Mathematical Relativity

Homework 13

Due on May 25

1. (a) Check that the Klein-Gordon equation in Minkowski spacetime, $\Box\phi - m^2\phi = 0$, can be derived from the Lagrangian density

$$\mathcal{L} = -\frac{1}{2}\left(\partial_\mu\phi\,\partial^\mu\phi + m^2\phi^2\right).$$

   (b) Show that the corresponding Hamiltonian is

$$H = \int_{\mathbb{R}^n} \frac{1}{2}\left((\partial_0\phi)^2 + \ldots + (\partial_n\phi)^2 + m^2\phi^2\right)\, dx^1 \wedge \ldots \wedge dx^n.$$

   (c) Starting with the Einstein-Hilbert-Klein-Gordon action

$$S = \int_M \left[R - 8\pi\left(g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi + m^2\phi^2\right)\right] dV_{n+1}$$

   obtain the energy-momentum tensor for ϕ:

$$T_{\mu\nu} = \partial_\mu\phi\,\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}\left(\partial_\alpha\phi\,\partial^\alpha\phi + m^2\phi^2\right).$$

   (d) Check that T_{00} coincides with the Hamiltonian density $\mathcal{H}$ (i.e. the integrand in the expression for the Hamiltonian H).

2. Let γ be the spherically symmetric Riemannian metric defined in $\mathbb{R}^3$ by

$$\gamma = \frac{dr^2}{1 - \frac{2m(r)}{r}} + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right),$$

where m is a smooth function whose derivative has compact support.

   (a) Check that in Cartesian coordinates we have

$$\gamma_{ij} = \delta_{ij} + \frac{\frac{2m(r)}{r^3}}{1 - \frac{2m(r)}{r}}x^i x^j.$$

   (b) Show that if the limit

$$M = \lim_{r\to\infty} m(r)$$

   exists then γ is asymptotically flat with ADM mass M (which in particular coincides with the Komar mass when appropriate).

(c) Check that γ has scalar curvature

$$R = \frac{4}{r^2}\frac{dm}{dr},$$

and use this to prove the Riemannian Positive Mass Theorem for γ.

(d) Show that $r = r_0$ is a minimal surface if and only if $m(r_0) = \frac{r_0}{2}$ (in which case r is a well-defined coordinate only for $r > r_0$), and use this to prove the Riemannian Penrose Conjecture for γ.