Mathematical Relativity

2021/2022

Mock Exam – 2 hours

1. Let (M, g) be a time-oriented spacetime. For each of the following statements, give a proof if true or a counter-example if false.

(2/20) (a) $I^+(p)$ is an open set for any $p \in M$.

(2/20) (b) $J^+(p)$ is a closed set for any $p \in M$.

(2/20) (c) If (M, g) is globally hyperbolic and $S \subset M$ is closed then $D^+(S)$ is also closed.

(2/20) (d) If for every $p, q \in M$ such that $q \in I^+(p)$ there exists a maximizing timelike curve connecting p and q then (M, g) is globally hyperbolic.

2. Let (S, h) be the Riemannian manifold given by $S = \mathbb{R} \times S^2$ and

$$h = dx^2 + d\theta^2 + \sin^2\theta d\varphi^2$$

(where (θ, φ) are the usual spherical coordinates on S^2), and let

$$K = -dx^2 + d\theta^2 + \sin^2\theta d\varphi^2.$$

(3/20) (a) Show that (S, h, K) is an initial data set for the vacuum Einstein equations.

(3/20) (b) Prove that the corresponding maximal globally hyperbolic development is geodesically incomplete.

3. Let (M, g) be an expanding spatially flat FLRW model, that is, $M = \mathbb{R}^+ \times \mathbb{R}^3$ and

$$g = -dt^2 + a^2(t)\left(dx^2 + dy^2 + dz^2\right)$$

with $\dot{a}(t) > 0$ for all $t \in \mathbb{R}^+$. Given $t_0 > 0$, consider the following Cauchy problem for the wave equation on (M, g):

$$\begin{cases} \Box\phi = 0 \\ \phi(t_0, x, y, z) = \phi_0(x, y, z) \\ \partial_t\phi(t_0, x, y, z) = \phi_1(x, y, z) \end{cases}$$

(where $\phi_0, \phi_1 : \mathbb{R}^3 \to \mathbb{R}$ are smooth functions).

(3/20) (a) Show that if $T_{\mu\nu} = \partial_\mu\phi\,\partial_\nu\phi - \frac{1}{2}\partial_\alpha\phi\,\partial^\alpha\phi\,g_{\mu\nu}$ is the energy-momentum tensor associated to the wave equation and $X = \frac{1}{a(t)}\frac{\partial}{\partial t}$ then the current $Y_\mu = T_{\mu\nu}X^\nu$ satisfies $\nabla_\mu Y^\mu \geq 0$.

(3/20) (b) Use the energy method to prove that any two smooth solutions of the Cauchy problem whose initial data coincide on a ball $B \subset \mathbb{R}^3$ are identical on $D^+(\{t_0\} \times B)$.