Mathematical Relativity

Homework 4

Due on March 16

Consider again the spherically symmetric Lorentzian metric given by

$$g = -(A(t,r))^2 dt^2 + (B(t,r))^2 dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2,$$

where A and B are positive smooth functions. Recall that in the ortonormal frame dual to

$$\omega^0 = A dt, \qquad \omega^r = B dr, \qquad \omega^\theta = r d\theta, \qquad \omega^\varphi = r \sin\theta d\varphi,$$

the Einstein tensor has components (using the notation $\dot{} = \frac{\partial}{\partial t}$ and $' = \frac{\partial}{\partial r}$)

$$\begin{aligned}
G_{00} &= \frac{2B'}{rB^3} + \frac{B^2-1}{r^2B^2} = \frac{2m'}{r^2};\\
G_{0r} &= G_{r0} = \frac{2\dot{B}}{rAB^2};\\
G_{rr} &= \frac{2A'}{rAB^2} - \frac{B^2-1}{r^2B^2};\\
G_{\theta\theta} &= G_{\varphi\varphi} = \frac{A''B - A'B'}{AB^3} + \frac{\dot{A}\dot{B} - A\ddot{B}}{A^3B} + \frac{A'}{rAB^2} - \frac{B'}{rB^3},
\end{aligned}$$

where

$$B(t,r) = \left(1 - \frac{2m(t,r)}{r}\right)^{-\frac{1}{2}}.$$

1. Assuming

   - $G_{0r} = 0$ (so that B, and hence m, do not depend on t);
   - $G_{00} + G_{rr} = 0$ (so that $A = \frac{\alpha(t)}{B}$ for some positive smooth function $\alpha(t)$);
   - $\alpha(t) = 1$ (which can always be achieved by rescaling t),

   show that

   $$G_{\theta\theta} = G_{\varphi\varphi} = \frac{1}{2}\left(A^2\right)'' + \frac{1}{r}\left(A^2\right)'.$$

2. Prove that the general spherically symmetric solution of the vacuum Einstein field equations with a cosmological constant Λ is the **Kottler metric**

   $$g = -\left(1 - \frac{2M}{r} - \frac{\Lambda}{3}r^2\right) dt^2 + \left(1 - \frac{2M}{r} - \frac{\Lambda}{3}r^2\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2.$$

3. Obtain the Penrose diagram for the maximal extension of the Kottler solution with $\Lambda > 0$ and $0 < M < \frac{1}{3\sqrt{\Lambda}}$.

4. Consider now the spherically symmetric electromagnetic field

$$F = E(t,r)\,\omega^r \wedge \omega^0.$$

Show that this field satisfies the vacuum Maxwell equations

$$dF = d^\star F = 0$$

in and only if

$$E(t,r) = \frac{e}{r^2},$$

where the constant $e \in \mathbb{R}$ is the electric charge in units where $4\pi\varepsilon_0 = 1$. (Here $\star$ is the Hodge star; you will need to use ${}^\star\omega^r \wedge \omega^0 = \omega^\theta \wedge \omega^\varphi$).

5. As we shall see, this electromagnetic field corresponds to the energy-momentum tensor

$$T = \frac{E^2}{8\pi}\left(\omega^0 \otimes \omega^0 - \omega^r \otimes \omega^r + \omega^\theta \otimes \omega^\theta + \omega^\varphi \otimes \omega^\varphi\right).$$

Prove that the general spherically symmetric solution of the Einstein field equations with an electromagnetic field of this kind is the **Reissner-Nordström metric**

$$g = -\left(1 - \frac{2M}{r} + \frac{e^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{e^2}{r^2}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2.$$

6. Obtain the Penrose diagram for the maximal extension of the Reissner-Nordström solution with $M > 0$ and $0 < e^2 < M^2$.