Mathematical Relativity

Homework 8

Due on April 20

1. Let (M, g) be the globally hyperbolic Lorentzian manifold corresponding to the exterior region of the Schwarzschild solution, that is, $M = \mathbb{R} \times \left(\mathbb{R}^3 \setminus \overline{B_{2m}(0)}\right)$ and

$$g = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right)$$

(with $m > 0$).

(a) Show that for any $r_0 > 2m$ the curve

$$c(t) = \left(t, r_0, \frac{\pi}{2}, \sqrt{\frac{m}{{r_0}^3}}t\right)$$

is a timelike, null or spacelike geodesic, according to whether $r_0 > 3m$, $r_0 = 3m$ or $r_0 < 3m$.

(b) Argue that the point $q = \left(\pi\sqrt{\frac{{r_0}^3}{m}}, r_0, \frac{\pi}{2}, \pi\right)$ is conjugate to the point $p = \left(0, r_0, \frac{\pi}{2}, 0\right)$ along c (note that this can be done without solving the Jacobi equation).

(c) Show explicitly that if $r_0 > 3m$ then c stops being maximizing for $t > \pi\sqrt{\frac{{r_0}^3}{m}}$.

2. Use ideas similar to those leading to the proof Hawking's singularity theorem to prove **Myers's theorem**: if $(M, \langle\cdot,\cdot\rangle)$ is a complete Riemannian manifold whose Ricci curvature satisfies $Ric(X, X) \geq \varepsilon\langle X, X\rangle$ for some $\varepsilon > 0$ then M is compact. Can these ideas be used to prove a singularity theorem in Riemannian geometry?

3. Explain why Hawking's singularity theorem does not apply to each of the following geodesically complete Lorentzian manifolds:

(a) Minkowski's spacetime;

(b) Einstein's universe;

(c) de Sitter's universe;

(d) Anti-de Sitter spacetime.