Mathematical Relativity
2021/2022
1^{st} Exam - 11 July 2022 - 10:00

1. Let (M, g) be a time-oriented spacetime and $p \in M$. For each of the following statements, give a proof if true or a counter-example if false.

(2/20) (a) $I^+(p) \neq M$.

(2/20) (b) $\partial I^+(p) = \partial J^+(p)$.

(2/20) (c) If $M = \mathbb{R} \times S$ and $g = -dt^2 + h$, where (S, h) is a geodesically incomplete Riemannian manifold, then (M, g) is not globally hyperbolic.

(2/20) (d) If $q \in I^+(p)$ is connected to p by a non-maximizing timelike geodesic c then there exists a conjugate point along c between p and q.

2. The Nariai solution is the Lorentzian manifold (M, g) given by $M = \mathbb{R}^2 \times S^2$ and

$$g = -dt^2 + \cosh^2(t)dx^2 + d\theta^2 + \sin^2\theta d\varphi^2$$

(where (θ, φ) are the usual spherical coordinates on S^2).

(2/20) (a) Sketch the Penrose diagram of the Nariai solution.

(2/20) (b) The Nariai solution solves the vacuum Einstein equations for a certain value of the cosmological constant Λ. Compute this value.

(2/20) (c) Show that the Penrose singularity theorem does not hold if we replace "trapped surface" by "marginally trapped surface" in its statement (a marginally trapped surface is a surface whose null expansions are both nonpositive, as opposed to negative for a trapped surface).

3. The Kerr metric is given by

$$\begin{aligned} ds^2 = &-\left(1 - \frac{2Mr}{\rho^2}\right)dt^2 - \frac{4Mar\sin^2\theta}{\rho^2}dtd\varphi + \frac{\rho^2}{\Delta}dr^2 \\ &+ \rho^2 d\theta^2 + \left(r^2 + a^2 + \frac{2Ma^2 r\sin^2\theta}{\rho^2}\right)\sin^2\theta d\varphi^2, \end{aligned}$$

where

$$\begin{aligned} \rho^2 &= r^2 + a^2\cos^2\theta, \\ \Delta &= r^2 - 2Mr + a^2, \end{aligned}$$

and $M, a \in \mathbb{R}$ are constants. Consider the region $r \gg M, a$.

(2/20) (a) Show that a positive orthonormal coframe is approximately given in this region by

$$\omega^0 \sim dt, \qquad \omega^r \sim dr, \qquad \omega^\theta \sim r d\theta,$$
$$\omega^\varphi \sim r \sin\theta d\varphi - \frac{2Ma\sin\theta}{r^2} dt.$$

(2/20) (b) Setting $X = \dfrac{\partial}{\partial t}$ and $Y = \dfrac{\partial}{\partial \varphi}$, establish the following asymptotic formulas:

$$X^\flat \sim -\left(1 - \frac{2M}{r}\right) dt - \frac{2Ma\sin^2\theta}{r} d\varphi;$$
$$Y^\flat \sim -\frac{2Ma\sin^2\theta}{r} dt + r^2 \sin^2\theta d\varphi;$$
$$dX^\flat \sim \frac{2M}{r^2} \omega^0 \wedge \omega^r + \dots ;$$
$$dY^\flat \sim -\frac{6Ma\sin^2\theta}{r^2} \omega^0 \wedge \omega^r + \dots .$$

(2/20) (c) Prove that the Komar mass and the Komar angular momentum of the Kerr solution are given by $M_{\text{Komar}} = M$ and $J_{\text{Komar}} = Ma$.