Mathematical Relativity

Homework 2

Due on March 9

Consider the spherically symmetric Lorentzian metric

$$ds^2 = -dt^2 + a^2(t)\left(\frac{1}{1-kr^2}dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\varphi^2\right),$$

where a is a postive smooth function.

1. Use the condition of compatibility with the metric and Cartan's first structure equations,

$$\begin{cases} \omega_{\mu\nu} = -\omega_{\nu\mu} \\ d\omega^\mu + \omega^\mu{}_\nu \wedge \omega^\nu = 0 \end{cases}$$

to show that the nonvanishing connection forms for the ortonormal frame dual to

$$\omega^0 = dt, \qquad \omega^r = a(t)\left(1-kr^2\right)^{-\frac{1}{2}}dr, \qquad \omega^\theta = a(t)rd\theta, \qquad \omega^\varphi = a(t)r\sin\theta d\varphi$$

are

$$\begin{aligned}
&\omega^0{}_r = \omega^r{}_0 = \dot{a}\left(1-kr^2\right)^{-\frac{1}{2}}dr;\\
&\omega^0{}_\theta = \omega^\theta{}_0 = \dot{a}rd\theta;\\
&\omega^0{}_\varphi = \omega^\varphi{}_0 = \dot{a}r\sin\theta d\varphi;\\
&\omega^\theta{}_r = -\omega^r{}_\theta = \left(1-kr^2\right)^{\frac{1}{2}}d\theta;\\
&\omega^\varphi{}_r = -\omega^r{}_\varphi = \left(1-kr^2\right)^{\frac{1}{2}}\sin\theta d\varphi;\\
&\omega^\varphi{}_\theta = -\omega^\theta{}_\varphi = \cos\theta d\varphi.
\end{aligned}$$

2. Use Cartan's second structure equations

$$\Omega^\mu{}_\nu = d\omega^\mu{}_\nu + \omega^\mu{}_\alpha \wedge \omega^\alpha{}_\nu$$

to show that the curvature forms on this frame are

$$\begin{aligned}
&\Omega^0{}_r = \Omega^r{}_0 = \frac{\ddot{a}}{a}\omega^0 \wedge \omega^r;\\
&\Omega^0{}_\theta = \Omega^\theta{}_0 = \frac{\ddot{a}}{a}\omega^0 \wedge \omega^\theta;\\
&\Omega^0{}_\varphi = \Omega^\varphi{}_0 = \frac{\ddot{a}}{a}\omega^0 \wedge \omega^\varphi;\\
&\Omega^\theta{}_r = -\Omega^r{}_\theta = \left(\frac{k}{a^2} + \frac{\dot{a}^2}{a^2}\right)\omega^\theta \wedge \omega^r;\\
&\Omega^\varphi{}_r = -\Omega^r{}_\varphi = \left(\frac{k}{a^2} + \frac{\dot{a}^2}{a^2}\right)\omega^\varphi \wedge \omega^r;\\
&\Omega^\varphi{}_\theta = -\Omega^\theta{}_\varphi = \left(\frac{k}{a^2} + \frac{\dot{a}^2}{a^2}\right)\omega^\varphi \wedge \omega^\theta.
\end{aligned}$$

3. Using

$$\Omega^\mu{}_\nu = \sum_{\alpha<\beta} R_{\alpha\beta}{}^\mu{}_\nu \omega^\alpha \wedge \omega^\beta$$

determine the components $R_{\alpha\beta}{}^\mu{}_\nu$ of the curvature tensor on this orthonormal frame, and show that the nonvanishing components of the Ricci tensor on this frame are

$$R_{00} = -\frac{3\ddot{a}}{a};$$
$$R_{rr} = R_{\theta\theta} = R_{\varphi\varphi} = \frac{\ddot{a}}{a} + \frac{2\dot{a}^2}{a^2} + \frac{2k}{a^2}.$$

Conclude that the nonvanishing components of the Einstein tensor on this frame are

$$G_{00} = \frac{3\dot{a}^2}{a^2} + \frac{3k}{a^2};$$
$$G_{rr} = G_{\theta\theta} = G_{\varphi\varphi} = -\frac{2\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - \frac{k}{a^2}.$$

4. Show that the Einstein equations with a cosmological constant Λ for a comoving pressureless perfect fluid of nonnegative density ρ, $G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi\rho\, dt^2$, are equivalent to the system

$$\begin{cases} \dfrac{\dot{a}^2}{a^2} + \dfrac{k}{a^2} = \dfrac{8\pi\rho}{3} + \dfrac{\Lambda}{3} \\ \\ \dfrac{2\ddot{a}}{a} + \dfrac{\dot{a}^2}{a^2} + \dfrac{k}{a^2} = \Lambda \end{cases}$$

Show that this system can be integrated to

$$\begin{cases} \dfrac{4\pi\rho}{3} a^3 = \alpha \\ \\ \dfrac{1}{2}\dot{a}^2 - \dfrac{\alpha}{a} - \dfrac{\Lambda}{6} a^2 = -\dfrac{k}{2} \end{cases}$$

where α is a nonnegative integration constant.

5. Draw the Penrose diagram of the solutions with $\alpha > 0$, $\Lambda > 0$ and $k = 0$ (currently believed to model the physical universe).