or a counter-example if false.

(2/20) (a) $I^+(p)$ is not closed.

(2/20) (b) $I^+(I^+(p)) = I^+(p)$.

(2/20) (c) If (M,g) is stably causal then there are no closed causal curves.

(2/20) (d) If (M,g) is stably causal then for each $q \in I^+(p)$ there exists a constant C_q such that any timelike curve connecting p and q satisfies $\tau(c) \leq C_q$.

2. Let (S,h) be the Riemannian manifold given by $S = \mathbb{R} \times S^2$ and

$$h = dx^2 + d\theta^2 + \sin^2\theta d\varphi^2$$

(where (θ,φ) are the usual spherical coordinates on S^2), and let

$$K = 2dx^2 + d\theta^2 + \sin^2\theta d\varphi^2.$$

(3/20) (a) Show that (S,h,K) is an initial data set for the vacuum Einstein equations for a certain value of the cosmological constant Λ, and compute this value.

(3/20) (b) Prove that the corresponding maximal globally hyperbolic development is geodesically incomplete.

3. Let h be the spherically symmetric Riemannian metric defined in $\mathbb{R}^3$ by

$$h = \left(1 - \frac{2m(r)}{r}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right),$$

where m is a suitable smooth function whose derivative has compact support.

(2/20) (a) Show that in Cartesian coordinates we have

$$h_{ij} = \delta_{ij} + \frac{2m(r)}{r^3}\left(1 - \frac{2m(r)}{r}\right)^{-1} x^i x^j,$$

and conclude that h is asymptotically flat with ADM mass

$$M = \lim_{r\to+\infty} m(r).$$

(2/20) (b) Check that the scalar curvature of h is given by

$$\bar{R} = \frac{4}{r^2}\frac{dm}{dr},$$

and prove the Riemannian positive mass theorem in this case.

(2/20) (c) Show that $r = r_0$ is a minimal surface if and only if $m(r_0) = \frac{r_0}{2}$ (in which case r is a well-defined coordinate only for $r > r_0$), and prove the Riemannian Penrose inequality in this case.