Mathematical Relativity

Homework 2

Due on March 2

Consider the spherically symmetric Lorentzian metric given by

$$g = -(A(t,r))^2 dt^2 + (B(t,r))^2 dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2,$$

where A and B are positive smooth functions.

1. Use the condition of compatibility with the metric and Cartan's first structure equations,

$$\begin{cases} \omega_{\mu\nu} = -\omega_{\nu\mu} \\ d\omega^\mu + \omega^\mu{}_\nu \wedge \omega^\nu = 0 \end{cases}$$

   to show that the nonvanishing connection forms for the ortonormal frame dual to

$$\omega^0 = Adt, \qquad \omega^r = Bdr, \qquad \omega^\theta = rd\theta, \qquad \omega^\varphi = r\sin\theta d\varphi$$

   are (using the notation $\dot{} = \frac{\partial}{\partial t}$ and $' = \frac{\partial}{\partial r}$)

$$\begin{aligned} \omega^0{}_r &= \omega^r{}_0 = \frac{A'}{B}dt + \frac{\dot B}{A}dr; \\ \omega^\theta{}_r &= -\omega^r{}_\theta = \frac{1}{B}d\theta; \\ \omega^\varphi{}_r &= -\omega^r{}_\varphi = \frac{\sin\theta}{B}d\varphi; \\ \omega^\varphi{}_\theta &= -\omega^\theta{}_\varphi = \cos\theta d\varphi. \end{aligned}$$

2. Use Cartan's second structure equations

$$\Omega^\mu{}_\nu = d\omega^\mu{}_\nu + \omega^\mu{}_\alpha \wedge \omega^\alpha{}_\nu$$

   to show that the curvature forms on this frame are

$$\begin{aligned} \Omega^0{}_r &= \Omega^r{}_0 = \left(\frac{A''B - A'B'}{AB^3} + \frac{\dot A \dot B - A\ddot B}{A^3 B} \right) \omega^r \wedge \omega^0; \\ \Omega^0{}_\theta &= \Omega^\theta{}_0 = \frac{A'}{rAB^2}\,\omega^\theta \wedge \omega^0 + \frac{\dot B}{rAB^2}\,\omega^\theta \wedge \omega^r; \\ \Omega^0{}_\varphi &= \Omega^\varphi{}_0 = \frac{A'}{rAB^2}\,\omega^\varphi \wedge \omega^0 + \frac{\dot B}{rAB^2}\,\omega^\varphi \wedge \omega^r; \\ \Omega^\theta{}_r &= -\Omega^r{}_\theta = \frac{B'}{rB^3}\,\omega^\theta \wedge \omega^r + \frac{\dot B}{rAB^2}\,\omega^\theta \wedge \omega^0; \\ \Omega^\varphi{}_r &= -\Omega^r{}_\varphi = \frac{B'}{rB^3}\,\omega^\varphi \wedge \omega^r + \frac{\dot B}{rAB^2}\,\omega^\varphi \wedge \omega^0; \\ \Omega^\varphi{}_\theta &= -\Omega^\theta{}_\varphi = \frac{B^2-1}{r^2B^2}\,\omega^\varphi \wedge \omega^\theta. \end{aligned}$$

3. Using

$$\Omega^\mu{}_\nu = \sum_{\alpha<\beta} R_{\alpha\beta}{}^\mu{}_\nu \omega^\alpha \wedge \omega^\beta$$

determine the components $R_{\alpha\beta}{}^\mu{}_\nu$ of the curvature tensor on this orthonormal frame, and show that the nonvanishing components of the Ricci tensor on this frame are

$$\begin{aligned}
R_{00} &= \frac{A''B - A'B'}{AB^3} + \frac{\dot{A}\dot{B} - A\ddot{B}}{A^3B} + \frac{2A'}{rAB^2}; \\
R_{0r} &= R_{r0} = \frac{2\dot{B}}{rAB^2}; \\
R_{rr} &= \frac{A'B' - A''B}{AB^3} + \frac{A\ddot{B} - \dot{A}\dot{B}}{A^3B} + \frac{2B'}{rB^3}; \\
R_{\theta\theta} &= R_{\varphi\varphi} = -\frac{A'}{rAB^2} + \frac{B'}{rB^3} + \frac{B^2-1}{r^2B^2}.
\end{aligned}$$

Conclude that the nonvanishing components of the Einstein tensor on this frame are

$$\begin{aligned}
G_{00} &= \frac{2B'}{rB^3} + \frac{B^2-1}{r^2B^2}; \\
G_{0r} &= G_{r0} = \frac{2\dot{B}}{rAB^2}; \\
G_{rr} &= \frac{2A'}{rAB^2} - \frac{B^2-1}{r^2B^2}; \\
G_{\theta\theta} &= G_{\varphi\varphi} = \frac{A''B - A'B'}{AB^3} + \frac{\dot{A}\dot{B} - A\ddot{B}}{A^3B} + \frac{A'}{rAB^2} - \frac{B'}{rB^3}.
\end{aligned}$$

4. Show that if we write

$$B(t,r) = \left(1 - \frac{2m(t,r)}{r}\right)^{-\frac{1}{2}}$$

for some smooth function m then

$$G_{00} = \frac{2m'}{r^2}.$$

Conclude that the vacuum Einstein equations $G_{00} = G_{0r} = 0$ are equivalent to

$$B = \left(1 - \frac{2M}{r}\right)^{-\frac{1}{2}},$$

where $M \in \mathbb{R}$ is an integration constant.

5. Show that the vacuum equation $G_{00} + G_{rr} = 0$ is equivalent to $A = \frac{\alpha(t)}{B}$ for some positive smooth function $\alpha(t)$.

6. Check that if A and B are as above then the remaining vacuum equation $G_{\theta\theta} = G_{\varphi\varphi} = 0$ is automatically satisfied.

7. Argue that it is always possible to rescale the coordinate t so that the any metric of the given form satisfying the vacuum Einstein field equations is written

$$g = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2.$$