Mathematical Relativity

Homework 6

Due on March 30

1. Consider the 3-dimensional Minkowski spacetime $(\mathbb{R}^3, g)$, where

   $$g = -dt \otimes dt + dx \otimes dx + dy \otimes dy.$$

   Let $c : \mathbb{R} \to \mathbb{R}^3$ be the curve $c(t) = (t, \cos t, \sin t)$. Show that although $\dot{c}(t)$ is null for all $t \in \mathbb{R}$ we have $c(t) \in I^+(c(0))$ for all $t > 0$. What kind of motion does this curve represent?

2. Let (M, g) be a stably causal Lorentzian manifold and h an arbitrary symmetric $(2,0)$-tensor field with compact support. Show that for sufficiently small $\varepsilon > 0$ the tensor field $g_\varepsilon := g + \varepsilon h$ is still a Lorentzian metric on M, and (M, g_ε) satisfies the chronology condition.

3. Let (M, g) be the quotient of the 2-dimensional Minkowski spacetime by the discrete group of isometries generated by the map $f(t, x) = (t + 1, x + 1)$. Show that (M, g) satisfies the chronology condition, but there exist arbitrarily small perturbations of (M, g) (in the sense of the exercise above) which do not.

4. Let (Σ, h) be an n-dimensional Riemannian manifold. Show that the Lorentzian manifold $(M, g) = (\mathbb{R} \times \Sigma, -dt \otimes dt + h)$ is globally hyperbolic with time function t if and only if (Σ, h) is complete.